WHEN DOES THIRD-DEGREE PRICE DISCRIMINATION REDUCE SOCIAL WELFARE, AND WHEN DOES IT RAISE IT?

Simon Cowan

Number 410
October 2008

Manor Road Building, Oxford OX1 3UQ
When does third-degree price discrimination reduce social welfare, and when does it raise it?

Simon Cowan

October 2008

Abstract

Sufficient conditions are developed for third-degree price discrimination by a monopolist serving all markets to reduce and raise social welfare. Welfare falls if the demand function in the market whose price is higher with discrimination is at least as convex as that in the other market (at the non-discriminatory price). Welfare rises if inverse demand in the low-price market is more convex (at the discriminatory price) than inverse demand in the high-price market and the discriminatory prices are close together, so the cost of misallocation is less than the benefit of higher output.

JEL classification: D42, L12, L13
Keywords: price discrimination, monopoly.
1. Introduction

This paper develops general sufficient conditions for third-degree price discrimination by a monopolist serving all markets to reduce and raise social welfare. A monopolist practising third-degree price discrimination uses an exogenous signal such as the age or location of the consumer, or the time of purchase, to divide customers into separate markets. The monopoly price can then be set in each market. A move from non-discrimination – with the same price in each market – to discrimination raises the firm’s profits, harms consumers in markets where prices increase and benefits consumers in the markets where prices fall. The overall welfare effect, when consumer surplus and profits have equal weight in social welfare, can be positive or negative. The aim of this paper is to provide conditions based on the shapes of the demand functions to determine the sign of the welfare effect.

When all demand functions are linear and all markets are served at the non-discriminatory price the welfare effect of discrimination is negative, since total output remains at the non-discrimination level (a fact proved by Pigou, 1920) and the given output is inefficiently distributed between consumers because they face different relative prices. Varian (1985), following Schmalensee (1981), shows generally that a necessary condition for welfare to rise is that total output with discrimination exceeds the no-discrimination level (see also Schwartz, 1990). The focus of the literature since Joan Robinson’s pioneering discussion (Robinson, 1933) has been on the output effect. The most general analysis of the output effect is by Aguirre (2008a), who discusses and extends all the known results. While the output test is useful it is inherently incomplete – when output does not rise it follows that welfare falls, but when output increases the effect on welfare can be positive or negative.

Instead in this paper the welfare effect is analyzed directly. The technique used by Schmalensee (1981), Holmes (1989) and Stole (2007), and earlier by Leontief (1940) and Silberberg (1970), to explore the output effect is applied to determine the sign of the welfare effect. It starts by supposing that the firm must set the same price in all markets, and then allows the amount of discrimination to increase. If the sign of the effect on welfare of a small increase in discrimination can be determined then under some conditions the total effect, which is the integral of the marginal effects, can also be signed. Simple conditions are presented that determine the sign of the marginal and total welfare effects using the properties of the demand functions. With demand at price \( p \) being \( q(p) \) the convexity (or curvature) of
direct demand can be defined, analogously to relative risk aversion for a utility function, as $\alpha \equiv -\frac{pq''}{q'}$. Similarly the convexity of inverse demand, $p(q)$, is $\sigma \equiv -\frac{qp''}{p'}$. Both measures, and how they change as more discrimination is allowed and thus prices diverge, determine the sign of the effect on social welfare. In particular what matters are the curvatures of direct demands at the non-discriminatory price and the curvatures of inverse demands at the discriminatory prices. In the language of Joan Robinson (1933), a market is called “weak” if the discriminatory price is below the non-discriminatory price and “strong” otherwise. In order to present the analysis without too much algebra it is assumed that there are two markets. The results can, however, be generalized straightforwardly to more than two markets.

An assumption is made that is satisfied by almost all demand functions, including linear, exponential and constant-elasticity demands, and which ensures that social welfare is a single-peaked function of the amount of discrimination. Throughout it is assumed that at the non-discriminatory price both markets are served with positive quantities, so price discrimination does not open up new markets (see Hausman and Mackie-Mason, 1988, and Layson, 1994, for analyses of price discrimination that opens new markets).

The main results are as follows. First, discrimination reduces social welfare if the direct demand function in the strong market, where the price rises with discrimination, is at least as convex as that in the weak market (measured at the non-discriminatory price). An example is when both markets have the same curvature, and a further special case is when the demand functions are linear so the curvatures are zero. Second, discrimination raises welfare when, at the discriminatory prices, the inverse demand function in the weak market is more convex than that in the strong market and the discriminatory prices are close together. Total output rises with discrimination, which is good for welfare, while the fact that the discriminatory prices are not far apart ensures that the negative effect on welfare of differing prices – the misallocation effect – is small and is outweighed by the benefit of the increased output. An example is when the weak market has an exponential demand function and the demand function in the strong market is linear.

The third result provides conditions for a small amount of discrimination to be good for welfare, while full discrimination goes too far in the sense that a reduction in discrimination would raise welfare. Starting at the non-discriminatory price a small amount of
discrimination raises welfare when direct demand in the weak market is more convex than direct demand in the strong market (so the condition in the first welfare result does not hold). Starting at the discriminatory prices a marginal reduction in discrimination raises welfare when the inverse demand function in the strong market is at least as convex as that in the weak market (which implies that the condition in the second welfare result does not hold). An example is when both demand functions are of the constant elasticity form. The overall effect on welfare of moving from no discrimination to full discrimination in this case can be positive or negative, and additional techniques must be used to say more about the welfare effect.

The intuitive reason that the difference between the curvatures of demand is important for welfare is as follows. A price increase when demand is concave has relatively little effect on welfare (the extreme form of concavity is when the demand function is rectangular and there is no deadweight loss from monopoly pricing). At the same time if the price falls in a market that is rather convex there is a large increase in output and thus in welfare in that market. This is the insight of Joan Robinson (1933), who showed that total output rises when discrimination causes prices to rise in markets with concave demands and prices to fall in markets with convex demands. Malueg (1994) presents a clear mathematical and graphical analysis of the relationship between the curvature of the demand function and the deadweight loss from monopoly pricing.

The condition that guarantees that welfare rises with discrimination is new. In the literature it has proved difficult to find general sufficient conditions for discrimination to raise welfare (at least when all markets are served). Cowan and Vickers (2007) find that discrimination raises welfare when each inverse demand function has constant curvature, $\sigma$, that is sufficiently above 1, the prices are close together and the values of $\sigma$ in weak markets are no smaller than those in strong markets. This finding seems to indicate that instances of welfare-increasing discrimination might be rare. The condition provided here shows that such pessimism about the welfare effects of discrimination may not be warranted. The condition for welfare to fall is a generalization of that of Cowan (2007), who assumes that demand in the strong market is an affine transformation of the demand function in the weak market (so the curvatures of direct demand are automatically equal at the non-discriminatory price). The case where a small amount of discrimination is good for welfare but full discrimination goes too far is related to the analysis of Cowan and Vickers (2007), who focus on inverse demands with
constant curvature and take tangencies of selected functions at the non-discriminatory and discriminatory prices and use the resulting inequalities to sign the effects on output and welfare. The results derived by Aguirre (2008a) on the output effect complement the welfare results in this paper. Aguirre derives general sufficient conditions for output to rise or to not increase in terms of the relative values of the curvature measures for both direct and inverse demand.

The paper is organized as follows. Section 2 presents the positive model of monopoly pricing with and without third-degree price discrimination. Section 3 contains the welfare analysis. Conclusions are in Section 4.

2. The model of discrimination and non-discrimination

A monopolist sells an identical product in two markets and has a constant marginal cost, \( c \geq 0 \). The assumption of two markets is made for simplicity – all the results can be generalized using the Lagrangian technique of Schmalensee (1981). Demand in a market with price \( p \) is \( q(p) \), which is twice-differentiable and decreasing and is independent of the price in the other market. Preferences are represented by quasi-linear utility functions so consumer surplus is a valid measure of consumer welfare. The price elasticity of demand is \( \eta = -pq'(p)/q \), with the prime denoting the derivative. The profit function for a market, \( \pi = (p-c)q(p) \), is assumed to be strictly concave i.e.

\[
\pi^* = 2q' + (p-c)q'' = q q'' + (p-c) \left[ 2 + (p-c) \frac{q''}{q'} \right] < 0
\]

and thus the expression in square brackets is positive.\(^1\) With strict concavity the first-order conditions for the maximization problems are necessary and sufficient. Define \( \alpha = -pq^*/q' \) as the convexity (or curvature) of direct demand and \( \alpha / p = -q'' / q' \) as the absolute convexity. These measures are analogous to relative and absolute risk aversion for a utility function, and their values at the non-discriminatory price will be important. The Lerner index, the mark-up of price over marginal cost, is \( L \equiv (p-c)/p \) so \( 2 + (p-c)q'' / q' = 2 - L \alpha > 0 \) by concavity of

\(^1\) See Nahata, Ostaszewski and Sahoo (1990) for an analysis of price discrimination when profit functions are not concave.
the profit function. A similar measure of curvature applies to the inverse demand function, \( p(q) \). The convexity (or curvature) of inverse demand is \( \sigma = -q p'' / p' = q q'' / [q']^2 \). The two convexity/curvature measures and the price elasticity are related by \( \sigma = \alpha / \eta \). The value of \( \sigma \) at the discriminatory price plays a key role.

When the firm discriminates the first-order condition for its problem in one market is \( q(p^*) + (p^* - c)q'(p^*) = 0 \) where \( p^* \) is the profit-maximizing discriminatory price (which exceeds \( c \)) and the star denotes the discriminatory regime. Rearranging the first-order condition gives \( L^* = 1 / \eta^* \), which is the standard Lerner condition for monopoly pricing, and thus \( L^* \alpha^* = \sigma^* \). Strict concavity of the profit function entails that \( 2 - \sigma^* > 0 \). The subscript \( w \) denotes the weak market, where the discriminatory price is below the non-discriminatory one, and subscript \( s \) denotes the strong market, where the price is higher with discrimination. When the firm may not discriminate it chooses the single price that maximizes aggregate profits \( \pi_w(p) + \pi_s(p) \). The non-discriminatory price \( \bar{p} \) is determined by \( \pi_w' (\bar{p}) + \pi_s' (\bar{p}) = 0 \) and satisfies \( p^*_w < \bar{p} < p^*_s \). It is assumed that both markets are served at the non-discriminatory price – a sufficient condition for this is that demand in the weak market is positive when the discriminatory price in the strong market is set, i.e. \( q_w(p^*_s) > 0 \). Social welfare is the sum of consumer surplus and producer surplus so the marginal effect of a price change on social welfare is \( (p - c)q'(p) \), which is the effect on the quantity multiplied by the price-cost margin.

When demand is strictly convex an additional assumption is required for the profit function to be concave. This holds if \( \alpha < 2 \) everywhere, so that \( 2 - L \alpha > 0 \) since the Lerner index is bounded above by 1. An alternative sufficient condition works when inverse demand has constant positive curvature, which includes the special cases of constant-elasticity and exponential demand functions. In this case if all prices are below \( 2 p^* - c \) the profit function is concave. In the strong market this holds automatically (since \( 2 p^* - c > p^* > \bar{p} \)) but it must be assumed for the weak market. This assumption implies that the non-discriminatory price is not too far above the discriminatory price in the weak market.

---

\[ 2 \text{ This can be shown using the generic inverse demand function with constant } \sigma, \quad p = a - bq^{1-\sigma}(1-\sigma) \text{ for } \sigma \neq 1 \text{ or } p = a - b \ln(q) \text{ for } \sigma = 1. \text{ The constant elasticity function is a special case where } a = 0 \text{ and the price elasticity is } 1/(\sigma - 1). \text{ It follows that } p^* = [(1-\sigma)a + c]/(2-\sigma) \text{ when } \sigma \neq 1 \text{ and } p^* = b + c \text{ when } \sigma = 1. \]
3. The effect of discrimination on welfare

The method is the one used by Schmalensee (1981), Holmes (1989) and Stole (2007) to consider the output effect. Initially the firm is not allowed (or is unable) to discriminate and thus sets a uniform price. Then the constraint that the prices must be equal is relaxed so that some discrimination is allowed. Eventually the firm can discriminate as much as it likes. The sign of the marginal effect on welfare of relaxing the constraint is determined and then the marginal effects are integrated. The firm chooses the two prices subject to the constraint that

\[ p_s - p_w \leq r \]

where \( r \geq 0 \) is the amount of discrimination allowed. With the constraint binding the firm’s objective function is \( \pi'_w(p_w) + \pi'_s(p_s + r) \) and the first-order condition is \( \pi'_w(p_w) + \pi'_s(p_s + r) = 0 \). As \( r \) rises more discrimination is allowed and the price in the weak market falls and that in the strong market rises:

\[
(1) \quad p'_w(r) = \frac{-\pi''_s}{\pi''_w + \pi''_s} < 0; \quad p'_s(r) = \frac{\pi''_w}{\pi''_w + \pi''_s} > 0.
\]

When \( r \geq r^* \equiv p'_s - p'_w \) the constraint does not bind and the firm sets the discriminatory prices. When \( r = 0 \) the firm sets the non-discriminatory price, \( \bar{p} \). The marginal change in social welfare, \( W \), as more price discrimination is allowed is

\[
(2) \quad W'(r) = (p_w - c)q'_w p'_w (r) + (p_s - c)q'_s p'_s (r).
\]

A relaxation of the constraint alters prices and thus the quantities demanded, and each additional unit of output has social value equal to the price-cost margin in that market. Expression (2) can be used to prove the Schmalensee-Varian result that an output increase is necessary for welfare to rise. Total output is \( Q = q_w(p_w) + q_s(p_s) \). Following Schmalensee (1981) add and subtract \( \bar{p} \) in both \( (p_w - c) \) and \( (p_s - c) \) in equation (2) so

\[
(3) \quad W'(r) = (\bar{p} - c)Q'(r) + (p_w - \bar{p})q'_w p'_w (r) + (p_s - \bar{p})q'_s p'_s (r).
\]
where \( Q'(r) = q_w'(p_w) + q_s'(p_s) \) is the marginal output effect. The second and third terms in equation (3) are negative since the price falls in the weak market, where the price is below the non-discriminatory price, with the opposite occurring in the strong market. Thus when (3) is integrated over \([0, r^*]\) the total welfare effect is \((\bar{p} - c)\) multiplied by the change in total output, plus two negative terms. It follows that if the welfare effect of full discrimination is positive, total output must increase and the benefit of the output effect must exceed the cost of the misallocation effect.\(^3\)

It turns out that much more can be said about the sign of the welfare effect. Using the comparative statics results for prices, (1), in (2) gives the marginal welfare effect:

\[
W'(r) = -\frac{\pi_w'\pi_r''}{\pi_w'' + \pi_r''} \left[ \frac{p_w - c}{2 - L_w\alpha_w} - \frac{p_s - c}{2 - L_s\alpha_s} \right],
\]

which has the same sign as the expression in square brackets. For \( r > r^* \) the marginal welfare effect is zero because the prices remain at the discriminatory levels. At \( r^* \) the welfare function is not differentiable (its left-derivative need not equal its right-derivative, which is zero), so \( W'(r^*) \) will be defined as the left-derivative. The following assumption is made throughout.

**Assumption 1.** \( \frac{p - c}{2 - L\alpha} \) is increasing in \( p \) in each market.

The expression in Assumption 1 is the ratio of the marginal effect of a price increase on social welfare, \((p-c)q'(p)\), to the second derivative of the profit function, \(2q' + (p-c)q''\).

Assumption 1 holds for a very large set of demand functions. These include: functions that are linear; inverse demands with constant positive curvature, including the exponential and constant-elasticity functions; direct demand functions with constant curvature (whether positive or negative); probits and logits (derived from the normal and logistic distributions respectively); and demand functions derived from the lognormal distribution. All are characterized by at least one of the three curvature measures, \( \alpha, \alpha/p \) and \( \sigma \), being non-

\(^3\) See Aguirre (2008b) for a decomposition of the welfare effect into an output effect and a misallocation effect.

decreasing in price. Appendix 1 presents the conditions for Assumption 1 to hold and gives a full list of the demand functions that satisfy the assumption. While it is very common for Assumption 1 to hold – and for all demand functions it holds locally in the region around marginal cost – it is not universally applicable. When inverse demand has constant negative curvature Assumption 1 does not hold for high enough prices. As will be shown, Assumption 1 implies that social welfare is single-peaked in the amount of discrimination, \( r \), and thus limits the possible outcomes to three: either welfare is everywhere decreasing in \( r \), or everywhere increasing, or it first rises then falls. These three possibilities are characterized in the propositions that follow.

The sufficient condition for discrimination to reduce welfare is now developed. It exploits the fact that if at \( r = 0 \) the marginal welfare effect is negative or zero, and the marginal welfare effect retains this sign as \( r \) rises, then the overall welfare effect of discrimination must be negative. In economic terms starting from no discrimination a small amount of discrimination either has no effect or is harmful and any additional discrimination reduces welfare.

**Proposition 1** If the direct demand function in the strong market is at least as convex as that in the weak market at the non-discriminatory price then discrimination reduces welfare.

Proof. The aim is to show that the derivative in (4) is non-positive when \( r = 0 \) and is negative for \( r > 0 \). At the non-discriminatory price \( p_w - c = p_s - c \) and \( L_w = L_s \). Since \( \alpha_s(\bar{p}) \geq \alpha_w(\bar{p}) \) the term in square brackets in (4) is non-positive so \( W'(0) \leq 0 \). Assumption 1 implies that \( (p - c)/(2 - L\alpha) \) falls in the weak market as \( r \) rises and the price falls, while the same term in the strong market increases. Thus \( W'(r) < 0 \) for \( 0 < r \leq r^* \). The total welfare effect, the integral of the marginal effects, is thus negative. \( \square \)

Starting from uniform pricing a small amount of discrimination reduces, or at best maintains, welfare. Assumption 1 then enables this local result to be extended to all additional increases in the amount of discrimination. As the two prices diverge the sign of the marginal welfare effect is certainly negative. The possibility that welfare first falls in \( r \) and then rises is eliminated by Assumption 1. The proposition encompasses the results of Cowan (2007), who assumes that demand in the strong market is an affine transformation of demand in the weak market, i.e. \( q_s(p) = M + Nq_w(p) \) where \( M \) and \( N \) are positive. At the same price the direct
demand functions, by construction, have the same curvature and thus $W'(0) = 0$. This is analogous to the result in expected utility theory that absolute and relative risk aversion, at a given income level, are invariant to positive affine transformations of the utility function. An example is when the direct demand functions have constant and common curvature, $\alpha$, and a special case is when both demand functions are linear ($\alpha = 0$). Proposition 1 is more general because it allows the demand functions to have different parameters or to have different functional forms altogether, as in the following example.

**Example 1: exponential and linear demands.** Demand in the strong market is $q_\text{s}(p) = Be^{-p/b}$ (for $B$ and $b$ positive) so $p_\text{s}^* = b + c$, $\sigma_\text{s} = 1$ and $\alpha_\text{s} = \eta_\text{s} = p/b > 0$. Demand in the weak market is $q_\text{w}(p) = a - p$ with $p_\text{w}^* = (a + c)/2$ and $\alpha_\text{w} = \sigma_\text{w} = 0$. Proposition 1 applies if $b > (a + c)/2$, which says that the margin in the exponential-demand market exceeds that in the market with linear demand. The weak market is served with non-discriminatory pricing if $a > b + c$.

In many cases where Proposition 1 applies, such as Example 1, output falls or remains constant with discrimination. The condition in the Proposition does not, however, imply that output falls or remains constant: for example Cowan (2007) shows that when $\alpha$ is negative, constant, and equal in the two markets – so Proposition 1 applies – total output rises with discrimination. Proposition 1 is thus more than a statement about when output does not rise.

Suppose that the condition in Proposition 1 does not apply, so direct demand in the weak market is more convex than that in the strong market and without further information the sign of the welfare effect is unknown. To develop some intuition for Proposition 1 it helps to borrow another device from expected utility theory. Imagine a strictly concave transformation of demand in the weak market that preserves its slope and position at the non-discriminatory price, which thus remains optimal. A transformation that reduces the convexity of direct demand in the weak market until it equals that in the strong market will entail that Proposition 1 applies. Formally the new demand function is $f(q_\text{w}(p))$ with $f(.)$ increasing and strictly concave and satisfying $f(q_\text{w}(\bar{p})) = q_\text{w}(\bar{p})$ and $f'(q_\text{w}(\bar{p})) = 1$ to keep the position constant.

---

4 Joan Robinson (1933) showed that output does not rise when the weak market is concave and the strong market is convex, which is the case in Example 1.

5 Aguirre (2008a, Theorem 1) shows that output rises if $\alpha_\text{w} \geq \alpha_\text{s}$ and $\sigma_\text{w} \geq \sigma_\text{s}$ (without equality everywhere), and the example in Cowan (2007) satisfies these conditions.
and slope of demand the same at the non-discriminatory price. At $\bar{p}$ the curvature of the new demand function is below that of the old one because $f(.)$ is strictly concave.

Standard monopoly theory helps to explain the effect of the concave transformation on marginal revenue and welfare. Figure 1 illustrates. The initial inverse demand is $p_1(q)$, while the transformed inverse demand is $p_2(q)$. At the non-discriminatory quantity $q_0$ the marginal revenues (MR) for the two demand functions are equal (because the price, demand slope and quantity are the same). For output levels above $q_0$, however, marginal revenue with $p_2(q)$ is everywhere below that for $p_1(q)$ because the price is lower and, in this region, a given output increase causes a larger price reduction. It follows that output with discrimination, determined by the intersection of marginal revenue with marginal cost, is lower with the transformed function ($q_2$ instead of $q_1$). The welfare gain from discrimination in the weak market is then lower with the transformed function for two reasons: the output increase is smaller, and in this region at every quantity the price is lower. The welfare gain from discrimination with the linear demand function is the area $MNq_1q_0$, while the welfare gain from discrimination with the concave demand function is the smaller area $MZq_2q_0$.

![Figure 1](image_url)

---

6 See Bulow and Pfleiderer (1983) for a discussion of the relationship between the marginal revenue curves associated with demand curves that have a point of tangency.
When Assumption 1 does not necessarily hold (for example when $\sigma$ is constant and negative) an alternative version of Proposition 1 applies in some circumstances. The welfare derivative is non-positive, whether or not Assumption 1 holds, if inverse demand in the weak market is concave and inverse demand in the strong market is at least as convex as that in the weak market at each value of $r$. Appendix 2 gives the details. Again higher convexity in the strong market tends to make discrimination bad for welfare.

When does discrimination raise welfare? The answer is that when inverse demand in the weak market is more convex than that in the strong market, and the discriminatory prices are not too far apart, welfare rises with discrimination. The trick is to start at $r^*$ and then to reduce the amount of discrimination, $r$. Using equation (4) and the fact that $L^* \alpha^* = \sigma^*$ the left-derivative of $W(r)$ at $r^*$ has the same sign as

$$\frac{p_w^*-c}{2-\sigma_w^*} - \left( \frac{p_s^*-c}{2-\sigma_s^*} \right).$$

**Proposition 2.** If the inverse demand function in the weak market is more convex at the discriminatory price than inverse demand in the strong market and the discriminatory prices are close, with

$$\frac{p_w^*-c}{2-\sigma_w^*} \geq \frac{p_s^*-c}{2-\sigma_s^*},$$

then welfare rises with discrimination.

Proof. The condition implies that $W'(r^*) \geq 0$. As $r$ falls below $r^*$, the price in the weak market rises and the price in the strong market falls. It follows from Assumption 1 that $(p_w - c)/(2 - L_w \alpha_w)$ rises and $(p_s - c)/(2 - L_s \alpha_s)$ falls so $W'(r)$ is positive for all $r < r^*$. \qed

Since welfare rises with discrimination total output must also increase. The condition in Proposition 2 ensures that the price difference with full discrimination is small enough that the benefit to social welfare from the output increase exceeds the cost of the inefficient distribution of output, because this holds for all marginal increases in discrimination. The proposition will be illustrated with three examples, two of which have inverse demand functions with constant curvature. The first uses the demand functions of Example 1. The exponential market, which has $\sigma = 1$, is now the weak market. Proposition 2 applies if $(a-c)/2 > b \geq (a-c)/4$. The first inequality states that the discriminatory margin in the exponential market, $b$, is below that in the linear market. The second inequality says that the
margin in the exponential market must be at least half that of the linear market and is the condition for $W'(r^*) \geq 0$ (it also guarantees concavity of the profit function in the exponential-demand market).

In the second example the demand function in the weak market has the constant-elasticity form $q_w = Ap^{-\eta}$, with market-size parameter $A$ and elasticity $\eta > 1$, so $p^*_w = \eta c / (\eta - 1)$, $p^*_w - c = c / (\eta - 1)$ and $\sigma = 1 + 1 / \eta$. The demand function in the strong market is exponential, $q_s = Be^{-p^*b}$. Proposition 2 applies if $\eta c / (\eta - 1)^2 \geq b > c / (\eta - 1)$. The second inequality states that the margin in the strong market exceeds that in the constant-elasticity market, while the first inequality implies that $W'(r^*) \geq 0$.

In the previous examples the demand functions are convex. The third example shows that Proposition 2 applies just as well when demand functions are concave. Suppose that both direct demand functions have the constant curvature form $q = A - p^{1-\alpha} / (1-\alpha)$ with $\alpha < 0$ so the demand functions are strictly concave. With marginal cost of zero the discriminatory price is $p^* = ((1-\alpha)A / (2-\alpha))^{1/(1-\alpha)}$ and $\sigma^* = \alpha$ because the price elasticity is unity at the monopoly price. By choosing $\alpha$ and $A$ appropriately for the two markets so that the discriminatory prices differ, but not by too much, the condition in Proposition 2 can be satisfied. For example if $A_w = A_s = 1$, $\alpha_s = -1$ and $\alpha_w = -0.5$ then discrimination raises welfare.

To check whether Proposition 2 applies requires only a small amount of calculation. The discriminatory prices and the curvatures of inverse demand have to be determined, Assumption 1 must be checked, and it has to be confirmed that both markets are served. The non-discriminatory price does not need to be calculated. The multiplicative market-size parameters, e.g. $B$ in the exponential case and $A$ in the constant-elasticity function, will determine the non-discriminatory price but do not affect the discriminatory prices and thus do not enter the statement of Proposition 2. The proposition is thus independent of relative market size.

The effect on the discriminatory price of an infinitesimal change in marginal cost is $1/(2 - \sigma^*)$. This suggests a possible empirical strategy for determining whether the condition in Proposition 2 holds when the firm initially discriminates. A regression of the monopoly
price on marginal cost yields a slope coefficient which is the estimate of \(1/(2-\sigma^*)\). The demand function itself does not need to be estimated in this reduced-form approach. Alternatively if the data are obtained when there is no discrimination the demand function could be estimated directly. Genesove and Mullin (1998) estimate four inverse demand functions with constant curvature.\(^7\)

Suppose that the condition for Proposition 2 does not hold and inverse demand in the weak market then is subject to a convex transformation while retaining its slope and position at the discriminatory quantity. The optimal discriminatory price is the same with both the transformed and the original demand functions, and the transformation raises \((p^*_w-c)/(2-\sigma^*_w)\) by increasing \(\sigma^*_w\). When the convexity of inverse demand in the weak market becomes sufficiently high (while remaining below 2) Proposition 2 will apply.

So far the conditions for welfare to be monotonically decreasing or increasing have been derived, providing unambiguous conclusions about the total effect on welfare of price discrimination. The next proposition gives conditions for welfare to rise initially, and then to fall, as discrimination increases. The conditions are simply the negation of those in the two previous propositions. This completes all the possible outcomes when Assumption 1 holds.

**Proposition 3.** If (i) direct demand in the weak market is more convex than demand in the strong market at the non-discriminatory price, and (ii) the discriminatory prices are sufficiently far apart with \(\frac{p^*_w-c}{2-\sigma^*_w} < \frac{p^*_s-c}{2-\sigma^*_s}\), social welfare rises initially as the amount of discrimination increases, and then falls.

Proof. Condition (i) implies that \(W'(0) > 0\) and (ii) implies that \(W'(r^*) < 0\). Assumption 1 implies that \((p_w-c)/(2-L_w\alpha_w) - (p_s-c)/(2-L_s\alpha_s)\) is decreasing in \(r\). It follows that there is a unique value, \(\hat{r} \in (0,r^*)\), at which \(W'(\hat{r}) = 0\). For \(r < \hat{r}\) \(W'(r) > 0\), while \(W'(r) < 0\) for \(r > \hat{r}\). \(\square\)

\(^7\) The special cases are linear demand (\(\sigma = 0\)), constant-elasticity demand (\(\sigma = 1 + 1/\eta\)), exponential demand (\(\sigma = 1\)) and quadratic demand (\(\sigma = 0.5\)).
Proposition 3 says that starting at the non-discriminatory price a small amount of price discrimination raises welfare, while starting at the discriminatory prices a small reduction in the amount of discrimination raises welfare. It does not determine whether the effect on welfare of full discrimination is positive or negative. The example with exponential and linear demands can again be used to illustrate. If the exponential market is the weak one and \( b < (a - c)/4 \) then the difference between the discriminatory prices is large enough for Proposition 3 to apply. The demand functions of Example 1 have now been used to illustrate all three propositions: taking the linear demand function and marginal cost as given, discrimination reduces welfare if the margin in the exponential market, \( b \), is high, raises welfare if \( b \) takes an intermediate value and has an uncertain effect on welfare when \( b \) is low.

Condition (ii) in Proposition 3 holds if \( \sigma^*_w \geq \sigma^*_s \). Two further applications of the proposition using this result are now discussed. First, let both markets have constant-elasticity demand functions, with \( \eta_w \) and \( \eta_s \) being the elasticities in the weak and strong markets respectively and \( \eta_w > \eta_s > 1 \). Demand in each market has to be elastic for an interior solution to the monopoly pricing problem to exist. The generic constant-elasticity demand function is \( q = Ap^{-\eta} \). The curvature of direct demand is \( \alpha = 1 + \eta \), which is higher in the weak market so condition (i) in Proposition 3 holds. The curvature of inverse demand is \( \sigma = 1 + 1/\eta \), which is higher in the strong market so (ii) also holds. Second, let \( \sigma \) be positive, constant and common to the two markets so the inverse demand functions are affine transformations of each other. At the non-discriminatory price \( \alpha \), which equals \( \sigma\eta \), is higher in the weak market (because the price elasticity in the weak market is higher when the same price is set in both markets). Since \( \sigma \) is the same in the two markets condition (ii) also holds.

When both demand functions have constant elasticities, and when \( \sigma \) is constant, common and positive, discrimination can cause welfare to fall or rise. Proposition 3 hints at this ambiguity. To say more about the overall effect on welfare of full discrimination different techniques must be used. For constant-elasticity demand functions Ippolito (1980) shows numerically that the welfare effect can have either sign. For this case Cowan and Vickers (2007) show analytically that if the difference between the elasticities in the two markets is no more than 1, i.e. \( \eta_w - \eta_s \leq 1 \), then discrimination reduces welfare. When this condition does not hold,
however, welfare can rise. Similarly when $\sigma$ is positive, constant and common Cowan and Vickers (2007) prove that if $\sigma > (\leq) 1$ welfare rises (falls) with discrimination, as long as the discriminatory prices are close together. If the discriminatory prices are far enough apart, though, the welfare effects can be reversed.

4. Conclusion
This paper has presented conditions that determine the sign of the welfare effect of price discrimination. The conditions rely on general properties of demand functions and are usually straightforward to apply. The main new result is that price discrimination raises welfare when inverse demand in the market with the lower price is more convex than that in the other market and the price difference is small. The paper also generalizes the negative welfare result of Cowan (2007) and provides a framework that nests all existing results. The standard presumption that the effect of price discrimination can be positive or negative for social welfare is confirmed, but some quite general conditions under which the sign of the welfare effect can be predicted have been provided and it has been shown that the conditions for discrimination to raise welfare are plausible.

There are two directions that might be followed in extending this work. First, there is scope for integrating the analysis where all markets are served with that where discrimination opens new markets. Convexity of demand in the market where the price falls tends to be good for welfare. The same idea applies when a new market is opened by discrimination, since there is a natural sense in which the demand function of a market that is opened has a region of convexity. Varian (1985, footnote 1) points out that “what economists call “linear” demand curves are not really linear functions”, because for prices above the choke price (the vertical intercept of inverse demand) the quantity demanded is zero. The demand curve thus has a sharp point at the choke price and is strictly convex around that point. A second direction is to see if the welfare approach can be applied to models of competitive price discrimination. Holmes (1989) uses the same technique to evaluate the output effect of discrimination in an oligopoly context, and an extension of this to examine welfare effects directly may be fruitful.
Appendix 1. Sufficient conditions for Assumption 1.

Assumption 1 states that \((p - c) / (2 - L\alpha)\) is increasing in \(p\). Calling this expression \(z(p)\), it may be written in three ways:

\[
(A1) \quad z(p) = \frac{1}{2} \frac{\frac{\alpha}{p - c} - 2 - L\alpha}{p - c} = \frac{p - c}{2 - L\alpha} = \frac{p - c}{2 - L\eta\sigma}.
\]

Using the first version of (A1) \(z(p)\) is increasing if the absolute curvature of direct demand, \(\alpha / p \equiv -q^\alpha / q'\), is non-decreasing in \(p\). This is equivalent to the slope of demand, \(-q'(p)\), being log-concave, and existing results on log-concave density functions in probability theory can be used, since the slope of the demand function can be thought of as a density function (see Caplin and Nalebuff, 1991, Bagnoli and Bergstrom, 2005, and Cowan, 2007). Many demand functions have log-concave slopes including linear demand, probit demand (from a normal distribution), demands derived from the extreme value and logistic distributions, the functions with constant negative \(\alpha\), and functions with constant \(\sigma\) in \([0, 1]\), including exponential demand. From the second version of (A1) \(z(p)\) is increasing if \(d(L\alpha)/dp > 0\). In turn there are two sets of sufficient conditions. First, if \(\alpha\) is non-decreasing in \(p\) and is positive then \(d(L\alpha)/dp > 0\). This is useful when the slope of demand is not log-concave. Examples are: \(\alpha\) that is constant and positive, a special case of which is the constant-elasticity demand curve with \(\alpha = 1 + \eta\) (where \(\eta > 1\) is the elasticity); demands derived from the lognormal, \(F\), Weibull and Gamma distributions (when their slopes are log-convex), and demand that comes from the \(t\) distribution with two or more degrees of freedom (see Cowan, 2007). Second, use the fact that \(L\alpha = L\eta\sigma\), which gives the third version in (A1). If \(\sigma\) is positive and non-decreasing, and \(d(L\eta)/dp > 0\), then \(z(p)\) is increasing. A sufficient condition for \(d(L\eta)/dp > 0\) is that \(\sigma \leq 1\). For constant \(\sigma > 1\) a sufficient condition for \(d(L\eta)/dp > 0\) is that demand is finite when price equals marginal cost (Cowan and Vickers, 2007, page 11). The derivative of \(z(p)\) evaluated at \(c\) is \(1/2\) for all demand functions, so \(z(p)\) cannot be everywhere decreasing. When \(\sigma\) is constant and negative \(z(p)\) is increasing when prices are close to marginal cost but is decreasing at high enough prices.
Appendix 2. Conditions for the welfare derivative to be non-positive whether or not Assumption 1 holds.

By rearranging the term in square-brackets in equation (4), and using the fact that $\alpha = \sigma \eta$, it can be seen that the welfare derivative $W'(r)$ has the same sign as

$$-2r + (p_w - c)(p_s - c) \left\{ \sigma_w \frac{\eta_w}{p_w} - \sigma_s \frac{\eta_s}{p_s} \right\},$$

which is non-positive if $\sigma_w \eta_w / p_w \leq \sigma_s \eta_s / p_s$ at each value of $r$. This holds when demand in the weak market is concave and demand in the strong market is convex, which together imply that total output does not rise (Joan Robinson, 1933). The sufficient condition also holds when $\sigma_w \leq \sigma_s \leq 0$ at each value of $r$. The first-order condition for the constrained maximization problem implies that $\eta_w > \eta_s$ for $r < r^*$ and $\eta_w \geq \eta_s$ at $r^*$. Since $p_w < p_s$ for $r > 0$ (with $p_w = p_s$ at $r = 0$) it follows that $\eta_w / p_w > \eta_s / p_s$ for all $r$, so if $\sigma_w \leq \sigma_s \leq 0$ then $\sigma_w \eta_w / p_w \leq \sigma_s \eta_s / p_s$. This condition also implies that total output does not rise (see Shih, Nai and Liu, 1988, Cheung and Wang, 1994 and Aguirre, 2008a). Putting the two conditions together the welfare derivative is non-positive if $\sigma_w \leq 0$ and $\sigma_w \leq \sigma_s$ at each value of $r$. 


References

Aguirre, I. (2008a) “Joan Robinson was almost right: output under third-degree price discrimination”, mimeo, University of the Basque Country.


