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OPTIMAL TIME-INVARIANT MONETARY POLICY

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Abstract: This paper investigates how best to determine time-invariant policy rules in macroeconomic models with forward-looking constraints, where fully optimal policy is known to be time-inconsistent. It proposes a new ‘coefficient optimisation’ approach that improves upon the timeless perspective method of Woodford (2003) in deterministic problems, and on average in stochastic problems, without resorting to asymptotic (‘unconditional’) loss comparisons.

1 I thank Simon Wren-Lewis for many useful discussions relating to this paper. Any remaining errors are my own.
1. Introduction

The time-inconsistency of optimal policy in environments characterised by forward-looking constraints was first brought to general attention by Kydland and Prescott (1977). These authors also highlighted the losses that would result if, in such settings, policy was determined by a ‘discretionary’ method: seeking the best strategy in each time period for given expectations of future policy, and then assuming expectations of an identical policy approach in the future. The twin observations led Kydland and Prescott to advocate a new approach to policymaking in forward-looking environments, the essence of which is captured in the title of their paper: ‘Rules rather than discretion’. If the best policy possible is time-inconsistent, whilst the (time-invariant) discretionary solution holds no natural claim to optimality, the natural response is to seek the policy that is, in some meaningful sense, best among the class of time-invariant ‘rules’.

In the past decade the question of optimal rules has motivated much investigation, notably on the part of Michael Woodford and co-authors. Woodford (1999) was first to advocate a new solution method to derive time-invariant policy in a forward-looking environment, the result of which could be considered a policy ‘rule’ in the sense intended by Kydland and Prescott. Woodford dubbed this method ‘policymaking from a timeless perspective’, and it has since become a standard method for deriving stationary policy solutions in forward-looking models.\(^2\) Loosely speaking, the timeless perspective approach exploits the fact that the time-inconsistent, fully optimal policy, which finds the best state-contingent path for target variables evaluated from a particular point in time onwards, becomes observationally equivalent to a time-invariant rule at horizons long after the initial period. This ‘limiting’ rule is then recommended as an alternative time-invariant solution to discretion, under the assumption that some credible mechanism exists by which the policymaker can bind its future self to act accordingly.

However, a central observation of this paper is that the timeless perspective approach generally yields sub-optimal outcomes among the set of time-invariant rules. The logic is very similar to that accounting for the sub-optimality of discretion. Both the discretionary and timeless perspective methods involve finding the best policy at a given point in time under the assumption of certain restrictions on present choices over target variables. But these restrictions are then determined endogenously by solving a fixed-point mapping between restrictions and chosen policy. In the case of discretion, the mapping is directly between current and expected future policy; in the case of the timeless perspective, it is between a (self-imposed) constraint generated to bind the next period’s policy, and a similar constraint assumed to bind today. We are able to show that the timeless

\(^2\) See, for instance, Sheedy (2007) for a recent application of the approach.
These arguments are presented formally below, along with a new approach to determining optimal time-invariant rules that follows from tackling directly the question: ‘What is the best policy to follow given that the *same* policy must be implemented at every point in the future?’ The starting point in this approach is that the set of rules should be suitably restricted to ensure time invariance *prior* to choice, not subsequent to it. The recommended policy does not generally coincide with the principal alternative to the timeless perspective in the literature, dubbed ‘unconditionally optimal’ policy by Damjanovic et al. (2008). This solution method delivers a stationary rule that minimises the value of loss assessed ‘asymptotically’ – that is, evaluated in *ex-ante* expectation assuming that the rule has been implemented for many periods. As discussed below, the most significant problem with this is its requirement that lagged endogenous variables (including any featuring in *structural* constraints) are treated as if their entire distribution is endogenous to the chosen policy. This feature ensures the approach fails to deal with ‘backward-looking’ constraints in a manner consistent with traditional optimal control methods, so can only be embraced if one simultaneously rejects established optimisation procedures even where time-inconsistency is not an issue. In contrast, our approach avoids dependence on lagged endogenous variables where this dependence is not already implied by the model’s structural equations, but can treat any initial conditions that *are* structurally relevant as given when solving for an optimal rule.

The rest of the paper is organised as follows. In section 2 we consider the derivation of timeless perspective rules, and illustrate the operational similarity between this procedure and the derivation of discretionary policy. In section 3 we present a general approach for deriving optimal time-invariant rules, stipulating two restrictions we believe ought to be placed on the set of rules considered, and two criteria that the procedure for choosing among the remaining rules ought to satisfy. In section 4 we implement this approach in three simple monetary policy models. Section 5 concludes.

2. A general problem

To illustrate the basic time-inconsistency problem and analyse existing stationary solution methods (defined as methods that ensure current endogenous variables can always be expressed as time-
invariant functions of lagged endogenous variables and lagged and current exogenous processes), we first focus on a problem with a very general structure as follows.

A policymaker is assumed to be pursuing a social objective (in discrete time), characterised by a loss function defined over the current and future values of endogenous (‘target’) variables and exogenous shocks; we assume geometric discounting and a time-separable objective function, so the objective at time \( t \) can be written as:

\[
\min_{x_{t+i}} E_t \sum_{i=0}^{\infty} \beta^i \pi_{t+i}(x_{t+i}, \varepsilon_{t+i}),
\]

(2.1)

where \( x_{t+i} \) is a \( k \times 1 \) vector of target variables realised at time \( t + i \), \( \varepsilon_{t+i} \) an \( l \times 1 \) vector of shocks that impact on the model’s structural equations at \( t + i \), the parameter \( \beta \) satisfies \( 0 < \beta < 1 \), and the period objective function \( \pi_{t+i} \) can be assumed continuously differentiable as necessary. If the model is derived from the behaviour of a representative agent, then it would be natural to invoke the Pareto criterion and base the \( \pi_{t+i} \) function on the expected (period) utility of this representative agent at time \( t + i \) as a function of \( x_{t+i} \) and \( \varepsilon_{t+i} \), following Rotemberg and Woodford (1998). In more general models that admit agent heterogeneity, stronger normative assumptions (such as utilitarianism) would usually be needed to allow outcomes that are not Pareto comparable to be ordered by the objective function in (2.1).

In solving (2.1), the policymaker is restricted by an \( m \times 1 \) vector of forward-looking constraints of the form:

\[
E_{t+i} g(x_{t+i}, \varepsilon_{t+i}, x_{t+i+1}) = 0,
\]

(2.2)

which must hold in every time period. An example of such a constraint, examined in detail in section 4, is the New Keynesian Phillips Curve that follows from Calvo pricing behaviour under monopolistic competition. We let \( m < k \) hold, so the policymaker is free to choose in at least one dimension. Note that it would be possible to replace some of the constraints in (2.2) with backward-looking restrictions of the form: \( f(x_{t+i}, \varepsilon_{t+i}; x_{t+i-1}) = 0 \), but doing so at this stage adds nothing to the general argument.4

As detailed by Kydland and Prescott (1977), a discretionary policymaker is unable to affect the policy choice made at time periods beyond the present, \( t \), so these future choices must be treated as given. The structural equations (2.2) and objective function (2.1) together imply that the state of the economy at any time \( t + i \) can be fully characterised by the vector of exogenous shocks \( \varepsilon_{t+i} \),5 so it

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4 Optimal rules under backward- and forward-looking constraints are considered in section 3.

5 Without loss of generality, this vector can be assumed to include any leading indicators of future shocks.
follows that the optimal discretionary policy choice made at some time \( t + i \) in the future will allow us to map from \( \varepsilon_{t+i} \) to \( x_{t+i} \) via a correspondence of the form:

\[
x_{t+i} = \psi(\varepsilon_{t+i}) \quad i \geq 1
\] (2.3)

(Note in particular that the absence of lagged target variables in (2.1) and (2.2) ensures the optimal discretionary choice cannot depend on \( x_{t+i-1} \).)

The exact form of the correspondence \( \psi \) remains to be inferred (its time-invariance reflects the stationary nature of the problem), but (2.3) allows us to capture the fact that future decisions must be treated as given by the current policymaker. Substituting this expression into (2.1) and (2.2), and writing the problem in Lagrangian form, the objective becomes:

\[
\min_{x_t} \left\{ \pi_t(x_t, \varepsilon_t) + \lambda_{t+i}'E_t g(x_t, \varepsilon_t, \psi(\varepsilon_{t+i})) + E_t \sum_{i=1}^\infty \beta^i \left[ \pi_{t+i}(\psi(\varepsilon_{t+i}), \varepsilon_{t+i}) + \lambda_{t+i}' E_{t+i} g(\psi(\varepsilon_{t+i}), \varepsilon_t, \psi(\varepsilon_{t+i+1})) \right] \right\},
\] (2.4)

where \( \lambda_{t+i}' \) is a \( 1 \times m \) vector of current-value Lagrange multipliers corresponding to the constraint vector (2.2) at time \( t + i \). Since, by assumption, \( \varepsilon_t \) contains all information relevant to the time-\( t \) prediction of \( \varepsilon_{t+i} \) (\( i \geq 1 \)), and assuming the problem has a unique solution, the vector \( x_t \) that solves (2.4) can be written as:

\[
x_t = \phi(\varepsilon_t; \psi(\cdot))
\] (2.5)

The notation here is intended to make explicit that the optimal solution will in general depend upon expected future policy – since the forward-looking constraints (2.2) mean expected future policy affects what is feasible in time \( t \). Following the approach of Kydland and Prescott, we then note from the stationarity of the problem that current expectations must be consistent with an identical policy solution being applied in all future periods; hence we have a fixed-point problem whose solution requires a mapping \( \psi^* \) that satisfies:

\[
\psi^*(\varepsilon_{t+i}) = \phi(\varepsilon_{t+i}; \psi^*(\cdot))
\] (2.6)

The resulting (stationary) correspondence \( \psi^* \) fully characterises the state-contingent evolution of the target variables when the policymaker acts with discretion.

As is well known, this discretionary solution is generally sub-optimal even among the set of stationary policy rules, and can be improved upon if the policymaker has some way to commit to a future policy strategy. We re-state the intuition behind this result here, as it is informative when attention turns to the timeless perspective method. Specifically, problem (2.4) requires the
policymaker to choose the best policy for the current period under the assumption that policy in all future periods cannot be affected by this current choice. Yet once the corresponding optimum is found, we impose the restriction (2.6) that whatever has been chosen in the current period will also be chosen in the future. So (mathematically) the current policy choice directly influences the expected value of $x_{t+i}$ (for all $i \geq 1$). Hence it also directly affects the set of policies feasible at time $t$, through the constraints in (2.2). But any marginal effects operating through this channel are not considered when determining $\phi$, which follows from taking first-order conditions with the expectation of functions of $x_{t+i}$ treated as exogenous (for $i \geq 1$). Consequently, $\phi$ will generally not be set such that the full marginal gains and losses from changing policy are equated with each other; there will remain scope for policy to be improved.

As noted above, when we do allow the policymaker to commit to a full (state-contingent) future action path, the optimal solution generally proves time-inconsistent. Writing the full problem in Lagrangian form, with no assumptions on the determination of future target variables, we have:

$$\min_{(x_{t+i})} E_t \sum_{i=0}^{\infty} \beta^i [\Pi_{t+i}(x_{t+i}, \epsilon_{t+i}) + \lambda_{t+i}' E_{t+i} g(x_{t+i}, \epsilon_{t+i}, x_{t+i+1})],$$

(2.7)

where once more we use $\lambda_{t+i}'$ as a $1 \times m$ vector of current-value Lagrange multipliers corresponding to constraint vector (2.2) at time $t + i$.

To address problems of the form (2.7), Marcet and Marimon (1999) develop a value function method in the spirit of Bellman’s (1962) theory of dynamic programming; equivalence between this function and (2.7) is established for problems with sufficient regularity. The authors demonstrate that the optimal vector of endogenous variables under a commitment path from some initial period $t$ can always be determined by a (time-invariant) correspondence defined on the current shock vector and the lagged vector of Lagrange multipliers; the state-contingent evolution of the multipliers is chosen to ensure optimality from the perspective of time $t$. It follows that one must only augment the vector of current exogenous shocks with the vector of lagged Lagrange multipliers to obtain the entire set of state variables required to characterise the optimal policy choice at time $t + i$ (with optimality defined from the perspective of time $t$). We thus know that the optimal choice of $x_{t+i}$ will be determined by a ‘policy correspondence’ of the form:

$$x_{t+i} = \tilde{\delta}(\lambda_{t+i-1}, \epsilon_{t+i}) \quad i \geq 0$$

(2.8)

The Lagrange multipliers are generated by a recursion of the form:

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6 A non-convex constraint set can provide difficulties. See Marcet and Marimon (1999).

7 Marcet and Marimon also allow for backward-looking constraints; we abstract from these for now.
\[ \lambda_{t+i} = \tilde{\theta}(\lambda_{t+i-1}, \epsilon_{t+i}), \quad i \geq 0 \]  

(2.9)

for optimal correspondences \( \tilde{\theta}, \tilde{\phi} \) to be found. Yet together (2.8) and (2.9) cannot determine the \( x_{t+i} \) unless augmented by an initial value for \( \lambda_{t-1} \) – the vector of lagged Lagrange multipliers when the optimal policy path is first chosen. Since the objective (2.7) implies forward-looking constraints that had to hold at \( t - 1 \) have no impact on the optimal policy problem at time \( t \), the only value for \( \lambda_{t-1} \) consistent with optimising behaviour is an \( m \times 1 \) vector of zeroes. Recursive substitution of (2.8) into (2.9) will thus give an optimal policy a correspondence of the form:

\[ x_{t+i} = \vartheta(\{\epsilon_{t+i-j}\}_{j=0}^{i}) \]  

(2.10)

As is well known,\(^8\) the initial condition \( \lambda_{t-1} = 0 \) implies the fully optimal policy is time-inconsistent. In general, equation (2.9) (which determines the optimal state-contingent values of the Lagrange multipliers from the perspective of time \( t \)) will generate a non-zero value for \( \lambda_{t+i} \) (\( i \geq 0 \)), but the fully optimal solution from the perspective of time \( t + i + 1 \) will require the policymaker to set \( \lambda_{t+i} = 0 \). A corollary of this is that the optimal choice of \( x_{t+i} \) from the perspective of time \( t \), given in (2.10), itself depends on \( t \): regardless of the value of \( i \), \( x_{t+i} \) will be fully determined by the exogenous shock vectors from \( t \) to \( t + i \), and thus will depend on the time elapsed from the ‘reference point’ \( t \) to the current period. Hence the full-commitment policy (2.10) cannot be considered a time-invariant rule, and our motivating question remains: what is the optimal time-invariant rule?

2.1. Optimal policy when past commitments are valued

Before outlining the timeless perspective stationary solution, it is worth exploring a little further the value function method of Marcet and Marimon (1999). Under this approach, the full-commitment problem (2.7) is first reformulated to resemble a standard (‘backward-looking’) optimal control problem.\(^9\) To achieve this, one must (a) augment the period objective function with terms in the current Lagrange multipliers, to give value to adhering to past commitments, and (b) add a law of motion for the Lagrange multipliers. Given this reformulation, the authors are able to write an equivalent to the standard recursive Bellman equation, which they dub the ‘saddle point functional

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\(^8\) See, for instance, the detailed discussions in Woodford (2003) and Marcet and Marimon (1999).

\(^9\) This is dubbed the ‘saddle point problem’ by Marcet and Marimon. The terminology derives from the treatment of forward-looking inequality restrictions, whose presence ensures the objective function requires maximisation with respect to the target variables but minimisation with respect to the costates. For ease of exposition, the setup here neglects the possibility of slack constraints.
equation.\textsuperscript{10} Hence the problem seems to admit standard recursive methods of the type popularised by Ljungqvist and Sargent (2000), with the lagged Lagrange multipliers $\lambda_{t+i-1}$ and current shock vectors $\varepsilon_{t+i}$ sufficient to summarise the state of the economy at any point in time $t + i$. By construction, the solution to the reformulated problem will coincide with the optimal policy from the perspective of time $t$ (with the initial condition $\lambda_{t-1} = 0$ imposed).

An important implication is that (2.8) will still deliver the best choice of endogenous variables at $t + i$ for any given (‘inherited’) $\lambda_{t+i-1}$ under the reformulated objective (the use of which will implement the time-$t$ optimal path, provided $\lambda_{t-1} = 0$). Re-optimisation subject to the inherited costates delivers the same policy choice as would have followed from an optimal state-contingent plan devised once and for all at time $t$.\textsuperscript{11} Similarly, (2.9) will continue to describe the optimal evolution of the costates for any given lagged vector thereof – whatever happens to be the value of that lagged vector. Hence we can interpret the Marcet-Marimon saddle point functional equation as an augmented objective function accounting for a desire to bind oneself to past commitments, and seek the optimal set of target variables subject to this self-imposed restriction. Equations (2.8) and (2.9) characterise the constrained optimal choice that follows, and so provided $\lambda_{t-1}$ is still set to zero we simply have an alternative approach for deriving state-contingent choice rules that admits re-application in subsequent periods, and will implement the full-commitment optimum from the perspective of time $t$.

This insight is crucial in understanding the timeless perspective approach. As outlined in Woodford (2003) and elsewhere, the timeless perspective method draws directly on the full-commitment solution, but replaces the condition $\lambda_{t-1} = 0$ with a requirement that $\lambda_{t-1}$ should be set in a ‘self-consistent’ way. The precise meaning of a self-consistent constraint in this context is a matter for legitimate debate. Woodford (2003) admits a broad definition that makes for simpler policy prescriptions at the expense of some analytical clarity. Specifically, he lets the initial Lagrange multiplier be given by any mapping from lagged and current endogenous variables and shocks, $\lambda_{t-1} = \chi(\{x_{t-1-j}\}_{j=0}^{\infty}, \{\varepsilon_{t-1-j}\}_{j=0}^{\infty})$, that satisfies the restriction:

$$
\chi(\{x_{t-j}\}_{j=0}^{\infty}, \{\varepsilon_{t-j}\}_{j=0}^{\infty}) = \tilde{\alpha} \left( \chi(\{x_{t-1-j}\}_{j=0}^{\infty}, \{\varepsilon_{t-1-j}\}_{j=0}^{\infty}), \varepsilon_t \right)
$$

\textsuperscript{10} We do not directly apply Marcet and Marimon’s recursive method here, in common with most of the optimal monetary policy literature; but the insight provided by its derivation, and its equivalence with the methods that are more frequently used, is vital.

\textsuperscript{11} Contrast with the usual treatment of Lagrange multipliers as endogenous to the optimal decision process rather than constraints upon it. Since the lagged multipliers will ordinarily only feature in first-order conditions taken with respect to variables whose prior expectation affected outcomes in the previous period, imposing a non-zero vector is equivalent to binding oneself to value past commitments in a (state-contingent) manner that is optimal from the perspective of some time period in the past.
In words, this equation requires: ‘if some correspondence $\chi$ is used to determine the value afforded to (hypothetical) commitments that would have been beneficial at $t - 1$ (brought into the optimisation problem through $\lambda_{t-1}$) then the same correspondence, led one period, should determine the value of current commitments to the policymaker at $t + 1$ (i.e., $\lambda_t$) – given that $\bar{g}$ will determine the evolution of the costates under optimal policy’.

Marcet and Marimon’s formulation is important here, since it implies (2.9) will give the optimal choice of self-imposed constraint for $t + 1$, from the perspective of time $t$, for any given $\lambda_{t-1}$ vector (regardless how the $\lambda_{t-1}$ vector was constructed). Hence (2.11) imposes self-consistency in the sense: if the self-constrained policymaker inherits a particular $\lambda_{t-1}$ vector given by $\chi$, then it will be optimal to bequeath a costate vector governed by precisely the same correspondence.

Given the interdependency that will exist between the different elements of $\chi_t$ and of $\varepsilon_t$ when a timeless perspective policy is implemented, and the lack of restrictions yet imposed on $\chi$, (2.11) will not generally be enough to impose a unique structure on $\chi$; it is for this reason that Woodford (2003) claims the timeless perspective policy is not unique. But at any given point in time, different ‘timeless’ policies will have different welfare implications. Unless further criteria are added to constrain choice among these policies, the policymaker acting at time $t$ will presumably select the ‘best’, assessed under (2.1). Yet this choice will depend on the lagged dependent variables $\{x_{t-j}\}_{j=0}^{\infty}$, which have been determined by an unknown policy regime operating prior to time $t$; and the best timeless perspective rule to implement at $t$ certainly need not remain the best to implement in period $t + i$. In such a scenario, switching policies would imply time-inconsistent behaviour, whilst failing to switch would imply a bias in favour of time-$t$ welfare assessments over assessments at time $t + i$ – precisely the same bias that we are seeking to avoid in ruling out the full-commitment solution. The problem is inherent in failing to narrow the set of admissible policies to a unique element; without further restrictions on this set, the sense in which one could claim to have overcome the time-inconsistency problem is weak.

A second, related problem with the timeless perspective as presented is the apparent arbitrariness in permitting $\lambda_{t-1}$ to be linked to lagged endogenous variables generated by an unknown prior policy regime. Though dependence on these variables will fade through time, their presence prevents timeless perspective rules from being considered stationary in the statistical sense: the initial vector $x_t$ will have different moments to its subsequent realisations precisely because

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12 In terms of the costates, a particular policy might imply a vector $\lambda_{t-1}$ very close to zero but, once implemented for $i$ periods, a significantly non-zero $\lambda_{t+i-1}$; it is quite possible that another admissible policy will then dominate it at time $t + i$. 

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dependence on the past regime is not time-invariant. This provides difficulties in interpreting the timeless perspective criterion. Woodford (2003) motivates timeless perspective policymaking through the claim that it sets target variables in a way that would have been considered optimal at some point in the distant past. Clearly correspondences (2.8) and (2.9) remain necessary conditions for this interpretation to be valid – leaving only the initial choice of the lagged costate. But strictly there is only one choice of $\lambda_{t-1}$ that will satisfy Woodford’s definition; this can be derived through infinite recursive substitutions into (2.9), to express the lagged costate vector as a mapping from the entire history of shocks. We can express the resulting correspondence as $q(\{e_{t-1-j}\}_{j=0}^\infty)$. Satisfying the requirement that timeless perspective policymaking should implement actions that would have been optimal from the perspective of a time in the distant past then requires:

$$\lambda_{t-1} = q(\{e_{t-1-j}\}_{j=0}^\infty)$$

(2.12)

(Notice that $q$ must satisfy the definition of the $\chi$ correspondence, so is within the set of timeless perspective rules.)

If policy were tied to lagged endogenous variables generated by a prior policy regime, the Woodford justification ceases to hold. Optimal policy for a given $\lambda_{t-1}$ vector is fully characterised by correspondences (2.8) and (2.9), and there is a unique $\lambda_{t-1}$ vector associated with an optimal plan proceeding from each point in the past. If we allow the initial optimality perspective to be time period $t - \tau$, then as $\tau \to \infty$ the appropriate $\lambda_{t-1}$ will approach the vector generated by $q$ (provided this infinite recursion on (2.9) converges to a well-defined random vector). This is the only choice of $\lambda_{t-1}$ that implements policy that would have been optimal from a point in the distant past.

These arguments suggest that there is a natural way to refine the timeless perspective criterion to deliver a single stationary policy rule, which sets current policy purely as a function of the history of past disturbances and is uniquely true to Woodford’s interpretation of the approach. In what follows we refer to a rule satisfying (2.8), (2.9) and (2.12) as the ‘strong’ timeless perspective solution, and those satisfying (2.8), (2.9) and (2.11) as ‘weak’ timeless perspective rules.

2.2. Timeless perspective policy as the solution to a fixed-point mapping

Having clarified the approach, we can now present the substantive critique of timeless perspective policymaking that is central to this paper.\(^{14}\)

\(^{13}\) We must assume that the resulting expression converges to a stable random vector.

\(^{14}\) Numerous other authors have voiced criticism of the timeless perspective principle, but rarely does this engage directly with the methodology outlined in Woodford (2003), Blake (2001) and Jensen and McCallum.
As outlined above, equations (2.8) and (2.9) together define the best continuation policy under the assumption that past commitments are treated as binding, and so the lagged costate vector affects policy in the first period in precisely the same way it is intended (under an optimal plan) to affect policy $i$ periods into the future. Let us first suppose that some arbitrary, given correspondence $v(\{\xi_{t-1-j}\}_{j=0}^{\infty})$ happens to be determining the value of the inherited lagged costate.\(^\text{15}\) That is:

$$\lambda_{t-1} = v(\{\xi_{t-1-j}\}_{j=0}^{\infty}) \quad (2.13)$$

Then (2.8) and (2.9) respectively imply that the policymaker ‘self-constrained’ by $\lambda_{t-1}$ will choose $x_t$ and $\lambda_t$ vectors that satisfy:

$$x_t = \hat{\psi}(v(\{\xi_{t-1-j}\}_{j=0}^{\infty}), \epsilon_t) \quad (2.14)$$

$$\lambda_t = \hat{\nu}(v(\{\xi_{t-1-j}\}_{j=0}^{\infty}), \epsilon_t) \quad (2.15)$$

Conceptually, the procedure that generates these expressions differs little from that which delivered equation (2.5) when solving under discretion. There, we found the best current vector of target variables under the constraint that future policy was to be set according to some arbitrary $\psi$ correspondence. Here, we are finding the best continuation policy under the restriction that past commitments must be valued according to some arbitrary $\nu$ correspondence. In the discretionary case we saw that solving for a fixed mapping between $\psi$ and the best policy given $\psi$ ensured a time-invariant (stationary) policy, but one that was sub-optimal since the marginal effects of current policy choices on future variables (and hence the current constraint set) were not considered. Analogously, we would expect that solving for a fixed correspondence between $\nu$ and the best choice of $\lambda_t$ given $\nu$ would deliver a policy rule that was stationary but sub-optimal – in this case, because the marginal effects of changing current policy on the self-imposed (generically non-zero) expression for $\lambda_{t-1}$ would not be considered, even though this initial condition would be affected by changes to current policy through the fixed-point mapping. Akin to $\psi^*$, we can define a fixed-point mapping $\nu^*$ implicitly by:

$$\hat{\nu}(v^* (\{\xi_{t-1-j}\}_{j=0}^{\infty}), \epsilon_t) = v^* (\{\xi_{t-j}\}_{j=0}^{\infty}) \quad (2.16)$$

\(^\text{15}\) It would be a trivial generalisation to allow $\nu$ also to vary in lagged endogenous variables, but ultimately we wish to focus on the strong timeless perspective solution – for precisely the reasons just outlined.
But repeated substitution at appropriate lags of the right-hand side into the left implies $v^*$ must be the same correspondence as $q$ — the correspondence determining $\lambda_{t-1}$ under the ‘strong’ timeless perspective method. Hence the timeless perspective policy, though stationary, will in general be sub-optimal for precisely the same reason as discretion: it requires the policymaker to optimise subject to a constraint that ultimately is endogenous to the chosen policy, determined ex-post by a fixed-point mapping between constraint and policy. Solving under discretion, the policymaker neglects the (mathematical) impact of changing current policy on expected future policy; solving under the timeless perspective, the policymaker neglects the (mathematical) impact of changing current policy on the ‘inherited’ vector of self-imposed constraints.

Notice that these arguments do not imply the timeless perspective solution can never deliver the best constant rule (and likewise for the discretionary solution). The randomness present in most models of interest ensures that the best stationary policy from the perspective of time $t$ will in general depend on the entire history of shocks $\{\epsilon_{t-j}\}_{j=0}^{\infty}$ up to and including time $t$. As section 3 makes clear, this means that further restrictions must be imposed if we are to obtain a unique time-invariant policy — in the form of a ‘best-on-average’ criterion. In particular, if the marginal effects of changing current policy on the inherited constraint happened always to be zero then it would be optimal to implement the timeless perspective solution (which equates marginal gains and losses for a given constraint). The simplest example is the case where $q$ is linear in past shocks, all of which have happened to equal zero — so changing the coefficients on these shocks in the $q$ correspondence cannot change $\lambda_{t-1}$. Yet in this case $\lambda_{t-1}$ under the timeless perspective policy happens to be zero, so timeless perspective is necessarily best just because of its coincidence with the full-commitment solution. Importantly, the timeless perspective method will not generally deliver the best constant rules for deterministic policy problems.

3. Deriving optimal time-invariant rules

We now turn to the general problem of how best to determine time-invariant policy rules in an intertemporal optimisation problem, subject to both forward-looking and backward-looking constraints. Since the aim of this paper is to establish precisely which time-invariant rules can justifiably be considered ‘best’, we proceed by defining explicitly two restrictions on the set of admissible policies (labelled T and M) and two restrictions on the subsequent choice procedure (labelled D and A) that we believe are sufficient together to capture the essence of the ‘optimal rules’ problem. We show how these restrictions can be incorporated into a conventional optimisation problem. Clearly the strength of the conclusions will depend heavily on the normative
acceptability of the restrictions T, M, D and A; but even if aspects of these are controversial, it is preferable for such ‘fundamental’ assumptions to be given clear prominence.

We retain an objective function of the form presented in section 2, given by:

\[ E_t \sum_{i=0}^{\infty} \beta^i \pi_{t+i}(x_{t+i}, \varepsilon_{t+i}) \]  \hspace{1cm} (3.1)

The vectors \( x_{t+i} \) and \( \varepsilon_{t+i} \) are defined as before (and retain dimensions \( k \times 1 \) and \( l \times 1 \) respectively), as are the other components of (3.1). As before, we assume a vector of forward-looking constraints of dimension \( m \times 1 \), which are defined by:

\[ E_{t+i}g(x_{t+i}, \varepsilon_{t+i}, x_{t+i+1}) = 0 \]  \hspace{1cm} (3.2)

However, we also now allow for conventional backward-looking restrictions – specifically, an \( n \times 1 \) vector of restrictions defined by the correspondence:

\[ f(x_{t+i}, \varepsilon_{t+i}; \bar{x}_{t+i-1}) = 0, \]  \hspace{1cm} (3.3)

where \( \bar{x}_{t+i-1} \) is a vector of dimension \( p \times 1 \), with \( p \leq k \), that contains only the elements of \( x_{t+i-1} \) strictly required for the backward-looking constraints to be expressed.\(^{16}\) To allow policy to play a role, we assume \( k > m + n \).

For the system to close and deliver unique outcomes for the target variables, it must be augmented by a policy procedure to span the remaining \( k - m - n \) dimensions.\(^{17}\) The only restrictions that this procedure must satisfy at present are given by (3.2) and (3.3). Formally, if \( S_{t+i} \) (with generic member \( s_{t+i} \)) is the entire set of mappings from any pair of arrays \( \{\varepsilon_{t+i-j}\}_{j=0}^{\infty} \) and \( \{x_{t+i-1-j}\}_{j=0}^{\infty} \) to a \( k \times 1 \) vector (interpreted as \( x_{t+i} \)), and \( \bar{S}_t \) – the ‘policy set’ (with generic element \( \bar{s}_t \)) – is defined by

\[ \bar{S}_t = \{s_{t+i} \in S_{t+i} : \bar{s}_{t+i} \in S_{t+i} \forall i\} \] (the set of arrays of correspondences with a unique element in \( S_{t+i} \) at every horizon \( t + i \)),\(^{18}\) we can define the set of feasible policies as follows.

**Definition 3.1:** The set of feasible policies, denoted \( \bar{S}_t^F \), is the set of policies in \( \bar{S}_t \) such that, when used to determine \( \{x_{t+i}\}_{i=0}^{\infty} \), (3.2) and (3.3) are satisfied at every \( i \) for any given pair of arrays \( \{\varepsilon_{t-j}\}_{j=0}^{\infty} \) and \( \{x_{t-1-j}\}_{j=0}^{\infty} \) and for every possible \( \{\varepsilon_{t+i}\}_{i=0}^{\infty} \).

\(^{16}\) It is a straightforward extension to admit more distant lags of the endogenous variables in \( f \). Note also that the framework will admit joint ‘backward-forward’ constraints, since it is always possible to define new \( t + i \)-dated variables (which can then be included in \( x_{t+i} \)) accounting for the value of any backward-looking components relevant at \( t + i \), with the definitions of these new variables then treated as purely backward-looking constraints, and the original constraints converted to purely forward-looking expressions.

\(^{17}\) This requirement is necessary but not sufficient for determinacy – even in linear systems, as the literature on sunspots makes clear (see, for instance, Farmer (1999)).

\(^{18}\) A point on notation: we write \( s_{t+i} \in \bar{S}_t \) if \( \bar{s}_{t+i} \) is the policy correspondence applied at \( t + i \) under \( \bar{s}_t \).
Without any restrictions preventing time inconsistency, the best possible policy to implement at \( t \) will be the member of \( \tilde{S}_t^F \) for which (3.1) attains the lowest value; this policy must be the full-commitment optimum from the perspective of time \( t \). However, we wish directly to rule out choosing time-inconsistent policies. To do so, we impose the following further restriction (Condition T) on the set of rules considered.

**Condition T (‘time-invariance’):** The policymaker at time \( t \) is restricted to choosing policies from the set \( \tilde{S}_t^T \subseteq \tilde{S}_t^F \), defined as follows:

\[
\tilde{S}_t^T = \{ \tilde{s}_t \in \tilde{S}_t^F : \forall s_{t+i}, s_{t+j} \in \tilde{s}_t, s_{t+i} \equiv s_{t+j} \}
\]

In words, \( \tilde{S}_t^T \) is the set of policies (from \( t \) onwards) such that the policy correspondence for each period, \( s_{t+i} \), takes an identical form for all \( i \) (this implies that the entire policy can be described by a single correspondence \( s \)).

Incorporating condition T explicitly into an optimisation problem and choosing directly among the elements of \( \tilde{S}_t^T \) is the clearest way to ensure time-invariant policy rules.\(^{19}\) But there remains a problem with considering all policies in \( \tilde{S}_t^T \): namely, for any given problem this set may admit a large number of policy rules that imply the same long-run outcome for target variables, but different outcomes over the immediate horizon.\(^{20}\) Since the sequence of lagged endogenous variables prior to time \( t \) has been generated by arbitrary, unknown processes, it is not desirable that policy choice should be swayed by this sequence if avoidable. Re-applying the argument of section 2.1, whenever there are two rules in \( \tilde{S}_t^T \), \( s \) and \( s' \), whose prescriptions differ only so long as endogenous variables from \( t - 1 \) or earlier have a non-negligible influence under \( s \), \( s' \) or both, optimal choice between these two must be influenced by lagged endogenous variables (there is no other non-arbitrary basis for choice). But this exposes choice even in the restricted set \( \tilde{S}_t^T \) to a time-inconsistency problem: there is no reason why the better of \( s \) and \( s' \) at \( t \) should remain the better at \( t + i \) (with \( i \) sufficiently small that \( s \) and \( s' \) still differ significantly at \( t + i \)). For this reason, \( \tilde{S}_t^T \) remains too large a set of policies to have eliminated time inconsistency fully.\(^{21}\)

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\(^{19}\) Note that the correspondence \( s_{t+i} \) can be interpreted as a constant policy rule – though strictly the policy itself only determines \( k - m - n \) dimensions of the mapping.

\(^{20}\) C.f. the discussion of alternative timeless perspective rules in section 2.1: these rules are all in \( \tilde{S}_t^T \) and imply the same long-run policy outcomes, but their initial prescriptions will differ.

\(^{21}\) We will see below that even within the restricted set of constant rules where dependence on lagged endogenous variables is limited as far as possible, time inconsistency may remain an issue – but because the statistical process generating endogenous variables at \( t - 1 \) and earlier must be treated as arbitrary, the type
Hence we add a further restriction to the set of policies considered, which we dub Condition M.

**Condition M (‘minimal use of lagged endogenous variables’):** The policymaker at time $t$ is restricted to choosing policies from the set $\tilde{S}_t^M \subseteq \tilde{S}_t^T$, defined as follows:

A policy $\tilde{s}_t \in \tilde{S}_t^T$, characterised by a time-invariant correspondence $s$ determining $x_{t+i}$ for all $i$, is a member of $\tilde{S}_t^M$ whenever $\lim_{t \to \infty} x_{t+i}$ is independent of $\{x_{t-1-j}\}_{j=0}^{\infty}$ under $\tilde{s}_t$, unless there exists an $\tilde{s}'_t \in \tilde{S}_t^T$ (characterised by the time-invariant correspondence $s'$) such that $\lim_{t \to \infty} x_{t+i}$ is always identical under $\tilde{s}_t$ and $\tilde{s}'_t$ and either (i) or (ii) holds:

1. If $\hat{x}_j$ is the shortest vector of endogenous variables (indexed to arbitrary time period $j$) such that knowledge of $\{x_{t+i-1-k}\}_{k=0}^{\infty}$ and $\{e_{t+i-k}\}_{k=0}^{\infty}$ is sufficient for $s$ to be used to determine $x_{t+i}$, and $\hat{x}'_j$ is the shortest vector of endogenous variables such that knowledge of $\{x'_{t+i-1-k}\}_{k=0}^{\infty}$ and $\{e_{t+i-k}\}_{k=0}^{\infty}$ is sufficient for $s'$ to be used to determine $x_{t+i}$, then $\dim(\hat{x}_j) > \dim(\hat{x}'_j)$.

2. If $\tau$ is the smallest integer such that knowledge of $\{x_{t+i-1-k}\}_{k=0}^{\infty}$ and $\{e_{t+i-k}\}_{k=0}^{\infty}$ is sufficient for $s$ to be used to determine $x_{t+i}$, and $\tau'$ is the smallest integer such that knowledge of $\{x'_{t+i-1-k}\}_{k=0}^{\infty}$ and $\{e_{t+i-k}\}_{k=0}^{\infty}$ is sufficient for $s'$ to be used to determine $x_{t+i}$, then $\tau' < \tau$. ■

Condition M requires that we ‘discard’ all elements of $\tilde{S}_t^T$ such that an alternative policy will deliver the same evolution of long-run target variables whilst depending on fewer lagged endogenous variables. (Notice that we are not concerned with the relative dependence of $s$ and $s'$ on exogenous variables.) It also introduces a technical requirement that we only consider policies for which dependence on lagged endogenous variables realised prior to time $t$ fades through time, and hence for which the long-run evolution of target variables can be analysed without any information on the endogenous variables realised prior to policy implementation. This limits our attention to policies that deliver stationary outcomes, so is certainly non-trivial, but it does not eliminate plausible policy solutions in the monetary policy models of interest.\(^{22}\)

Crucial to its normative appeal is that M rules out all time-invariant policies that link outcomes at $t+i$ to (‘arbitrary’) endogenous variables not in $\tilde{x}_{t+i-1}$ (the vector of endogenous variables affecting the backward-looking constraints (3.3)). This is demonstrated in Proposition 3.1.

---

\(^{22}\) Recall that monetary policy problems generally study target variables defined *relative* to steady state; the steady state itself can be non-stationary without affecting the analysis.
Proposition 3.1: A policy \( \bar{s}_t \in \bar{S}_t^T \) (with time-invariant policy rule \( s \)) will only be contained in \( S_t^M \) if a sufficient set of variables required for \( s \) to determine \( x_{t+i} \) at all horizons \( i \) is \( \bar{x}_{t+i-1} \in \{ \varepsilon_{t+i-j} \}_{j=0}^{\infty} \).

Proof: Take a candidate rule \( \bar{s}_t \in \bar{S}_t^T \) that uses lagged endogenous variables other than those in \( \bar{x}_{t+i-1} \). We aim to show that \( \bar{s}_t \) is inadmissible in \( S_t^M \). Under \( \bar{s}_t \) we have:

\[
x_{t+i} = s(\bar{x}_{t+i-1}, \bar{x}_{t+i-1}, \{x_{t+i-j}\}_{j=2}^{\infty}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty}),
\]

where \( \bar{x}_j \) is defined as the vector of variables in \( x_j \) but not in \( \bar{x}_j \) (for arbitrary \( j \)).

By the stationarity assumption contained within it, \( M \) will certainly rule out the use of \( \bar{s}_t \) unless

\[
limit_{i \to \infty} s(\bar{x}_{t+i-1}, \bar{x}_{t+i-1}, \{x_{t+i-j}\}_{j=2}^{\infty}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) \text{ can be defined on } \{\varepsilon_{t+i-j}\}_{j=0}^{\infty} \text{ alone (dependence on endogenous variables from } t-1 \text{ and earlier fades through time, and the exogenous disturbances are the only terms affecting endogenous variables thereafter). Hence for } \bar{s}_t \text{ to be admissible there must exist a unique correspondence } s_t \text{ that satisfies:}
\]

\[
s_t(\{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) = \lim_{i \to \infty} s(\bar{x}_{t+i-1}, \bar{x}_{t+i-1}, \{x_{t+i-j}\}_{j=2}^{\infty}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty})
\]

Defining \( \bar{s} \) as the part of \( s \) that determines \( \bar{x}_{t+i} \) alone at \( t+i \), and similarly for other ‘partitions of rules’ \( \bar{s}, \bar{s}_t \) etc, \( s_t \) must satisfy:

\[
s_t(\{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) = s(\bar{s}(\{\varepsilon_{t+i-1-j}\}_{j=0}^{\infty}), \bar{s}(\{\varepsilon_{t+i-1-j-k}\}_{k=0}^{\infty}))_{j=2}^{\infty}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty})
\]

Now, consider the alternative rule \( s' \), defined only on \( \{\bar{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty}\} \), given by:

\[
s'(\bar{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) \equiv s(\bar{x}_{t+i-1}, \bar{s}_t(\{\varepsilon_{t+i-1-j}\}_{j=0}^{\infty}), \{\varepsilon_{t+i-j-k}\}_{k=0}^{\infty})_{j=2}^{\infty}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty})
\]

Since this correspondence just implements in every period \( t+i \) the same outcome that \( s \) would have determined had different values of the lagged endogenous variables not in \( \bar{x}_{t+i-1} \) been realised, and since only the variables in \( \bar{x}_{t+i-1} \) can constrain outcomes at \( t+i \) (so for given \( \bar{x}_{t+i-1} \), \( s_t \) will be both defined and implementable for any assumed values of other lagged endogenous variables), the \( \bar{s}'_t \) that mandates its repeated use is certainly in \( \bar{S}_t^F \leq S_t^F \).

If \( \lim_{i \to \infty} s'(\bar{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) = s_t(\{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) \), we can rule out \( s \) under \( M \) (using parts (i) and (ii) in the definition of \( M \)). Suppose first that \( s' \) satisfies the stationarity assumption in \( M \) (this is confirmed below), so we can find a correspondence \( s'_t(\{\varepsilon_{t+i-j}\}_{j=0}^{\infty}) \) that coincides with
\[ \lim_{i \to \infty} s'(\tilde{x}_{t+i-1}, \{e_{t+i-j}\}_{j=0}^\infty) \] for any \( \{e_{t+i-j}\}_{j=0}^\infty \). Then from the definitions of \( s' \) and \( s'_i \) we have the following:

\[
s'_i(\{e_{t+i-j}\}_{j=0}^\infty) = s'_{i-1}(\{e_{t+i-j}\}_{j=0}^\infty), \tilde{s}_i(\{e_{t+i-j}\}_{j=0}^\infty), \{s_i(\{e_{t+i-j-k}\}_{k=0}^\infty)\}_{j=0}^\infty, \{e_{t+i-j}\}_{j=0}^\infty)\]  

(3.8)

Comparison with (3.6) makes it apparent that the ‘stochastic stationary state’ reached by implementing \( s \) will also be a stochastic stationary state under \( s' \). It remains only to confirm that repeated application of \( s' \) will indeed deliver this stationary state. We can deduce the long-run evolution of target variables under \( s' \) by assuming a large value for \( i \) and substituting for \( \tilde{x}_{t+i-1} \) in the right-hand side of (3.7) the expression

\[
s(\tilde{x}_{t+i-2}, \tilde{s}_i(\{e_{t+i-2-j}\}_{j=0}^\infty), \{s_i(\{e_{t+i-1-j-k}\}_{k=0}^\infty)\}_{j=0}^\infty, \{e_{t+i-1-j}\}_{j=0}^\infty) \]  

and repeating recursively. By the stationarity of \( s \), this implies that for large enough \( i \) we must approach the limiting result:

\[
s'(\tilde{x}_{t+i-1}, \{e_{t+i-j}\}_{j=0}^\infty) = s(\tilde{s}_i(\{e_{t+i-1-j}\}_{j=0}^\infty), \{s_i(\{e_{t+i-j-k}\}_{k=0}^\infty)\}_{j=0}^\infty, \{e_{t+i-j}\}_{j=0}^\infty) \]  

(3.9)

This completes the proof.

Armed with Proposition 3.1, we are now in a position to show how \( T \) and \( M \) can be incorporated directly into an optimisation problem, which can potentially be used to deliver the best policy rule within the set \( \tilde{S}_t^M \). By the result just stated, we need only consider time-invariant rules that determine \( x_{t+i} \) by an expression of the form:

\[
x_{t+i} = s(\tilde{x}_{t+i-1}, \{e_{t+i-j}\}_{j=0}^\infty) \quad \forall i \geq 0 \]  

(3.10)

The method we propose for choosing an optimal \( s \) correspondence has the restricting feature that it does not allow us to search across all members of \( \tilde{S}_t^M \), instead restricting consideration to a parametric family of correspondences (i.e., a subset of \( \tilde{S}_t^M \)) within which the optimal rule may be nested. This is certainly a non-trivial methodological limitation, but its significance ought not to be exaggerated. In particular, if it is possible to find the (strong) timeless perspective rule in any given situation, it should certainly be possible to search for the best rule within a broader set of correspondences that nests this rule. Hence we should always be able at least to improve on the timeless perspective. Moreover, in linear-quadratic policy problems of the type commonly studied in the monetary policy literature, optimal policy will always be linear in all the arguments of the \( s \) correspondence, and it is to these cases that our method is most readily applicable.
Formally, we let $s$ be parameterised by a $q \times 1$ real-valued vector $\theta$ (where $q$ need not be finite\textsuperscript{23}) – so we compare over different values of $\theta$ the anticipated outcomes at $t$ of policy rules:

$$x_{t+i} = s(\vec{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta) \quad \forall i \geq 0$$  \hspace{1cm} (3.11)

For instance, in systems with $n = 0$ there will be a $\theta^{TP}$ such that $s(\{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta^{TP})$ is always equal to $\vec{\theta}(\{\varepsilon_{t-1-j}\}_{j=0}^\infty, \varepsilon_t)$ – the strong timeless perspective policy as defined in section 2.1.

To ensure the chosen policy is in $\vec{S}_t^\sigma$, restrictions on $\theta$ will be necessary (so that conditions (3.2) and (3.3) are not violated), and these restrictions must be derived explicitly, and adhered to, when ultimately we optimise over $\theta$. We should further emphasise that the values of $\vec{x}_{t+i-1}$ will be endogenous to the chosen value for $\theta$ whenever $i > 0$. Hence $\vec{x}_{t+i-1}$ can only depend on the history of past shocks, $\{\varepsilon_{t+i-j}\}_{j=0}^\infty$, and the initial vector of ‘structural’ lagged endogenous variables, $\vec{x}_{t-1}$; noting this, it is useful to eliminate $\vec{x}_{t+i-1}$ from our representations (where $i > 0$). But dependence of outcomes on $\vec{x}_{t-1}$ relative to the subsequent shocks cannot possibly be stable as time advances from period $t$, so if we wish to express $x_{t+i}$ in terms of $\{\varepsilon_{t+i-j}\}_{j=0}^\infty$ and $\vec{x}_{t-1}$ alone then we must additionally parameterise by $i$.\textsuperscript{24} Hence:

$$x_{t+i} = \check{s}(\vec{x}_{t-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta, i),$$  \hspace{1cm} (3.12)

where for every $i$ the $\check{s}$ correspondence must satisfy:

$$\check{s}(\vec{x}_{t-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta, i) = \check{s}(\vec{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta, 1) = \check{s}(\vec{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta)$$  \hspace{1cm} (3.13)

to maintain time consistency.\textsuperscript{25}

We can now substitute this $\check{s}$ correspondence into (3.1) to obtain a loss function that can be evaluated for given $\vec{x}_{t-1}$ and $\{\varepsilon_{t-j}\}_{j=0}^\infty$ for any admissible $\theta$:

$$E_t \sum_{d=0}^\infty \beta^d \pi_{t+i}(\check{s}(\vec{x}_{t-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta, i), \varepsilon_{t+i})$$  \hspace{1cm} (3.14)

The restrictions on $\theta$ needed to satisfy (3.2) and (3.3) can be written implicitly as:

$$E_t+i g \left( s(\vec{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta), \varepsilon_{t+i}, s(\check{s}(\vec{x}_{t+i-1}, \{\varepsilon_{t+i-j}\}_{j=0}^\infty; \theta), \{\varepsilon_{t+i+1-j}\}_{j=0}^\infty; \theta) \right) = 0, \hspace{1cm} (3.15)$$

\textsuperscript{23} Indeed, if policy depends on an infinite history of shocks one might expect a distinct element in $\theta$ corresponding to each of these shocks.

\textsuperscript{24} This additional step will not be necessary if $n = 0$ a – i.e., if there are no backward-looking constraints (since then the $\check{s}$ and $s$ correspondences are identical in what follows).

\textsuperscript{25} To paraphrase a point made in Svensson (1999), there is nothing special about time $t$. 

18
where the correspondence \( \tilde{s} \) remains as defined in the proof of Proposition 3.1, and:

\[
f(s(\tilde{x}_{t+i-1}, \{\epsilon_{t+i-j}\}_{j=0}^{\infty}; \theta), \epsilon_{t+i}; \tilde{x}_{t+i-1}) = 0
\]

(3.16)

Clearly we require that the subset of rules in \( \tilde{S}_t^M \) considered is sufficiently large to allow these restrictions to be satisfied by a non-trivial set of parameter vectors for all feasible \( \tilde{x}_{t+i-1} \).\(^{26}\)

We are now in a position to outline condition D – the first of two normative requirements that are desirable features of any optimisation procedure that might be applied to the policy problems studied in this paper. This condition (and condition A that follows it) is best understood as refining the properties of the choice procedure to be applied to the elements of \( \tilde{S}_t^M \), rather than as a direct restriction on the set of admissible rules itself.\(^{27}\)

**Condition D (‘preservation of discounting in backward-looking systems’):** If a choice procedure for selecting within \( \tilde{S}_t^M \) is defined for all problems characterised by an objective of the form (3.1) and constraints of the form (3.2) and (3.3), then when applied to a system with \( m = 0 \) (i.e., a purely backward-looking model), the results of this procedure must coincide with standard optimal control methodology; in particular, the effects of intertemporal discounting for pure time preference must be preserved.\(^{28}\)

It is well known that optimal policy in purely backward-looking systems does not suffer from a time-inconsistency problem (unless geometric discounting is relaxed), and only depends on variables that characterise the ‘state’ of the economy (the exogenous and lagged endogenous variables in (3.3)) – an implication being that fully optimal policy (allowing for pure time preference) must always be in the set \( \tilde{S}_t^M \). Hence it would certainly be odd for a method established to select the ‘best’ policy in \( \tilde{S}_t^M \) to recommend anything else. But methods such as that proposed by Damjanovic et al. (2008), which give a central role to an ‘asymptotic’ loss criterion, violate condition D.\(^{28}\) In our setup, these methods do not evaluate outcomes under loss criterion (3.14), but rather under the augmented function:

\[
E \sum_{i=0}^{\infty} \beta^i \pi_{t+i}\left(s_t\left(\{\epsilon_{t+i-j}\}_{j=0}^{\infty}; \theta\right), \epsilon_{t+i}\right),
\]

(3.17)

\(^{26}\) Generally, we would expect this set to include a particular \( \theta \) that corresponded with the (strong) timeless perspective rule, unless there was a good reason to believe the optimal time-invariant rule was in a different parametric class.

\(^{27}\) The distinction is subtle but not redundant: clearly the aim of any choice procedure is to reduce the set of admissible rules to a single element, but there may be multiple candidate approaches for carrying out this final reduction from \( \tilde{S}_t^M \) to chosen rule. Alone, neither condition D nor condition A will generally suffice to isolate a unique reduction procedure – so by themselves these conditions cannot be considered to reduce the set of admissible policies.

\(^{28}\) The method of Damjanovic et al., like Woodford’s timeless perspective, also violates condition M – ensuring choice may still be time-inconsistent among rules that guarantee the same long-run outcome.
where $s_t$ is the correspondence that $s$ approaches as dependence on $\tilde{x}_{t-1}$ fades (see the proof of Proposition 3.1 – note that we now make dependence on $\theta$ explicit). The use of the unconditional expectations operator captures the assumption that loss is being assessed without any knowledge of the $e_{t-j}$ terms. More importantly, though, the use of $s_t$ implies we are acting as if the choice of $\theta$ will impact on the value of $\tilde{x}_{t-1}$. This endogeneity of initial conditions to chosen policy is certainly not a feature of optimal control methodology, where the best strategy is found for any given $\tilde{x}_{t-1}$. Since optimal control methods deliver a solution in $\tilde{S}_t^M$ for purely backward-looking models, which is the best state-contingent policy path to follow from $t$ onwards given $\tilde{x}_{t-1}$, it follows that $D$ will be satisfied so long as the choice procedure selects the best admissible value for $\theta$ under loss function (3.14) (i.e., for given $\tilde{x}_{t-1}$), and provided that choice is within a sufficiently large subset of $\tilde{S}_t^M$ for one particular value of $\theta$ to imply the same policy as may derived by optimal control. In section 4.3 we give an example of our method coinciding with optimal control. The point here is that loss function (3.14) should be preferred to (3.17) if we wish to retain consistency with well-established intertemporal optimisation procedures under pure time preference.

So within the class of admissible rules parameterised by $\theta$, and for given $\tilde{x}_{t-1}$ and $(e_{t-j})_{j=0}^\infty$, we find the optimal rule by solving:

$$\min_\theta E_t \sum_{i=0}^\infty \beta^i \pi_{t+i}(\tilde{s}(\tilde{x}_{t-1}, (e_{t+i-j})_{j=0}^\infty; \theta, i), e_{t+i})$$

subject to:

(a) $E_{t+i} g \left(s(\tilde{x}_{t+i-1}, (e_{t+i-j})_{j=0}^\infty; \theta), e_{t+i}, s(\tilde{s}(\tilde{x}_{t+i-1}, (e_{t+i-j})_{j=0}^\infty; \theta), (e_{t+i+1-j})_{j=0}^\infty; \theta) \right) = 0$

(b) $f(s(\tilde{x}_{t+i-1}, (e_{t+i-j})_{j=0}^\infty; \theta), e_{t+i}; \tilde{x}_{t+i-1}) = 0$ (3.18)

Where the problem is sufficiently tractable, a solution can be found through a straightforward Lagrangian method, as the examples of section 4 demonstrate.

Now, in the event that the solution to this problem is characterised by first-order conditions, these conditions will generally be functions of $\tilde{x}_{t-1}$ and $(e_{t-j})_{j=0}^\infty$. As discussed above, in the event that

29 Note also that policies assessed under (3.17) will still be implementable provided $\theta$ is still required to satisfy (3.15) and (3.16) – we have not eliminated dependence on $\tilde{x}_{t+i-1}$ in $s$ itself.

30 Allowing for the $\tilde{x}_{t-1}$ terms to be endogenous is rightly noted by Damjanovic et al. as equivalent to evaluating, under alternative candidate policies, the expectation at time $t$ of the value of (3.1) will take at a time period long after $t$. Since conditions prevailing at $t$ are irrelevant to this welfare criterion by construction, the policies that are optimal under it are dubbed ‘unconditionally optimal’ by these authors. The use of such an objective is a contentious step. A significant contribution of this paper is to show that one need not reject the immediate perspective of time $t$ to justify alternative time-invariant rules to those that follow from the timeless perspective approach.
the economy is entirely backward-looking \((m = 0)\), the fully optimal policy is well-known to be time-invariant and independent of \(\{\epsilon_{t-j}\}_{j=1}^{\infty}\) – hence in these cases the realised values of \(\tilde{x}_{t-1}\) and \(\epsilon_t\) cannot affect the analytics. A simple example is provided in section 4.3. But if there are some forward-looking constraints, and provided the model is not entirely deterministic, the first-order conditions will generally imply different optimal choices for \(\theta\) depending on the inherited conditions \(\{\epsilon_{t-j}\}_{j=0}^{\infty}\). \(^{31}\) This implies the ‘optimal time-invariant rule’ from the perspective of time \(t\) will in general not be the same as the optimal rule, identically derived, from the perspective of time \(t + i\).

Given that our search for time-invariant rules was motivated by a desire to overcome the time-inconsistency inherent in time-varying policy rules, stationary rules that would be changed if the method used to derive them were reapplied in the future do not provide a satisfactory resolution. If implemented, they would require precisely the same adoption of a particular temporal perspective as does the full-commitment optimum – in which case, one might just as well implement this latter optimum instead. Hence we must impose one further restriction that guarantees a time-invariant choice criterion, whilst still motivated by the basic problem: what is the best rule to select from \(S_t^{SM}\)? To this end, we define condition A.

**Condition A (‘best on average criterion’):** If the best rule to select from the set \(S_t^{SM}\), as assessed under (3.1) at time \(t\), depends on the values realised by \(\{\epsilon_{t-j}\}_{j=0}^{\infty}\) then we select the rule that would have been chosen in the event that any functions of \(\{\epsilon_{t-j}\}_{j=0}^{\infty}\) affecting optimal choice had taken their unconditional expected values – as defined under the joint distribution of \(\{\epsilon_{t-j}\}_{j=0}^{\infty}\).

Applied to the optimisation method proposed here, condition A requires that we take expectations across the first-order conditions to (3.18) under the joint distribution of the \(\{\epsilon_{t-j}\}_{j=0}^{\infty}\) terms, and repeatedly implement the rule implied by the resulting \(\theta\) vector. Hence it can be interpreted as a requirement that the chosen rule should be the best possible for an average set of circumstances.

We leave unresolved the important issue of whether the optimal choice of \(\theta\) under A will generally depend on \(\tilde{x}_{t-1}\) in ‘backward-forward’ models. If so, A would have to be augmented by a further restriction; but we refrain from stipulating such a restriction, as it remains unclear whether it would ever be required.

There is an alternative condition to A that one could conceivably invoke to deliver time-invariance, and we refer to this as A*.

---

\(^{31}\) As subsequently explained, it is an important (and open) question whether the optimal rule in general varies in \(\tilde{x}_{t-1}\) for given \(\{\epsilon_{t-j}\}_{j=0}^{\infty}\), or whether invariance to changes in the ‘structural’ lagged endogenous variables is retained from purely backward-looking systems.
**Condition A* (‘best under undisturbed history’):** If the best rule in the set $\mathcal{S}_t^M$, assessed under (3.1) at time $t$, varies in the values realised by $\{\varepsilon_{t-j}\}_{j=0}^\infty$, then we select the rule that would have been best in the event that $\{\varepsilon_{t-j}\}_{j=0}^\infty = \{E[\varepsilon_{t-j}]\}_{j=0}^\infty$.

As section 4.1.2 makes clear, we do not believe that this criterion should be favoured over A. The basic reason is that the optimal choice of rules will generally depend on non-linear functions of the lagged shock vectors – and simply assuming that these vectors have been equal to their expected values may imply very extreme assumptions about the non-linear functions. For instance, if the expected value of a shock is zero but optimal policy varies in its squared value, A* implies that we must assume the lowest possible squared value (zero). However, we state the condition explicitly, as it proves insightful in section 4.1.

In summary: unlike the timeless perspective and ‘unconditionally optimal’ approaches, we explicitly restrict consideration to time-invariant rules that make minimal use of lagged endogenous variables (imposing M); we require that any rule selection procedure should preserve discounting for pure time preference (D), which is violated by the ‘unconditionally optimal’ method; and, through A, we are explicit about the set of conditions for which we wish choice to be best. On the last point, notice that an unnecessary dependence on lagged endogenous variables could well make it impossible to state the precise circumstances in which a chosen rule will be best within a given parametric class: absent any knowledge of the process generating these variables prior to the implementation of the chosen rule, ‘best on average’ criteria lack applicability. We now show how to implement the methods of this section in three simple models.

### 4. Determining optimal rules: three examples

In this section we determine optimal policy rules in the set restricted by T and M, and allowing the choice procedure to satisfy D and A, for the following environments: first, a linear stochastic New Keynesian Phillips Curve (NKPC) model; second, a deterministic version of the same model, but in which aggregate demand externalities (due to market power) constrain the steady-price level of output below its optimum; third, an entirely backward-looking Phillips curve model, studied purely to demonstrate the distinction between our approach and methods for deriving ‘optimal’ rules that minimise asymptotic loss. In the first case, the optimal rule is a close cousin of the ‘unconditionally optimal’ policy derived by Damjanovic et al. (2008), though Condition M ensures our method delivers a unique rule. In the second case, our results contrast with those of Woodford (2003), who argued that the optimal constant rate of inflation in an identical model was zero. Instead, we show that a constant positive inflation rate is superior to steady prices (except in limiting parametric
cases). In the third example, we confirm that our parametric optimisation method does not conflict with conventional optimal control methods, as required by Condition D.

4.1. Stabilisation bias revisited

We first turn to a simple inflation stabilisation problem, of the type popularised by Clarida et al. (1999). As there, the argument makes use of the NKPC, together with a quadratic loss function that is a valid second-order approximation to the welfare of a representative agent in the chosen setup (provided market power inefficiencies have been overcome).\(^{32}\)

The basic policy objective is to choose a state-contingent optimal rule that will minimise the loss function \(L_t\):

\[
L_t = \mathbb{E}_t \left[ \sum_{i=0}^{\infty} \beta^i (\pi_{t+i}^2 + \omega y_{t+i}^2) \right]
\]

subject to the forward-looking constraint:

\[
\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \gamma y_t + \xi_t,
\]

where the mark-up shock term \(\xi_t\) is again assumed to follow an AR(1) process for simplicity:\(^{33}\)

\[
\xi_{t+i} = \rho \xi_{t+i-1} + v_{t+i},
\]

with the \(v_{t+i}\) disturbances assumed iid and mean-zero.

Since there is no reason for output or inflation to deviate from their optimal levels of zero in the absence of disturbances, it follows that shock stabilisation is the sole function of monetary policy. Moreover, the form of the loss function prevents the policymaker from desiring randomness for its own sake; so there is no reason for optimal policy to link the values of output and inflation at time \(t\) to terms other than the \(\{\xi_{t-j}\}_{j=0}^{\infty}\) shocks. In the absence of any structural linkage between lagged and current endogenous variables, we also rule out any explicit link between the values of the current target variables and their lagged values, consistent with Proposition 3.1. Again, the purpose of policy is simply to stabilise the target variables in response to shocks, so explicit dependence of future outcomes on these shocks, as opposed to implicit dependence via lagged endogenous

\(^{32}\) The derivation of this linear-quadratic problem from a Calvo pricing model characterised by Dixit-Stiglitz preferences need not be replicated here. See Clarida et al. (1999) for detailed coverage. It should be apparent that the arguments presented in this paper retain their validity irrespective of the manner in which the policymaker’s objective function and any constraints upon it are constructed.

\(^{33}\) The precise shock dynamics play no important role in what follows.
variables, is sufficient to achieve any desired stabilisation gains.\textsuperscript{34} Given the linear-quadratic nature of the problem, we can restrict our attention to the subset of policies in $S^M$ that imply a linear relationship between target variables and lagged shocks (which nests both the timeless perspective and discretionary solutions). We allow $\pi_t$ and $y_t$ to be expressed as follows:

$$\pi_t = A^*(L)e_t = \sum_{j=0}^{\infty} A^*_j e_{t-j}$$

(4.4)

$$y_t = B^*(L)e_t = \sum_{j=0}^{\infty} B^*_j e_{t-j}$$

(4.5)

where the values of the $A^*_j$ and $B^*_j$ coefficients remain to be determined. Since each $e_{t-j}$ term is itself formed of lagged $v_{t-j}$ terms, in accordance with (4.3), we can rewrite these expressions:

$$\pi_t = \sum_{j=0}^{\infty} A_j v_{t-j}$$

(4.6)

$$y_t = \sum_{j=0}^{\infty} B_j v_{t-j},$$

(4.7)

and it is straightforward to show that the $A_j$ and $B_j$ coefficients are linked to their counterparts in (4.4) and (4.5) by the identities:

$$A_j \equiv \sum_{i=0}^{j} \rho^i A^*_{j-i}, \quad A^*_j \equiv A_j - \rho A_{j-1} \quad \forall j \geq 0$$

(4.8)

$$B_j \equiv \sum_{i=0}^{j} \rho^i B^*_{j-i}, \quad B^*_j \equiv B_j - \rho B_{j-1} \quad \forall j \geq 0$$

(4.9)

(Clearly $A_{-1}$ and $B_{-1}$ must equal zero, since current endogenous variables cannot depend upon future shocks.) These representations imply values of $A^*_j$ and $B^*_j$ can be recovered directly once optimal values for $A_j$ and $B_j$ have been determined at all lags $j$.\textsuperscript{35}

Now, since the random disturbance term $v_{t+1}$ is mean-zero in every period, it follows that $E_t \pi_{t+1}$ must be given by $\sum_{j=0}^{\infty} A_{j+1} v_{t-j}$. From this, we can deduce a set of necessary restrictions on the coefficients that follows from the Phillips Curve (4.2):

$$A_j = \beta A_{j+1} + \gamma B_j + \rho^j \quad \forall j \geq 0$$

(4.10)

Substituting (4.6) and (4.7) into the objective (4.1) and adding constraint (4.10) allows the problem to be written in Lagrangian form:

\textsuperscript{34} This does not preclude the possibility that a single (index) variable could be used to store all policy-relevant information on past shocks once the optimal rule is determined – playing a role akin to that of the costates in Marcet and Marimon’s (1999) ‘saddle point problem’ formulation. Hence dependence on the entire history of lagged shocks need not imply a significant computational problem at every time period.

\textsuperscript{35} It proves simpler to determine optimal coefficients on the ‘pure’ disturbance terms than on the $e_{t-j}$.  

24
Next, we take first-order conditions with respect to the generic coefficients $A_j$ and $B_j$ in turn:

$$E_t \left[ \sum_{i=0}^{\infty} \beta^i \left\{ \sum_{j=0}^{\infty} A_j v_{t+i-j} \right\} \right] - \lambda_j + \beta \lambda_{j-1} = 0 \quad \forall j \geq 0 \quad (4.12)$$

$$E_t \left[ \sum_{i=0}^{\infty} \beta^i \left\{ \sum_{k=0}^{\infty} B_k v_{t+i-j} v_{t+i-k} \right\} \right] + \gamma \lambda_j = 0 \quad \forall j \geq 0 \quad (4.13)$$

Note that the value of $\lambda_{-1}$ is zero, as there can be no coefficients linking endogenous variables to the values of future disturbances – and hence no associated constraints. Importantly, though, the costates are no longer time-specific, so this condition does not imply time-inconsistency.

It is apparent from (4.12) and (4.13) that optimal rules will in general depend on the history of lagged shocks when optimisation is carried out. This is because all terms in the summations in (4.12) and (4.13) that involve interactions between two disturbances indexed at time $t$ or earlier are known at time $t$, and these terms (when considered relative to $t$) are not the same as they would be if optimising at $t+i$ instead. Hence the problem of finding optimal constant rules in $S^M_t$ here appears to run into a time-inconsistency problem itself. For instance, what appears to be the best rule at a time when recent pricing disturbances have not been particularly severe will no longer dominate when there is a clearer incentive to let bygones be bygones. Indeed, since implementing the rule that follows from (4.12) and (4.13) (assessed at time $t$) in every period could itself be improved upon by implementing the full commitment solution from time $t$ onwards, there seems little to commend such a rule – the method used to derive it is not time-consistent, and there are strictly better time-inconsistent procedures available.

4.1.1. Solving with condition A

Yet the fact that no rule is optimal irrespective of the history of recent disturbances should not be a reason to cease searching for a rule that is, in some meaningful sense, ‘best on average’. Hence we impose condition A – the best on average requirement. That is, we look for a policy that satisfies the first-order conditions after expectations have been taken across (4.12) and (4.13) under the entire joint distribution of the disturbance terms $v_{t+i-j}$. Since the disturbance terms $v_t$ are iid (with variance $\sigma^2$), this implies setting $v_{t+i-j} v_{t+i-k}$ to $\sigma^2$ where $j = k$ and zero otherwise. (4.12) and (4.13) then become:

$$\min_{(A_j, B_j)} E_t \left[ \sum_{i=0}^{\infty} \beta^i \left\{ \left( \sum_{j=0}^{\infty} A_j v_{t+i-j} \right)^2 + \omega \left( \sum_{j=0}^{\infty} B_j v_{t+i-j} \right)^2 \right\} \right] - \sum_{j=0}^{\infty} \lambda_j \left( A_j - \beta A_{j+1} - \gamma B_j - \rho \right)$$

(4.11)
\[
\frac{\sigma^2}{1-\beta} A_j - \lambda_j + \beta \lambda_{j-1} = 0
\]  
(4.14)

\[
\frac{\alpha \sigma^2}{1-\beta} B_j + \gamma \lambda_j = 0
\]  
(4.15)

Substituting into (4.10) the implied values for \(A_j\) and \(B_j\) then delivers a second-order difference equation in the costates:

\[
\lambda_{j+1} - \left( \frac{1}{\beta} + \beta + \frac{\gamma^2}{\beta \alpha} \right) \lambda_j + \lambda_{j-1} = -\frac{\sigma^2}{\beta (1-\beta)} \rho^j
\]  
(4.16)

Comparison with results featured in Woodford (2003), and elsewhere, reveals that the left-hand side is the same as follows when solving for the ‘full-commitment’ evolution of the costates in a conventional optimisation problem over target variables, but under the assumption that the policymaker’s discount factor approaches one.\(^{36}\) A policymaker that did not discount the future would never trade off an expected increase in loss in stochastic steady state for a finite-horizon gain, so we immediately observe similarity between the policy rule that is best on average and that which minimises ‘asymptotic’ loss. But again we emphasise that there is no need to embrace the ‘unconditional’ loss criterion of Damjanovic et al. (2008) to justify use of this policy. Of the time-invariant policies considered, it just happens that that which minimises loss asymptotically also stands the greatest chance of minimising discounted loss from the perspective of time \(t\), when the values taken by lagged shocks are unknown. To embrace any other policy, including the timeless perspective rule, would be welfare-reducing on average, even from this temporal perspective.

‘Overcoming’ time inconsistency via criterion A is not entirely satisfactory, but the approach does have the merit of transparency. In particular, we need not arrive at time-invariance through a fixed-point mapping imposed only after optimality conditions have been derived, as under the timeless perspective approach. Moreover, whilst condition A is certainly not proposed as a uniquely ‘correct’ resolution to the time-inconsistency of choice within \(\bar{S}_t^M\), it is at least clear from its character how one might construct a normative justification. Specifically, since doing what is best for a policymaker at time \(t\) will generally conflict with doing what is best at time \(t + i\), in choosing among the rules in \(\bar{S}_t^M\) we must find a compromise solution to an intertemporal conflict of interests. Always doing what would have been best in the event that terms in past shocks in (4.12) and (4.13) had taken their expected values seems a plausible way forward – it gives special weight neither to the policymaker at the point in time at which optimisation first takes place, nor a policymaker who happens to have inherited a notably benign (or malign) set of initial conditions.

\(^{36}\) The private-sector discount factor, which features in (4.2), must be kept at \(\beta\).
Returning to the specific problem at hand, our main result follows.

**Proposition 4.1:** In the stabilisation bias model characterised by loss function (4.1) and Phillips Curve (4.2), the best policy rule under assumption A in the set that satisfies T and M delivers outcomes for output and inflation in accordance with the following expressions:

\[
\gamma_{t+i} = -\frac{\gamma}{\omega \beta (\varphi_2 - \rho)} \sum_{j=0}^{\infty} \varphi_1 j \varepsilon_{t+i-j} \tag{4.17}
\]

\[
\pi_{t+i} = \frac{1}{\beta (\varphi_2 - \rho)} \varepsilon_{t+i} - \frac{(\beta - \varphi_1)}{\beta (\varphi_2 - \rho)} \sum_{j=1}^{\infty} \varphi_1^{j-1} \varepsilon_{t+i-j} \tag{4.18}
\]

where \( \varphi_1 \) and \( \varphi_2 \) take the values:

\[
\varphi_1 = \frac{(1+\beta^2 + \beta^2 \varphi_2^2) \sqrt{(1+\beta^2 + \beta^2 \varphi_2^2)^2 - 4\beta^2}}{2\beta}, \quad \varphi_2 = \frac{(1+\beta^2 + \beta^2 \varphi_2^2) \sqrt{(1+\beta^2 + \beta^2 \varphi_2^2)^2 - 4\beta^2}}{2\beta} \tag{4.19}
\]

**Proof:** We can factorise (4.16) using the lag operator to obtain:

\[
(1 - \varphi_1 L) (L^{-1} - \varphi_2) \lambda_j = - \frac{\sigma^2}{\beta (1-\beta)} \rho^j \tag{4.20}
\]

where \( \varphi_1 \) and \( \varphi_2 \) take the values stated in the proposition:

From this we obtain:

\[
\lambda_j = \varphi_1 \lambda_{j-1} + \frac{1}{\beta (\varphi_2 - \rho)} \frac{\sigma^2 \rho^j}{(1-\beta)} \tag{4.21}
\]

Adding to this the initial condition \( \lambda_{-1} = 0 \) and simplifying yields:

\[
\lambda_j = \frac{1}{\beta (\varphi_2 - \rho)} \frac{\sigma^2}{(1-\beta)} \frac{(\rho^{j+1} - \varphi_1^{j+1})}{(\rho - \varphi_1)} \tag{4.22}
\]

Using this solution in (4.14) and (4.15) then delivers the desired expressions for \( A_j \) and \( B_j \):

\[
A_j = \frac{1}{\beta (\varphi_2 - \rho)} \frac{(\rho^j - \varphi_1^j)}{(\rho - \varphi_1)} \tag{4.23}
\]

\[
B_j = \frac{1}{\omega \beta (\varphi_2 - \rho)} \frac{(\rho^j - \varphi_1^j)}{(\rho - \varphi_1)} \tag{4.24}
\]

Expressions for \( A_j^* \) and \( B_j^* \) then follow from (4.8) and (4.9) respectively:

\[\text{An additional condition that the coefficients in the optimal rule do not become unbounded as } j \text{ becomes large must be imposed to assign } \varphi_1 \text{ and } \varphi_2 \text{ in this way. In essence this is a standard transversality requirement.}\]
\[
A_j^* = \frac{\varphi_j^{j-1}(\varphi_j-\rho)}{\beta(\varphi_j-\rho)} \quad \forall j > 0
\]

\[
A_0^* = \frac{1}{\beta(\varphi_j-\rho)}
\tag{4.25}
\]

\[
B_j^* = -\frac{\gamma \varphi_j^{j}}{\omega \beta(\varphi_j-\rho)} \quad \forall j \geq 0
\tag{4.26}
\]

These immediately deliver the stated result. ■

As we anticipated from the form of (4.16), (4.25) and (4.26) correspond exactly to the coefficients on the \( \varepsilon_{t-j} \) shock terms that would follow when applying the timeless perspective methodology and letting the policymaker’s discount factor approach unity.\(^{38}\) Hence it happens that the implied policy must minimise asymptotic loss – as one would expect, since ‘steady state’ is achieved immediately under the types of rules considered. Yet in reality, the policymaker’s discount factor equals \( \beta \). So provided \( \beta \) does not equal one, the ‘timeless perspective’ solution cannot be justified on the grounds that the constant rule it generates is ‘best on average’ as we have defined the term. The intuition for this result is as follows: if we seek a constant rule of the form taken by (4.8) and (4.9), and we seek the best such rule under A, then any marginal change with effects on the inflation-output trade-off at time \( t \) must have identical effects on the equivalent trade-off at \( t + 1 \). Hence for every change in \( t \)-dated inflation, there is an equal change in expectations of inflation at \( t + 1 \). Since expectations of inflation at \( t + 1 \) affect the policy trade-off at \( t \) (c.f. equation (4.2)), there is no reason for the marginal effects that operate through this channel to be discounted or augmented by the rate of pure time preference relative, say, to the direct marginal effects of changing the rule on the rate of inflation in \( t \).\(^{39}\)

When optimisation under full-commitment takes place (considering the entire set \( S_{\text{FC}} \)), changes that affect the current policy trade-off need not be replicated in future periods – and, indeed, in the specific case of the initial output-inflation choice they are not. This means the policymaker must consider the specific welfare effects of each output-inflation choice through time, equating marginal gains and losses with due concern for the time at which they are realised. Hence in this environment the policymaker should value the welfare effects of changes in \( t + 1 \)-dated inflation on the inflation-output trade-off at \( t \) (through the expectations channel) at a rate \( 1/\beta \) greater than the effects on the trade-off at \( t + 1 \). With no need for constant rules or time-consistency imposed, exploiting the fact that changes to policy in the current period cannot have any effects on yesterday’s expectations of current inflation is an inherent part of the optimal strategy.

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\(^{38}\) Confirming this is an algebraic exercise left to the interested reader. C.f. Woodford (2003).

\(^{39}\) Mathematically, there is no reason for the \( \beta \) in equation (4.14) to be multiplied or divided by a further \( \beta \).
Now, as explored in section 2, the strong interpretation of timeless perspective policy sees target variables *immediately* take the values that would have followed if a full-commitment strategy had been pursued since a point infinitely far in the past. But, by the logic just stated, this long-run evolution is part of an optimal solution only if it is possible that the marginal effects of changing policy in one period can conceivably differ from the marginal effects of changing policy in another. Or, put differently, the long-run evolution under full commitment is optimal only if we have set ourselves the full-commitment problem. If instead the problem works under the constraint that changes to policy today are always matched by equivalent changes to policy tomorrow,\(^{40}\) and this constraint is implicit in the basic desire for a time-invariant rule of some kind, then the case for weighting due to time preference alone evaporates.

4.1.2. When is the timeless perspective best?

There is one way in which we might justify the timeless perspective solution within our framework. Return to equations (4.12) and (4.13) – the general first-order conditions for the choice of an optimal constant rule. Now, suppose that instead of A we impose A* on the choice procedure – that is, we find the best rule assuming all past (and current) shocks have taken their expected value, which is zero. Then we have the following result.

*Proposition 4.2:* In the stabilisation bias model characterised by loss function (4.1) and Phillips Curve (4.2), the best policy under Condition A* in the set of policies satisfying T and M is the strong timeless perspective rule.

*Proof:* With Condition A*, the first-order conditions become:

\[
\frac{\sigma^2 \beta^{j+1}}{1-\beta} A_j - \lambda_j + \beta \lambda_{j-1} = 0 \quad \forall j \geq 0 \tag{4.27}
\]

\[
\frac{\omega \sigma^2 \beta^{j+1}}{1-\beta} B_j + \gamma \lambda_j = 0 \quad \forall j \geq 0 \tag{4.28}
\]

Defining \(\tilde{\lambda}_j \equiv \lambda_j / \beta^{j+1}\), substitution into (4.10) then gives:

\[
\tilde{\lambda}_{j+1} - \left( \frac{1}{\beta} + 1 + \frac{\gamma^2}{\beta \omega} \right) \tilde{\lambda}_j + \frac{1}{\beta} \tilde{\lambda}_{j-1} = -\frac{\sigma^2}{\beta(1-\beta)} \rho^j \tag{4.29}
\]

Invoking the lag operator in the usual way, we have:

\(^{40}\) Note the ‘best on average’ criterion abstracts from differences due to realised values of the disturbance terms here. It is this that means changes in the policy trade-off at time \(t\) are assessed as having exactly the same effects on the policy trade-off at \(t + 1\).
\[(1 - \phi_1 L)(L^{-1} - \phi_2) \tilde{x}_{t+i} = -\frac{\sigma^2}{\beta (1-\beta)} \rho^i \]  

(4.30)

where:\(^{41}\)

\[
\phi_1 = \frac{(1+\beta + \frac{\gamma^2}{\omega}) - \sqrt{(1+\beta + \frac{\gamma^2}{\omega})^2 - 4\beta}}{2\beta} \quad \phi_2 = \frac{(1+\beta + \frac{\gamma^2}{\omega}) + \sqrt{(1+\beta + \frac{\gamma^2}{\omega})^2 - 4\beta}}{2\beta}
\]  

(4.31)

Hence we can solve for \(\tilde{x}_j\) as before:

\[
\tilde{x}_j = \frac{1}{\beta (\phi_2 - \rho)} \frac{\sigma^2}{(1-\beta)} \frac{(\rho^{i+1} - \phi_1 \rho^i)}{(\rho - \phi_1)}
\]  

(4.32)

Substituting into (4.27) and (4.28) we have:

\[
A_j = \frac{1}{\beta (\phi_2 - \rho)} \frac{(\rho^i (\rho - \phi_1) \rho^i)}{(\rho - \phi_1)} \quad \forall j \geq 0
\]  

(4.33)

\[
B_j = -\frac{\gamma}{\omega} \frac{1}{\beta (\phi_2 - \rho)} \frac{(\rho^{i+1} - \phi_2 \rho^{i+1})}{(\rho - \phi_2)} \quad \forall j \geq 0
\]  

(4.34)

which then delivers new values for \(A_j^*\) and \(B_j^*\) via (4.8) and (4.9):

\[
A_j^* = \frac{\phi_1^{i-1} \rho^{i-1} (\phi_1^{i-1})}{\beta (\phi_2 - \rho)} \quad \forall j > 0
\]  

(4.35)

\[
A_0^* = \frac{1}{\beta (\phi_2 - \rho)}
\]

\[
B_j^* = -\frac{\gamma}{\omega} \frac{\phi_1^i}{\beta (\phi_2 - \rho)} \quad \forall j \geq 0
\]  

(4.36)

These expressions immediately deliver the result.\(^{42}\)

Proposition 4.2 should come as no surprise: if all realised disturbance terms have taken values of zero then nothing is lost by tying current policy to these disturbances, given the linearity of the rule sought. Loss will depend entirely on the expected response to shocks realised at \(t + 1\) and later. But the full-commitment solution involves a constant response to all (equally-valued) shocks that are realised subsequent to time \(t\). Hence the best constant rule determined at \(t\) need not differ from the full-commitment solution except in the way it would have dealt with shocks that, in the event, are assumed to have taken a value of zero (and thus are considered irrelevant to loss). \(A^*\) is equivalent to acting as if initial conditions are always sufficiently benign to permit the full-commitment

\(^{41}\)Again, we allocate the roots under the assumption coefficients remain bounded.

\(^{42}\)C.f. Woodford (2003).
solution, notwithstanding restrictions T and M. Given this, the rule chosen under A* should be one that replicates the way the full-commitment optimum deals with any relevant shocks – that is, those realised from t + 1 onward. The strong timeless perspective solution provides just such a rule.

It may be possible to find other assumptions regarding past shocks that could deliver the timeless perspective policy as a conditionally optimal constant rule. But our objective is not to find the best rule for any arbitrary set of initial conditions, since this approach is not time-consistent, and there are superior time-inconsistent methods. Rather, it is to find a constant rule that can be justified as best under an ‘average’ set of initial conditions. Now, one might argue that an average set of initial conditions would be a history of disturbances whose first moments coincided with ex-ante expectations – which would justify A*. And by this criterion, the timeless perspective would be ‘best on average’. But re-examining the first-order conditions (4.12) and (4.13), slightly re-written, makes it clear that this implies asymmetric treatment of past and future shock terms:

\[ E_t\left[ \sum_{i=0}^{\infty} \beta^i \left\{ \sum_{k=0}^{\infty} A_k \pi_{t+i-j} \pi_{t+i-k} \right\} \right] + \sum_{i=0}^{l} \beta^i \left\{ \sum_{k=0}^{\infty} A_k \pi_{t+i-j} \pi_{t+i-k} \right\} - \lambda_j + \beta \lambda_{j-1} = 0 \quad \forall j \geq 0 \quad (4.12') \]

\[ E_t\left[ \sum_{i=0}^{\infty} \beta^i \left\{ \sum_{k=0}^{\infty} B_k \pi_{t+i-j} \pi_{t+i-k} \right\} \right] + \sum_{i=0}^{l} \beta^i \left\{ \sum_{k=0}^{\infty} B_k \pi_{t+i-j} \pi_{t+i-k} \right\} + \gamma \lambda_j = 0 \quad \forall j \geq 0 \quad (4.13') \]

When solving for an optimal rule at time t, the policymaker must consider the marginal effects of any increase in some A_j or B_j on the values of all \( \pi_{t+i}^2 \) and \( \pi_{t+i}^2 \) (respectively) both through components known only in expectation at t (represented by the first summation terms in (4.12') and (4.13')) and through those known fully at t (represented by the second summation terms). The first of these depend on expected values of interacted disturbances, the second on their actual values. There is no reason for optimal policy to depend on actual or expected values of these disturbances in isolation – so the first moments of the \( \pi_{t+i-j} \) terms are not relevant. Since assuming that all past disturbances have been equal to zero (their expected values) implies a very extreme assumption about their past squared values (namely, that these have also been equal to zero), and squared values (along with cross-products) are of the greater significance to the marginal requirements for optimality, A* delivers rules that are best for extreme circumstances, not average ones. In contrast, assuming that past disturbances have interacted in accordance with their second moments, as A does in this case, ensures symmetric treatment of the forward- and backward-looking components of (4.12') and (4.13'); it is equivalent to assuming that, as we change coefficients A_j and B_j, any expected marginal effects on \( \pi_{t+i}^2 \) and \( \pi_{t+i}^2 \), with i > j, due to their dependence on the squared
value of \( v_{t+k-j} \) should be considered equal to the marginal effects on \( \pi_{t+k}^2 \) and \( y_{t+k}^2 \), with \( k < j \), due to their dependence on the squared value of \( v_{t+k-j} \).

### 4.1.3. The nature of the optimal time-invariant policy

We can provide some intuition to account for the superiority (on average) over the timeless perspective method of the policy derived in section 4.1.1 by comparing the impulse response functions of output and inflation to a one-off shock at time \( t \) (whose magnitude would be sufficient to induce a unit percentage deviation in inflation if output and expected future inflation were held constant) under the two policies. Figure 4.1 displays these functions, assuming values for the parameters: \( \beta = 0.95, \omega = 0.0625, \gamma = 0.02 \) and \( \rho = 0.43 \).

**Figure 4.1** – Impulse response functions of output and inflation under optimal time-invariant and timeless perspective rules (percentage deviations in response to a one per cent shock)

![Impulse response functions](image)

The basic form of these functions is familiar from the stabilisation bias literature (see, for instance, Clarida et al. (1999)): in response to a shock, a negative output gap is engineered that only decays

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43 See McCallum and Nelson (2004) for a short discussion of appropriate calibration values in this model \((\beta = 0.95 \) is used to enhance the observable difference between the timeless perspective and optimal constant policy\). We set \( \rho = 0 \) to prevent assumed dynamics for the shock term from clouding interpretation.

44 To be clear, the impulse response functions associated with timeless perspective policy in Figure 4.1 are identical to the time-zero full-commitment responses, precisely because we have assumed away any lagged
gradually through time. As a result, inflation turns negative after the initial period – the result of which is to temper the immediate impact of the shock, through the expectations channel.

When discussing the timeless perspective approach in section 2.2, we argued that it would generally deliver sub-optimal time-invariant rules because the post-optimisation fixed-point mapping by which it was derived failed to consider the marginal effects of changing policy on ‘inherited’ constraints, even though these constraints were ultimately endogenous to policy. This is reflected here in the longer time it takes for output and inflation to revert to steady state under the timeless perspective rule, relative to the optimal rule. This implies immediate inflation expectations are higher when the optimal rule is used, and so the impact of the shock on initial inflation is slightly greater (this is not readily discernable in Figure 4.1, but the inflation deviation in period zero with the timeless perspective rule is 0.942 per cent, and is 0.956 per cent with the optimal rule): the optimal rule delivers less immediate ‘stabilisation’ as the price of a less protracted adjustment process.

4.2. A deterministic problem

The logic of our arguments carries over very neatly from stabilisation bias to another familiar setting – one in which the policymaker desires an incompatible inflation-output combination, as in the famous model analysed by Kydland and Prescott (1977). Moreover, this problem has the advantage of admitting a deterministic setup, demonstrating that the merits of our solution method relative to the timeless perspective do not rely heavily on the ‘best on average’ assumption, Condition A; this condition is not required at all here.

Suppose we replace (4.1) with the following loss function, which is implied by a representative agent’s utility function in the basic monopolistic competition/Calvo pricing model if price-setting power has not been offset by an output subsidy to firms:

\[
L_t = E_t \left[ \sum_{i=0}^{\infty} \beta^i (\pi_{t+i}^2 + \omega (y_{t+i} - k)^2) \right],
\]

whilst the stochastic term in the Phillips curve (4.2) is dropped, so this becomes:

\[
\pi_t = \beta E_t \pi_{t+1} + \gamma y_t
\]

shocks (c.f. the discussion in section 4.1.2). But note that it is the ‘overlaid’ responses to many shocks that determines output and inflation, and some of these shocks will have taken place much earlier than time zero – hence the entire impulse response function is relevant to the immediate policy problem.

\[^{45}\text{See Woodford (2003) for the microfoundations.}\]
Since constraint (4.38) remains purely forward-looking, fully optimal policy is undermined by precisely the same time-inconsistency problem as before – a point that has been studied at great length in the literature. A brief sketch of this follows.

Solving directly for optimal choices of target variables, via the standard Lagrangian method delivers a pair of first-order conditions in terms of the costate $\lambda_{t+i}$ (now used to incorporate constraint (4.38)):

$$\pi_{t+i} - \lambda_{t+i} + \lambda_{t+i-1} = 0$$  \hspace{1cm} (4.39)

$$\omega(y_{t+i} - k) + \gamma \lambda_{t+i} = 0,$$  \hspace{1cm} (4.40)

which hold together with the restriction $\lambda_{t-1} = 0$ when full-commitment optimisation takes place at time $t$. Substituting into (4.38) to deliver a second-order difference equation on the costates, and factorising as usual, we have:

$$(1 - \mu_1 L)(L^{-1} - \mu_2)\lambda_{t+i} = -\frac{\gamma k}{\beta},$$  \hspace{1cm} (4.41)

with:

$$
\mu_1 = \frac{(1 + \beta + \frac{\gamma^2}{\omega}) - \sqrt{(1 + \beta + \frac{\gamma^2}{\omega})^2 - 4\beta}}{2\beta}
$$

$$
\mu_2 = \frac{(1 + \beta + \frac{\gamma^2}{\omega}) + \sqrt{(1 + \beta + \frac{\gamma^2}{\omega})^2 - 4\beta}}{2\beta}
$$  \hspace{1cm} (4.42)

This solves to give:

$$\lambda_{t+i} = \frac{\gamma k}{\beta(\mu_2 - 1)} \frac{(1 - \mu_1^{-1})}{1 - \mu_1}$$  \hspace{1cm} (4.43)

From (4.42) it can be shown that:

$$(\mu_2 - 1)(1 - \mu_1) = \frac{\gamma^2}{\omega\beta}$$  \hspace{1cm} (4.44)

Hence we have:

$$y_{t+i} = k\mu_1^{i+1}$$  \hspace{1cm} (4.45)

$$\pi_{t+i} = k\frac{\omega}{\gamma} (1 - \mu_1)\mu_1^i$$  \hspace{1cm} (4.46)

Since $\mu_1$ lies within the unit circle, the solution has the property that as $i$ becomes large both output and inflation approach zero, irrespective of the value of $k$. Moreover, since the (strong) timeless perspective solution immediately implements this long-run outcome, it is argued by Woodford

$^{46}$ Again, Woodford (2003) gives a comprehensive treatment and further references.
that setting output and inflation to zero in every period is the best time-invariant strategy that the policymaker can pursue. Loss functions similar to (4.37) have been used extensively to demonstrate and analyse the possibility of 'inflation bias' that follows as the policymaker seeks an unfeasibly high level of output,\footnote{The classic reference is Barro and Gordon (1983).} so initially this result seems to provide weight to the timeless perspective approach. Surely any method to overcome time inconsistency that did not eliminate inflation 'bias' would have to be flawed?

Unfortunately, the logic does not go through. To demonstrate this, we again take the approach of optimising over coefficients rather than target variables. This gives Proposition 4.3.

**Proposition 4.3**: In the deterministic inflation bias model characterised by loss function (4.37) and Phillips Curve (4.38), the optimal policy among the set satisfying T and M yields constant levels of inflation and output that are weakly positive for all admissible parameter values, and strictly positive except in limiting parametric cases. Specifically, we have:

\[
\pi_{t+i} = \frac{\omega y(1-\beta)}{\gamma^2+\omega(1-\beta)^2} k \quad \forall i \geq 0 \tag{4.47}
\]

\[
\gamma_{t+i} = \frac{\omega(1-\beta)^2}{\gamma^2+\omega(1-\beta)^2} k \quad \forall i \geq 0 \tag{4.48}
\]

**Proof**: Since there are no exogenous shocks and no structural dependence on lagged target variables exists in this model, rules in the set \(S_t^M\) need only specify constant levels of inflation and output to be implemented each period:

\[
\pi_{t+i} = A \tag{4.49}
\]

\[
\gamma_{t+i} = B \tag{4.50}
\]

Constraint (4.38) implies a simple restriction on these coefficients:

\[
A = \beta A + \gamma B \tag{4.51}
\]

Note that \(A = B = 0\) corresponds to the timeless perspective solution, which is consistent with (4.51) and thus within the set of rules considered. In Lagrangian form, the problem is:

\[
\min_{A,B} \sum_{i=0}^{\infty} \beta^i \cdot \frac{1}{2} \{A^2 + \omega (B-k)^2\} - \lambda[A - \beta A - \gamma B] \tag{4.52}
\]

The first-order conditions with respect to \(A\) and \(B\) respectively are:
Together with (4.51), these conditions give three equations in three unknowns \( (A, B \text{ and } \lambda) \). We solve to find:

\[
A = \frac{\omega \gamma (1 - \beta)}{\gamma^2 + \omega (1 - \beta)^2} k, \tag{4.55}
\]

\[
B = \frac{\omega (1 - \beta)^2}{\gamma^2 + \omega (1 - \beta)^2} k, \tag{4.56}
\]

as required.\[\Box\]

These results are significant: the best constant levels of output and inflation in a deterministic Phillips curve model based on Calvo pricing are not zero, as suggested by the timeless perspective solution, but strictly positive. This is true whenever \( k \) is strictly positive (i.e., some increase in output above capacity is desired), \( \omega \) is strictly positive (so the policymaker has some concern for the output objective) and \( \beta \) is less than one.\[48\] This last requirement plays a crucial role, as it states that the long-run Phillips curve must not be vertical. As McCallum (2004) emphasises, the New Keynesian Phillips Curve (4.38) does not exhibit the NAIRU property, since a trade-off between inflation and output exists even in the long run. If we define \( \bar{\pi} \) and \( \bar{y} \) to be the long-run constant levels of inflation and output respectively, the long-run equivalent of (4.38) is:

\[
\bar{\pi} = \frac{\gamma}{1 - \beta} \bar{y}, \tag{4.57}
\]

With \( \beta < 1 \) and \( \gamma \) finite, this implies a permanent output-inflation trade-off. The existence of this trade-off gives an intuitive explanation for (4.55) and (4.56): the loss function (4.37) implies that, ceteris paribus, zero inflation is fully optimal for the policymaker. But then, appealing to the envelope theorem, the losses from increasing inflation above zero are themselves zero at the margin. On the other hand, a positive value of \( k \) implies non-negligible gains at the margin from increasing output above zero – hence the marginal gains from moving ‘up’ the long-run Phillips curve away from a zero-zero inflation-output combination will be strictly positive, and hence strictly positive values of the two target variables must be optimal. Indeed, if we consider the microfoundations of the NKPC relationship, the optimal policy rule is an exemplary ‘second-best’

\[48\] For inflation to be strictly positive we also require \( \gamma \) to be strictly positive – that is, the Phillips curve should not be a horizontal line through the origin (\( \gamma = 0 \) would allow the bliss point of \( \pi_t = 0, \ y_t = k \) to be attained in every period).
strategy: permitting positive inflation ensures a dispersion of prices among (Calvo-constrained) firms, which is inefficient in isolation, but also induces increased output – which is desirable since prices exceed marginal cost in equilibrium. At the margin, the inefficient allocation of production among firms induced by inflation is a price worth paying to mitigate the effects of market power.

To confirm that (4.55) and (4.56) do indeed improve upon the timeless perspective, we note that loss for any given $A$ and $B$ will be given by:

$$L_t = \frac{A^2 + \omega(B-k)^2}{1-\beta} \tag{4.58}$$

Hence, defining $L_t^{TP}$ to be the value of (4.58) when the timeless perspective solution is used, and $L_t^{OP}$ its value under the optimal solution given by (4.55) and (4.56), we have (after some manipulation):

$$L_t^{TP} = \frac{\omega}{1-\beta} k^2 \geq \frac{\nu^2}{\gamma^2 + \omega(1-\beta)^2} \frac{\omega}{1-\beta} k^2 = L_t^{OP}, \tag{4.59}$$

where the inequality is strict whenever $k$ and $\omega$ are strictly positive and $\beta$ is strictly less than one.

In light of these results, it is apparent that the timeless perspective policy implies a disinflationary bias – though the detrimental consequences of this are principally due to associated output restraint. A similar observation has already been made in the context of forward-looking Phillips curves that do have the NAIRU property by Kirsanova, Vines and Wren-Lewis (2006) – where the bias is purely disinflationary. These authors also provide intuition for the result, which is identical in essence to that given above for the stabilisation problem. In a forward-looking environment, optimising over specific choices of the target variables implies giving additional weight to the $t$-dated disinflationary effect of a reduction in inflation at time $t + 1$ – operating through the expectations channel – relative to effects felt directly at time $t + 1$, simply because the former effect is experienced sooner. Solving under full-commitment, there are no expectation effects to consider in the initial period, so it is desirable for the target variables to vary through time to take advantage of this asymmetry; this is why optimising over target variables is appropriate. But if we impose a need for constant outcomes then the effects of an increased level of output and inflation in the first period cannot be considered in isolation from the implied increase in these variables in all subsequent periods. So among the marginal welfare effects of increasing output and inflation above zero initially (and thus in each subsequent period in turn) must be included a contemporaneous worsening of the initial output-inflation trade-off due to an increase in the $E_t \pi_{t+1}$ term in (4.38). This means the relative weightings of marginal effects used in deriving the full-commitment solution
are no longer appropriate, and a ‘constant’ solution method under which they are still applied, such as the timeless perspective approach, will yield disinflationary bias.

A second, more commonly studied form of bias in this context comes not from an inappropriate weighting of the marginal effects of policy changes, but a failure even to consider some such effects. This occurs when the discretionary solution is applied. Since discretionary optimisation treats expectations as given, and (4.38) implies no structural dependence on lagged outcomes in the given problem, the approach is equivalent to solving a distinct static constrained optimisation problem at each point in time – and thus the outcome is a pair of constant values for output and inflation. As has been shown by Clarida et al. (1999) and elsewhere, these values are given by:

\[ \pi_t = \frac{\omega \gamma (1 - \beta)}{\omega (1 - \beta)^2 + \gamma^2 (1 - \beta)} k \quad \forall i \geq 0 \]  

\[ \gamma_t = \frac{\omega (1 - \beta)^2}{\omega (1 - \beta)^2 + \gamma^2 (1 - \beta)} k \quad \forall i \geq 0 \]  

(4.60)

(4.61)

These expressions are directly comparable with (4.55) and (4.56), and it is apparent by inspection that output and inflation both exceed their optimal constant levels, provided the policymaker has some concern for the future (that is, \( \beta > 0 \)). We have an inflationary – and excessive output – bias.

The value of the loss function is then given by \( L^\text{Dis}_t \):

\[ L^\text{Dis}_t = \frac{\gamma^2 (\gamma^2 + \omega)}{[\gamma^2 + \omega (1 - \beta)]^2} \frac{\omega}{1 - \beta} k^2 \]  

(4.62)

It is useful to depict the outcomes of all four policies graphically. For this purpose, we assume values for the parameters of: \( \omega = 0.0625 \), \( \gamma = 0.05 \), \( \beta = 0.95 \), \( k = 0.2 \). Figure 4.2 then illustrates the (deterministic) time-path of output and inflation under each policy.

Figure 4.2 – Time-paths of output and inflation under different optimisation procedures

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49 Again, see McCallum and Nelson (2004) for a short discussion of appropriate calibration values (once more, \( \beta = 0.95 \) is used to allow the difference between the timeless perspective and optimal constant policy to be clearly observable). The value \( k = 0.2 \) is also used in Woodford (2003), based on a calibrated Calvo pricing model.
As expected, the optimal constant rule yields output and inflation values in between the timeless perspective and discretionary solutions. It delivers a 5.9 per cent reduction in loss relative to the timeless perspective in this case. Moreover, the graphic makes clear exactly how the full-commitment solution achieves gains relative to the best constant rule: by allowing a short-run boost to output (and, to a lesser degree, inflation) at the expense of long-run restraint. The timeless perspective policy enforces this restraint immediately – rather like a perpetual hangover, it requires one to pay the appropriate price for an extravagance that was never enjoyed.

As a slight aside, the results of this section have important implications for the treatment of ‘distorted steady state’, as analysed by Benigno and Woodford (2005, 2007). In forward-looking models that do not permit a microfounded loss function to attain a global minimum in the absence of shocks, the particular time-invariant solution method used to determine policy will generally shift the location of steady state (in the example just studied, for instance, applying the optimal time-invariant solution took us to a more inflationary steady state than the timeless perspective). Moving to a stochastic environment, whether target variables fluctuate in the neighbourhood of the timeless perspective steady state or the steady state that follows from optimal time-invariant policy will affect both the dynamics and the welfare implications of economic outcomes. In contrast with the approach recommended by Benigno and Woodford, our results suggest that a policymaker would do
best to implement, and consider fluctuations about, the ‘distorted’ steady state that follows from a rule determined through parametric optimisation.

4.3. Backward-looking constraints

By framing the issue as a choice between using a ‘conditional’ and an ‘unconditional’ or ‘asymptotic’ loss function, authors including Sauer (2007) and Blake (2001) have unwittingly exposed sceptics of the timeless perspective approach to claims they have arbitrarily neglected the loss functions of actual economic agents – who do discount the future and are not concerned merely for the asymptotic value of loss. But here we have shown that the timeless perspective is not generally the best approach to finding time-invariant rules even under a perfectly conventional loss function: for a given welfare criterion – microfounded or otherwise – we can do better.

We have demonstrated that parametric optimisation within a plausible subset of $S^M_t$ yields constant rules unambiguously superior to the timeless perspective in a deterministic environment and superior by the most appealing ‘best on average’ criterion in a stochastic environment. But it remains important to emphasise that our method is not simply an alternative way to minimise asymptotic loss. Specifically, we want to confirm that our approach does not prevent condition D (the preservation of discounting in backward-looking systems) from being satisfied. For although there are strong normative arguments for giving an equal weighting to all time periods when setting policy, these are best treated as essentially-contestable ethical claims – and must be separated from the demonstrable point that parametric optimisation improves upon the timeless perspective for average past exogenous disturbances in a forward-looking model with a given intertemporal preference structure, and when choice is over the set of rules in $S^M_t$.

The issue becomes significant when we work with a model that implies current values of inflation and output depend structurally upon lagged target variables – the most straightforward example being a purely ‘backward-looking’ Phillips curve. Here, we demonstrate that the coefficient optimisation method outlined in section 3 will justify the optimal policy from the perspective of the current time $t$, consistent with condition D.

Reverting to loss function (4.1), we now further assume that the Phillips curve takes the form:

$$
\pi_t = \pi_{t-1} + \gamma y_t + \epsilon_t
$$

(4.63)

The shock term $\epsilon_{t+i}$ is now allowed, for simplicity, to be iid for each $i$. No claim is made that this is a realistic relationship to expect – suggesting, as it does, an extreme lack of forward-looking behaviour.

50 See, for instance, Benigno and Woodford (2008) for a critique of the ‘unconditional’ approach.
among price-setters (it would follow as a limiting case of the Steinsson (2003) model in which all price-setters adopt 'rule of thumb' behaviour); (4.63) is used only to make the necessary point.

We start with a brief derivation of optimal policy assuming that the policymaker directly chooses values for output and inflation, using a method very similar to that deployed in the forward-looking cases above. To this end, the appropriate Lagrangian formulation is:

$$\min_{(\pi_{t+i}, y_{t+i})} E_t \left[ \sum_{i=0}^{\infty} \beta^i \left( \frac{1}{2} (\pi_{t+i}^2 + \omega y_{t+i}^2) - \lambda_{t+i}(\pi_{t+i} - \pi_{t-1+i} - \gamma y_{t+i} - \epsilon_{t+i}) \right) \right]$$  \hspace{1cm} (4.64)

The first-order conditions for an optimal state-contingent path are:

$$E_t \pi_{t+i} - E_t \lambda_{t+i} + \beta E_t \lambda_{t+i+1} = 0 \hspace{1cm} \forall i \geq 0$$  \hspace{1cm} (4.65)

$$\omega E_t y_{t+i} + \gamma E_t \lambda_{t+i} = 0, \hspace{1cm} \forall i \geq 0$$  \hspace{1cm} (4.66)

Applying conventional methods, one can readily show that the optimal state-contingent paths for output and inflation are given by:

$$\pi_{t+i} = \phi_1 \pi_{t+i-1} + \phi_1 \epsilon_{t+i} \hspace{1cm} \forall i \geq 0$$  \hspace{1cm} (4.67)

$$y_{t+i} = -\frac{\gamma}{\omega - \beta \phi_1} \pi_{t+i-1} - \frac{\gamma}{\omega - \beta \phi_1} \epsilon_{t+i} \hspace{1cm} \forall i \geq 0$$  \hspace{1cm} (4.68)

with $\phi_1$ taking the value given in (4.31).\(^{51}\) The solution takes a predictable form: inflation increases in response to a shock, but by less than the shock’s value. To achieve this effect, output is constrained below zero – where it remains as long as lagged inflation remains positive. Inflation gradually decays back to zero from above, in contrast with its behaviour under optimal policy in the forward-looking model (see Figure 4.1).

It is a standard result that optimisation under geometric discounting in purely backward-looking models yields time-consistent solutions, and this case is no exception. Given this, it follows immediately that any coefficient optimisation method treating $\pi_{t-1}$ as fixed must yield (4.67) and (4.68) as optimal policy rules for this model (provided these are nested within the set of coefficients considered). This result is summarised in Proposition 4.4.

**Proposition 4.4:** In the purely backward-looking optimal stabilisation model characterised by loss function (4.1) and Phillips curve (4.63), the optimal policy satisfying T and M for given $\pi_{t-1}$ (i.e., from the perspective of time t) is the rule characterised by (4.67) and (4.68) – and hence an appropriate coefficient optimisation method applied to derive the optimal rule in $\tilde{\sigma}_t^H$ must satisfy condition D.

\(^{51}\) Note a necessary condition for optimality is that the target variables are bounded.
Proof: We have established that (4.67) and (4.68) give the best policy in the ‘feasible set’ \( \hat{S}_t^f \) from the perspective of time \( t \). But this policy is also a member of \( \hat{S}_t^M \), and \( \hat{S}_t^M \subseteq \hat{S}_t^f \). So provided the set of rules considered when optimising over coefficients nests this particular policy, and provided one seeks the best rule in the coefficient ‘family’ from the perspective of time \( t \) (treating \( \pi_{t-1} \) as fixed), the result must be the rule characterised by (4.67) and (4.68).

With some work, it can be verified that coefficient optimisation under loss function 4.1 (subject to the appropriate constraints) within the set of rules:

\[
\pi_{t+i} = A \pi_{t+i} + B \pi_{t+i-1} \tag{4.69}
\]

\[
y_{t+i} = C \pi_{t+i} + D \pi_{t+i-1} \tag{4.70}
\]

treating \( \pi_{t-1} \) as given, indeed delivers values for \( A, B, C \) and \( D \) consistent with (4.67) and (4.68).

As suggested by the discussion in section 3, when using coefficient optimisation the crucial step in ensuring condition D remains satisfied is to avoid replacing \( \pi_{t-1} \) in the objective function with the value (as a function of lagged shocks) that it would have taken had the chosen rule been operational for many periods. Inflation and output subsequent to time \( t \) must of course be treated as endogenous to any chosen policy, but to allow the same of \( \pi_{t-1} \) when optimising is implicitly to reject the temporal perspective inherent in objective function 4.1, and instead embrace the ‘asymptotic’ loss criterion of Damjanovic et al. (2008). Notice the difference with cases featuring purely forward-looking constraints: there, the key flaw in the timeless perspective approach was its failure to allow for the endogeneity of lagged costates when selecting time-invariant rules. As we have seen, the optimal time-invariant policy under purely forward-looking constraints (and condition A if required) does constitute an ‘unconditionally optimal’ policy according to Damjanovic et al.’s terminology – the intuition being that there it is quite appropriate for the only relevant lagged variables (in this case the costates) to be treated as endogenous. But more generally, in cases where lagged dependent variables feature in structural equations, the unconditional approach will be too blunt a tool to for extracting optimal rules from the perspective of time \( t \).

Trivially, the optimal policy in the purely backward-looking case can be considered a ‘timeless perspective’ rule, since it is a time-invariant strategy, and cannot but yield the same long-run distribution of target variables as follows from ‘full commitment’ (another name for the same policy). So we have a neat asymmetry: the best rule in \( \hat{S}_t^M \) (under A, if necessary) in a purely forward-looking model is a policy that minimises asymptotic loss, whilst the best rule in \( \hat{S}_t^M \) in the
purely backward-looking model is a timeless perspective policy; together, these results caution against a naïve application of either timeless perspective or ‘unconditional’ optimisation methods.

5. Conclusion

The question that motivated this paper follows directly from the title of Kydland and Prescott (1977): if policymakers in forward-looking systems should favour rules over discretion, what are the best rules to implement? In our view, the voluminous literature on time-inconsistency in forward-looking systems has yet to resolve this issue satisfactorily. An important reason for this has been a reluctance to consider methods for deriving optimal rules that do not simply augment the (time-inconsistent) full-commitment solution in a minimal, but ultimately *ad hoc*, manner to ensure time-invariance. Accordingly, the timeless perspective method has recently gained adherents, due principally to the work of Michael Woodford; but we have demonstrated that this approach suffers from a methodological problem very similar to that which renders the straightforward discretionary method sub-optimal. It finds the best policy subject to constraints that are subsequently rendered endogenous to the policy choice (and thus are not incorporated into the optimisation problem appropriately). Paradoxically, we find the principal alternative to the timeless perspective – so-called ‘unconditional optimisation’, as detailed in Damjanovic et al. (2008) – problematic for entirely the opposite reason: it recommends treating as endogenous certain state variables that in fact do serve as fixed constraints on any policymaker with a concern for immediate loss. Moreover, neither of these methods delivers a unique rule – raising questions about the extent to which they have truly overcome time-inconsistency.

We have sought to address the problem directly. First, we have proposed two restrictions on the set of rules within which the policymaker chooses – ensuring time-invariance and minimising policy dependence on lagged endogenous variables (whose values cannot constitute an appropriate basis for choice *per se*). Second, we have proposed two conditions we believe should be satisfied by any criterion for choosing among the remaining rules: that the choice is best for an average set of circumstances, and that the method should deliver standard optimal control solutions in purely backward-looking systems. We have shown how these requirements can be incorporated into an optimisation problem – albeit one limited to considering a specific parametric family of rules.

Applying this methodology, we have demonstrated that the best rule ‘on average’ in a simple stabilisation bias model does not follow from applying the timeless perspective method. Similarly, in a deterministic inflation bias model we found that the optimal constant levels of output and inflation are generically positive; again, the timeless perspective method delivers an inferior outcome.
Marcet and Marimon (1999) demonstrated how to recover the full-commitment solution to problems with forward-looking constraints through recursive methods familiar from backward-looking control problems, by reformulating the objective function to account for lagged commitments and treating lagged costates as ‘state’ variables. In a sense, the timeless perspective is an attempt to develop this approach further, augmenting it with restrictions against time-inconsistency. Unfortunately, the arguments contained in this paper suggest the endeavour is misconceived – since it is central to the recursive approach that optimisation takes place treating the ‘state variables’ (including lagged costates) as given, when in forward-looking systems they are not. We conclude that Marcet and Marimon’s methods are not an appropriate starting point for deriving optimal time-invariant rules.

We leave for further research two important issues. First, it remains unclear whether different values for lagged endogenous variables that feature in backward-looking constraints will generally imply different optimal rules, when forward-looking constraints are also present; if so, further restrictions on choice among candidate rules will be necessary. Second, it would be desirable to find a methodology that does not require us to choose only among a parametric family of rules within $\tilde{S}^M$, but instead optimises across the entire set. It is yet unclear whether such a method can be found – but since we are always able to derive the (strong) timeless perspective solution, we can start by asking whether the best time-invariant rule will always lie in the same parametric family as this rule. If so, optimising within this parametric family will always be sufficient.

**Bibliography**


