Climate Tipping and Economic Growth:

Precautionary Capital and the Price of Carbon*

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Abstract

The optimal reaction to a productivity shock which becomes more imminent with global warming is to price carbon (proportional to the marginal hazard of a catastrophe) to curb the risk of climate change, but also to accumulate precautionary capital to facilitate smoothing of consumption and curb the adverse effects of the calamity on the economy. We allow for conventional marginal climate damages as well and decompose the optimal carbon price in two catastrophe components and a conventional component. Moreover, the productivity catastrophe is compared with recoverable catastrophes and with a shock to the temperature response.

Key words: non-marginal climate shock, tipping point, precautionary capital accumulation, economic growth, risk avoidance, social cost of carbon.

JEL codes: D81, H20, O40, Q31, Q38.

This draft: April 2015, first draft: September 2013

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* We are grateful to John Hassler, Niko Jaakkola, Larry Karp, Per Krusell, Derek Lemoine, Matti Liski, Thomas Lontzek, Antony Millner, Christian Traeger and Sweder van Wijnbergen for helpful comments on an earlier draft and thank Xiaoli He for help with preparing the figures. We also benefited from many comments at presentations in UCL, Amsterdam, Bergen, Berlin, Modena, Oxford, Paris, Stockholm and Tilburg, at the EAERE conference in Toulouse, 2013, and at a workshop at the Beijer Institute, Stockholm, 2014, and the Royal Economic Society Conference, Manchester, 2015. Van der Ploeg is also affiliated with the VU University Amsterdam and grateful for support from the ERC Advanced Grant 'Political Economy of Green Paradoxes' (FP7-IDEAS-ERC Grant No. 269788) and the BP funded OXCARRE. De Zeeuw is grateful for support from the European Commission under the 7th Framework Programme (Socioeconomic Sciences and Humanities - SSH.2013.2.1-1 – Grant Agreement No. 613420).

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1. Introduction

Global warming calls for a global price of carbon corresponding to the present value of all future marginal damages arising from emitting one ton of carbon (e.g., Nordhaus, 2008; Stern, 2007; Golosov et al., 2014). Although it is acknowledged that there may be catastrophic damages associated with global warming, the problem is usually resolved by assuming convex damages[1]. Our main contribution is to investigate the consequences for climate policy of a pending non-marginal calamity which becomes more imminent with global warming. Uncertainty about the time, impact and type of such a climate catastrophe is a key feature of our analysis. We show that the economy needs to react in two ways to such catastrophes. First, carbon needs to be priced (via a global carbon tax or emissions market) to curb the risk of climate calamities as the hazard of a catastrophe increases with global mean temperature or the stock of atmospheric carbon. Second, precautionary saving is needed to cope with the chance of an abrupt downward drop in consumption at some point in the future.

Policy must thus deal with the possibility of large, abrupt and persistent changes in the climate system, called regime shifts in the ecological literature (e.g. Biggs et al., 2012). A point where such a regime shift occurs is called a tipping point. The idea that the prime role of climate policy is to deal with a small risk of abrupt and often irreversible climate disasters and tipping points at high temperatures rather than internalizing smooth global warming damages at low and moderate temperatures is gaining traction (e.g., Lenton and Ciscar, 2013; Kopits et al., 2013; Pindyck, 2013). These sudden shifts in damage might show up as shocks to total factor productivity arising from flooding of cities, sudden increased occurrence of storms and droughts, abrupt desertification of agricultural land, or reversal of the Gulf Stream and change in the thermohaline circulation (e.g., Alley et al., 2003).

Another example of a regime shift is a sudden acceleration of global warming associated with a reduction in cooling when ice sheets such as the Western Antarctic Ice Sheet melt away (e.g., Oppenheimer, 1988).[2] A related example occurs if rising sea

[1] E.g., Golosov et al. (2014) find the rule for the optimal carbon tax by using a weighted average of low and high damages and implement this rule in an integrated assessment model.
[2] The ice-albedo effect acts more quickly over oceans than land as sea ice melts faster than continental ice sheets. Also, the demise of rain forests curbs transpiration as plants have lower reflectivity than soil.

temperatures and sea levels trigger the sudden release of methane buried in sea-beds and permafrost (e.g., Dutta et al., 2006).[3] Since methane is itself a powerful (albeit shorter lived) greenhouse gas, release of it increases global warming and sets in motion further destabilization. Such tips can induce positive feedback in the carbon cycle; we view them as a sudden increase in the climate sensitivity or stock of greenhouse gases.

We also highlight how the optimal response depends on what type of catastrophe one has to deal with, how imminent it is, and how sensitive the arrival time is to changes in global warming. We analyse capital accumulation and fossil fuel use in a Ramsey growth model where global warming makes a regime shift more imminent. We start with the possibility that a catastrophe might trigger a sudden drop in total factor productivity. We show that the economy then aims for a higher steady-state stock of capital than without an imminent disaster. This precaution facilitates consumption smoothing over the whole time horizon. Since the hazard of a regime shift increases with the stock of greenhouse gases, it is optimal to price carbon in order to curb emissions and the probability of a regime shift. We show that the marginal hazard of a catastrophe plays a key role in setting the carbon price.

We offer expressions for the precautionary return on capital and the optimal price of carbon. We also give simulation results both before and after the catastrophe based on a calibrated Ramsey model of the global economy with catastrophic climate shocks. We extend our core analysis in three directions. First, we allow for *gradual* marginal climate damages as in the integrated climate assessment literature (e.g., Tol, 2002; Nordhaus, 2008; Stern, 2007; Golosov et al., 2014). We show that the optimal carbon price consists of a catastrophe component and the usual component internalizing marginal damages. The catastrophe component consists of a risk-averting part and a raising-the-stakes part, where the latter equals the expected present value of future increases in after-calamity welfare from curbing carbon emissions with one unit. Second, we derive the optimal response to a catastrophic, sudden increase in climate sensitivity. Global warming after the tip then becomes more severe (for the productivity shock the opposite occurs as economic activity and carbon emissions fall

[3] If sea temperatures rise above a critical level, bleaching of coral reefs as systems shift from a coral- to an algae-dominated state destroys fish populations and tourist infrastructure (Hughes et al., 2003).

after a productivity tip). Third, we consider the implications of *recoverable* catastrophes such as a sudden rise in the carbon stock or destruction of part of capital.

Our contribution is to emphasize the need for precautionary capital as well as an appropriate carbon price in the face of tipping points driven by global warming in a transparent framework. The previous literature has attended to the hazard of climate catastrophes but does not provide the full picture. Clarke and Reed (1994), Tsur and Zemel (1996) and Gjerde et al. (1999) analyse catastrophes with hazard rates, but they adopt a partial equilibrium perspective and do not have Ramsey growth and thus no need for precautionary capital accumulation. Gjerde et al. (1999) allow for serious carbon cycle and temperature modules with 17 different greenhouse gases, gradual and catastrophic damages, and give for the first time detailed simulation studies of the effects of pending catastrophes on temperature and the economy, but do not focus on the optimal price of carbon or the need for precautionary saving. Smulders et al. (2014) do discuss the need for precautionary saving to deal with an impending disaster, but their analysis uses a constant hazard rate whereas our analysis highlights temperature-dependent hazard rates for climate policy. Engström and Gars (2014) study catastrophes within the model of Golosov et al. (2014), but the special assumptions needed to get a tractable solution of this model remove the incentive to have precautionary saving to cope with a pending catastrophe. Keller et al. (2004) study the combined effects of a climate threshold, a potential ocean thermohaline circulation collapse, and learning in the DICE model, an integrated assessment model for climate change (Nordhaus, 2008). Cai et al. (2012), Lontzek et al. (2014) and Lemoine and Traeger (2014) extend the DICE model to numerically analyse the effect of tipping points on the optimal carbon tax. Cai et al. (2012) and Lontzek et al. (2014) use a hazard rate for an upward shift in the damage function whilst Lemoine and Traeger (2014) consider a temperature threshold that is uncertain and add learning by formulating a hazard rate that is zero at temperature levels that have proven to be safe. Our study focuses on theory first to highlight the need for precautionary saving as well as pricing carbon, and then presents illustrative quantitative results in a simple calibrated model of the world economy.

Other studies have focused on catastrophes and extreme events but not in the context of general equilibrium models of Ramsey growth and climate change. Barro (2013) studies the optimal investment needed to curb the probability of an environmental disaster but abstracts from precautionary accumulation of capital. Weitzman (2007) highlights the fundamental uncertainty at the upper end of the probability distribution of possible increases in temperature and corresponding damage, and shows that the policy consequences of a fat instead of a thin tail can be dramatic. Martin and Pindyck (2014) study the 'strange' implications for cost-benefit analysis of a cascade of catastrophes within a partial equilibrium context. Pindyck and Wang (2013) use a general equilibrium framework to obtain a price of insurance against catastrophic risks.

Section 2 presents our model of growth with tipping points. Section 3 shows that the optimal response to climate tipping requires precautionary capital accumulation and, if the hazard increases with atmospheric carbon, pricing carbon to curb the risk of tipping. It also shows the effects of a change in the elasticity of intertemporal substitution. Section 4 allows for gradual as well as catastrophic damages from global warming and decomposes the optimal carbon tax into the usual social cost of carbon (i.e., the present value of marginal gradual damages) and the extra tax that is needed to curb the risk of climate tipping. It also discusses regime shifts with a sudden increase in climate sensitivity and recoverable catastrophes that destroy part of the capital stock or lead to a sudden release of atmospheric carbon. Section 5 concludes.

2. Ramsey growth with climate tipping

Consider a continuous-time Ramsey growth model with constant population. Fossil fuel E is input into the production process, has constant marginal cost $d > 0$, and (given abundance of coal and shale gas) is in abundant supply. There is a carbon-free imperfect substitute for fossil fuel R, renewable energy, with constant cost $c > 0$. Capital K, fossil fuel and the renewable substitute are cooperative factors of production. Total factor productivity is A before the regime shift and drops to $(1-\pi)A < A$ afterwards, where $0 < \pi < 1$ is the size of the climate disaster. Utility is denoted by U, consumption by C, the production function by A times F before and $(1-\pi)A$ times F after the regime switch, the depreciation rate of capital by $\delta > 0$ and the pure rate of time preference by $\rho > 0$. We ignore population growth and technical progress. Use of fossil fuels leads to emissions of carbon dioxide with the emission rate $\psi > 0$. The stock of atmospheric carbon P decays naturally at the rate $\gamma > 0$ (typically about 1/300).[4] To focus on the policy implications of tipping points, we abstract from carbon capture and carbon sequestration, learning-by-doing in the renewable sector, and other forms of technical progress.

We assume that the magnitude of the potential drop in total factor productivity is known but that it is not known *when* the climate regime shift will take place. The hazard rate h gives the conditional probability of the tipping point T. Formally,

$$(1) \qquad h(t) = \lim_{\Delta t \to 0} \frac{\Pr[T \in (t, t+\Delta t) \mid T \notin (0,t)]}{\Delta t},$$

so $h(t)\Delta t$ is the probability that the regime shift takes place between t and $t + \Delta t$, given that it has not occurred before t. A constant hazard rate h has the exponential density function $f(t) = he^{-ht}$ with mean $1/h$ and cumulative density function $F(t) = 1 - e^{-ht}$. The probability of "survival" is e^{-ht}. If the hazard rate h is not constant, ht is replaced by $\int_0^t h(s)ds$. A large stock of carbon P increases the probability of climate change, hence $h(t) = H\left(P(t)\right)$ with $H'(P) > 0$. With global warming the expected duration before the regime shift occurs, $1/H(P)$, decreases over time. Hence, failing climate policy makes the shock to productivity more imminent.[5]

The social planner maximize the expected value of social welfare

$$(2) \qquad \max_{C,E,R} \mathrm{E}\left[\int_0^\infty e^{-\rho t} U(C(t))dt\right]$$

subject to the accumulation of capital and greenhouse gases

[4] About a fifth of carbon emissions remain in the atmosphere for thousands of years or forever, but in contrast to Golosov et al. (2014) we suppose that all of it decays eventually. We also ignore temperature lags, but see Gerlagh and Liski (2012) for that. Our calibration of (3) below is close to that of the seminal contribution of Nordhaus (1991), who uses $\psi = 0.5$ and $\gamma = 0.005$. We set $\psi = 0.5$ and $\gamma = 0.003$.
[5] This type of modelling allows the stochastic dynamic optimization to be transformed into a deterministic one (cf. Clarke and Reed, 1994).

$$\dot{K}(t) = \tilde{A}(t)F\left(K(t), E(t), R(t)\right) - dE(t) - cR(t) - C(t) - \delta K(t),$$
(3)
$$\dot{P}(t) = \psi E(t) - \gamma P(t), \quad K(0) = K_0, \quad P(0) = P_0,$$

with total factor productivity given by

(4) $$\tilde{A}(t) = A, \quad 0 \le t < T, \qquad \tilde{A}(t) = (1-\pi)A < A, \quad t \ge T, \quad 0 < \pi < 1,$$

where the tipping point T is driven by the hazard rate $H(P)$.

We solve this problem by backward induction. After the climate regime shift, the problem is a standard Ramsey growth model with total factor productivity $(1-\pi)A$. The maximum level of output *net* of input costs and capital depreciation increases with the capital stock and decreases with the size of the climate disaster:

$$Y(K,\pi) \equiv \underset{E,R}{\text{Max}} \left[(1-\pi)AF(K,E,R) - dE - cR - \delta K\right],$$
(5)
$$Y_K = (1-\pi)AF_K - \delta > 0, \quad Y_\pi = -AF < 0.$$

For brevity, we suppress the arguments A, c and d in the net output function but $Y_A = (1-\pi)F > 0$ and energy use is given by $E^A = -Y_d > 0$ and $R^A = -Y_c > 0$, where after-calamity values are denoted with superscript A. Energy use increases with the capital stock K and decreases in the size of the disaster π and the own price, so $Y_{iK} < 0$ and $Y_{i\pi} > 0, i = d, c$. The after-calamity Ramsey problem yields the manifold for consumption $C^A = C^A(K,\pi), C^A_K > 0, C^A_\pi < 0,$ and value function $V^A(K,\pi)$ where $U'(C^A) = V^A_K(K,\pi) > 0$ (see Appendix 1 for details).

3. Precautionary actions needed before the regime shift

The before-tip problem is to choose the rate of consumption and energy use that

(6) $$\max_{C,E,R} \text{E}\left[\int_0^T e^{-\rho t} U\left(C(t)\right) dt + e^{-rT} V^A\left(K(T),\pi\right)\right]$$

subject to the capital and carbon stock dynamics (3). The question is how the prospect of a climate regime shift and the fact that this prospect becomes more imminent as temperature and atmospheric carbon rise affect the optimal growth path before the tip occurs. We denote before-event values with superscript *B*. Since atmospheric carbon increases the hazard rate, $H(P)$, the before-tip value function $V^B(K,P)$ depends on *P*. The Hamilton-Jacobi-Bellman equation for the before-tip problem is then

$$
\begin{aligned}
&\rho V^B(K,P) = \underset{C,E,R}{\text{Max}} \Big[U(C) - H(P)\left\{ V^B(K,P) - V^A(K) \right\} \\
&+ V_K^B(K,P)\left\{ AF(K,E,R) - dE - cR - C - \delta K \right\} + V_P^B(K,P)(\psi E - \gamma P) \Big],
\end{aligned}
\tag{7}
$$

with the optimality conditions

$$
U'(C^B) = V_K^B(K,P), \quad AF_E(K,E^B,R^B) = d + \tau, \quad AF_R(K,E^B,R^B) = c,
\tag{8}
$$

where the *social cost of carbon* is the marginal disvalue of carbon emissions expressed in final goods units: [6]

$$
\tau = \psi \frac{-V_P^B(K,P)}{V_K^B(K,P)}.
\tag{9}
$$

The second term on the right-hand side of (7) captures the expected capitalized loss from a regime shift that occurs at some unknown future date. With a zero hazard rate, the standard Ramsey growth model results, but with total factor productivity equal to *A*. This is the *naive* solution, since the potential regime shift is effectively ignored.

We define $Y^B(.)$ as the maximum level of output *net* of total input costs and capital depreciation that can be attained given the capital stock and the costs of energy:

$$
Y^B(K,\tau) \equiv \underset{E,R}{\text{Max}} \left[AF(K,E,R) - (d+\tau)E - cR - \delta K \right], \qquad Y_\tau^B = -E < 0,
\tag{10}
$$

so that the efficiency conditions for energy use are $AF_E = d + \tau$ and $AF_R = c$. The cost of fossil fuel is thus augmented with the social cost of carbon. If carbon is priced at the social cost of carbon, this condition for the social optimum is replicated in the market.

Differentiating (7) with respect to *K* and *P* using (8) gives the Pontryagin conditions:

$$
\begin{aligned}
-\dot{V}_K^B &= \left[Y_K^B(K,\tau) - \rho - H(P) \right] V_K^B + H(P) V_K^A(K), \\
\dot{V}_P^B &= \left[\rho + \gamma + H(P) \right] V_P^B + H'(P) \left[V^B(K,P) - V^A(K) \right].
\end{aligned}
\tag{11}
$$

Using (9) and (11), we get a differential equation for the price of carbon τ:

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[6] For simplicity, we suppress the dependence of the value functions on π from now on.

(12) $$\dot{\tau} = \left[Y_K^B(K,\tau) + \gamma + H(P) + \theta \right] \tau - \frac{\psi H'(P)\left[V^B(K,P) - V^A(K) \right]}{U'(C^B)}.$$

Using the first parts of (11) and (8), we get the modified Keynes-Ramsey rule

(13) $$\dot{C}^B = \sigma \left[Y_K^B(K,\tau) + \theta - \rho \right] C^B, \qquad \theta \equiv H(P)\left[\frac{V_K^A(K)}{U'(C^B)} - 1 \right].$$

Consumption growth is thus proportional to the marginal net product of capital *plus* the precautionary return θ *minus* the pure rate of time preference.

With a constant hazard, $H'(P) = 0$, the pre-tip value does not depend on the carbon stock and $\tau = 0$, so that the social optimum only requires precautionary saving and no carbon tax. With $H'(P) > 0$, there is a positive price of carbon τ to encourage substitution away from fossil fuel.

The accumulation of capital and the stock of carbon can be written as

(14) $$\begin{aligned} \dot{K} &= Y^B(K,\tau) - \tau Y_\tau^B(K,\tau) - C^B, \qquad K(0) = K_0, \\ \dot{P} &= -\psi Y_\tau^B(K,\tau) - \gamma P, \qquad P(0) = P_0. \end{aligned}$$

The capital dynamics include lump-sum rebates of carbon taxes in a market economy.

The steady state of (12)–(14) is indicated with an asterisk and follows from the modified golden rule of capital accumulation $Y_K^B(K^{B*}, \tau^{B*}) = \rho - \theta^{B*}$ with the precautionary return on capital $\theta^{B*} = H(P^{B*})\left[\frac{V_K^A(K^{B*})}{U'(C^{B*})} - 1 \right]$, the price of carbon

(15) $$\tau^{B*} = \frac{\psi H'(P^{B*})\left[V^B(K^{B*}, P^{B*}) - V^A(K^{B*}) \right]}{\left[\rho + \gamma + H(P^{B*}) \right] U'(C^{B*})} = \frac{\psi H'(P^{B*})\left[U(C^{B*}) - \rho V^A(K^{B*}) \right]}{\left[\rho + H(P^{B*}) \right]\left[\rho + \gamma + H(P^{B*}) \right] U'(C^{B*})},$$

the rate of consumption $C^{B*} = Y^B(K^{B*}, \tau^{B*}) - \tau^{B*} Y_\tau^B(K^{B^*}, \tau^{B*})$ and the carbon stock $P^{B*} = -\psi Y_\tau^B(K^{B*}, \tau^{B*}) / \gamma$. This gives a *target* steady state as after the regime shift the system moves to the after-calamity steady state K^{A*}. The long-run price of carbon τ^* depends on the gap between the before-disaster and after-disaster values. The precautionary return θ^* pushes up the long-run capital stock K^{B*} to be prepared for a possible tip but the optimal price of carbon τ^{B*} pushes it down again to reduce the risk.

3.1. The precautionary return on capital

The precautionary return on capital θ defined in (13) increases in the hazard rate and the drop in consumption from $C^B(T)$ to $C^A(T)$ at the time of the climate calamity. It is proportional to the hazard of a climate calamity and through this channel rises with temperature and the carbon stock. This precautionary return, if necessary forced on the market by a capital subsidy, induces precautionary capital accumulation and softens the blow to consumption when the calamity strikes. If the calamity strikes early enough, capital at the time of the tip is still below the after-tip steady state $K(T) < K^{A*}$ and afterwards capital accumulates further towards K^{A*}; if the calamity strikes relatively late, $K(T) > K^{A*}$ and thus capital moves afterwards down towards K^{A*}.

Notice that in a doomsday scenario all value is destroyed, $V^A(K) = 0$, in which case the hazard rate $H(P)$ must be added to the discount rate ρ and $\theta = -H(P) < 0$. Before tipping, consumption is then higher and capital accumulation lower than in the naive outcome (cf. Polasky et al., 2011). However, with life after the shock, discounting increases but this is more than offset by the precautionary effect, so $\theta > 0$.

3.2. The price of carbon

The optimal price of carbon depends on the *marginal* hazard of a calamity. The rationale for carbon pricing is thus different from most integrated assessment models (see Section 4). Integration of (12) gives the optimal carbon price as the present value of the expected loss in welfare from a future disaster at some unknown date:

$$\tau(t) = \psi \int_t^\infty e^{-\int_t^s r^Y(s')ds'} \frac{H'(P(s))\left[V^B(K(s),P(s)) - V^A(K(s))\right]}{U'(C^B(s))} ds \tag{16}$$
$$= \frac{\psi \int_t^\infty e^{-\int_t^s r^U(s')ds'} H'(P(s))\left[V^B(K(s),P(s)) - V^A(K(s))\right] ds}{U'(C^B(t))},$$

where $r^Y \equiv Y_K^B(K,\tau) + \gamma + H(P) + \theta$ denotes the rate used to discount utility in final goods units and $r^U \equiv \rho + \gamma + H(P)$ the rate to discount utils. The rate used to discount the expected drop in welfare resulting from a climate calamity includes the rate of decay of atmospheric carbon γ and the hazard rate itself. The rate used to discount final goods units includes the precautionary return on capital whereas the rate used to discount utility units does not. Hence, the price of carbon is large if the drops in future welfare from climate calamities and the marginal hazard are large. The *convexity* of the hazard function thus pushes up the carbon price but the *level* of the hazard rate depresses it (via the higher discount rate).

3.3. Simulations with temperature-dependent hazard rates

To illustrate these results, we assume that at the initial carbon stock, P_0 = 826 GtC, the hazard rate is $H(826) = 0.025$ and that it increases to $H(1652) = 0.067$. So, as the stock of atmospheric carbon doubles and global warming increases by 3 degrees Celsius (using a climate sensitivity of 3), the average time it takes for the tip to occur drops from 40 to 15 years. We calibrate both a linear and a quadratic hazard function:

$$(17) \quad H_1(P) = 0.025 + 5.04\times10^{-5}(P-826), \quad H_2(P) = 0.025 + 6.11\times10^{-8}(P-826)^2.$$

Both are depicted in figure 1. At higher stocks of atmospheric carbon the quadratic hazard function leads to higher hazard rates than the linear one. For example, if the carbon stock quadruples to 3304 GtC (and global warming increases with an additional 6 degrees Celsius), the hazard rate increases to 40 percent per annum with the quadratic and 15 percent per annum with the linear hazard function.

Figure 1: Different specifications for the hazard function

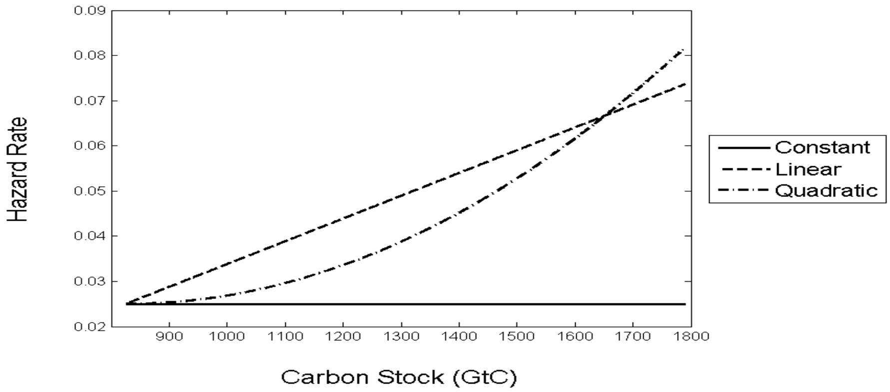


Table 1 reports the steady states for different scenarios.[7] With a constant hazard rate, the capital stock is pushed up to enable smoothing of consumption in the event of a sudden drop in factor productivity, but fossil fuel use and the stock of carbon in the

[7] Calibration details are in Appendix 2. In the simulations we set θ and τ to their steady-state levels.

atmosphere are boosted too. This is a Green Paradox effect (cf., Sinn, 2008). But if the risk of catastrophe increases with global warming, this induces a target carbon tax of 22 US$/tCO2 with a linear and 57 $/tCO2 with a quadratic hazard to curb emissions and the risk of disaster. This mitigates Green Paradox effects.

Table 1: After-disaster, naive and before-disaster target steady states

(20% TFP shock)

	After disaster	Naive solution	Constant hazard	Linear hazard	Quadratic hazard	EIS = 0.8
Capital stock (T $)	276	392	472	530	486	436
Consumption (T $)	41.3	58.6	59.4	59.6	59.2	58.9
Fossil fuel use (GtC/year)	7.3	10.4	11.0	9.7	7.7	7.7
Renewable use (million GBTU/year)	8.2	11.7	12.4	12.7	12.2	11.8
Carbon stock (GtC)	1218	1731	1838	1623	1281	1279
Precautionary return (%/year)	0	0	0.76	1.24	0.99	0.57
Price of carbon ($/tCO2)	0	0	0	22.4	56.9	51.0

The quadratic hazard has a higher target carbon tax so that the target carbon stock is cut substantially below that with the linear hazard (1281 instead of 1623 GtC). As a result, the corresponding hazard under the quadratic specification is lower (3.8 instead of 6.5 percent per annum) and the target precautionary return on capital is lower under the quadratic than under the linear hazard (0.99 versus 1.24 percent per annum). Hence, the target before-catastrophe steady-state capital stock is only 486 trillion dollars whilst under the linear hazard it is 530 trillion dollars. In both cases the target precautionary return on capital is higher than with a constant hazard rate (0.76 percent per annum) and hence, the target capital stock is higher. The reason is that the higher probability of a tip increases the target level of precautionary capital accumulation.

With the quadratic hazard the marginal hazard rate is slightly larger at the target level of the carbon stock than with the linear hazard ($H_2'(1281) = 5.56\times10^{-5} > H_1'(1623) = 5.04\times10^{-5}$).[8] This pushes up the carbon tax. Since the hazard rate itself is lower with the quadratic hazard and thus the discount rate used to calculate the present value of the drop in value after a calamity is lower, the carbon tax is further pushed up.

[8] With a 30% shock to total factor productivity the marginal hazard for the quadratic hazard function is smaller. Still, the optimal target carbon tax is bigger due to the lower hazard rate and thus lower discount rate used to discount expected climate damages.

The optimal time paths of capital, consumption and the accumulated carbon stock for the linear and quadratic hazard functions are plotted in figure 2. With global warming the hazard rates go up and expected time of the regime shift is brought forward. This is especially the case for the linear hazard rate and thus precautionary saving is higher to mitigate the effect of the shock. Hence, consumption is lower in the beginning and only catches up if the regime shift happens to occur late. In case of a quadratic hazard function, the hazard rate goes up more slowly in the relevant range. Moreover, the high carbon tax keeps the stock of atmospheric carbon down. The expected date of the regime shift occurs later than for the linear hazard rate and precautionary saving is not as high. If the linear hazard function is the more realistic one, substantial precautionary saving and a moderate carbon tax are required. If the quadratic hazard function is the more realistic one, precautionary saving is lower[9] but a higher carbon tax is required. For both hazard functions fossil fuel use is less after than before the calamity as a result of a lower level of economic activity (even though the carbon is no longer priced) and thus carbon accumulation occurs less rapidly.

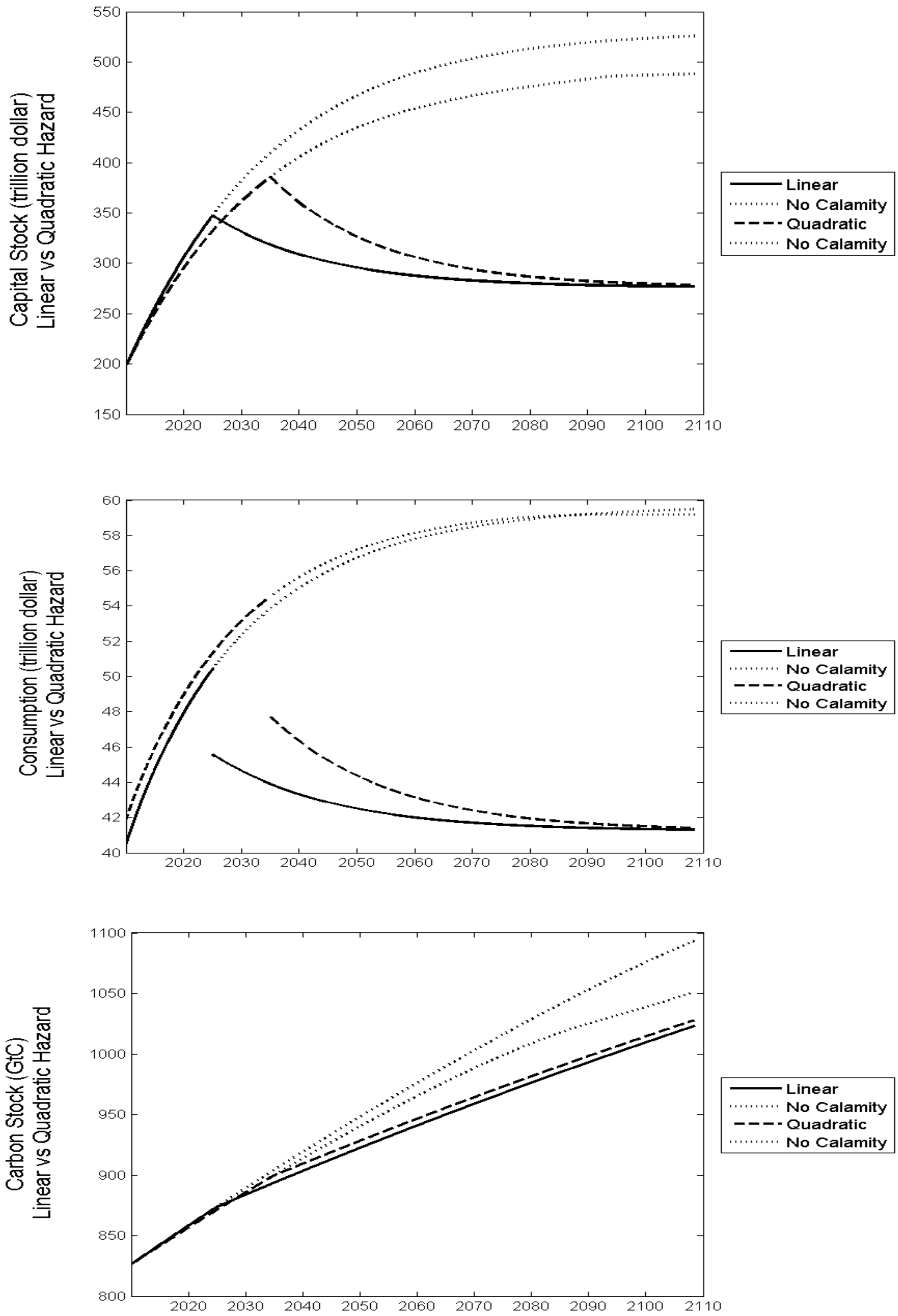
3.4. Risk aversion and intergenerational inequality aversion

Much attention has been given to the role of time preference for precautionary climate policy. It has been argued that this should not be deduced from market outcomes but from ethical considerations (e.g., Stern, 2007; Broome, 2012) or prudence considerations resulting from a positive third derivative of the utility function (e.g., Kimball, 1990; Gollier, 2012). We put forward another rationale which results from the need to be better prepared when a large-scale climate shock with an uncertain arrival time eventually hits the world.

Another important parameter indicating society's attitude to intergenerational welfare comparisons is the coefficient of relative intergenerational inequality aversion (*CRIA*) which is also the coefficient of relative risk aversion (*CRRA*). A higher elasticity of substitution σ corresponds to both a lower *CRIA* and a lower *CRRA* and thus has two effects. A higher elasticity of substitution σ induces a lower precautionary return θ (as in equations (13)) and thus less precautionary capital accumulation. Effectively,

[9] For much more convex hazard functions precautionary saving can be swamped completely, so capital is lower than in the naive outcome to curb fossil fuel emissions and global warming.

Figure 2: Rational outcomes with linear and quadratic hazard functions



society has a lower *CRIA* and is therefore less willing to sacrifice consumption and accumulate precautionary capital to be prepared for the eventual shock. As a result of using less capital in production, there will be less use of fossil fuel and thus carbon emissions will be lower. This yields a lower price of carbon. Furthermore, a lower *CRRA* implies that society is less willing to avert risk of a climate catastrophe and this depresses the price of carbon and the carbon tax τ too. The net effect on the stock of atmospheric carbon is ambiguous, since a lower *CRIA* induces less precautionary capital accumulation and thus less fossil fuel use and carbon emissions whereas a lower *CRRA* reduces the optimal price of carbon which means that carbon emissions are reduced less and the carbon stock is higher.

To illustrate these arguments, we increase the elasticity of intertemporal substitution from σ = 0.5 to σ = 0.8, and thus reduce the *CRIA* and the *CRRA* from 2 to 1.25: see last column of table 1. As a result of the lower *CRIA*, we see substantial reductions in the precautionary return (from 1.24 to 0.73 percent per annum under the linear hazard function and from 0.99 to 0.57 percent per annum under the quadratic hazard function). This results in substantial drops in the target before-calamity capital stocks from 530 to 462 trillion dollars under the linear hazard and from 486 to 436 trillion dollars for the quadratic hazard function. As a result of the lower *CRRA*, we see only modest cuts in the carbon tax (from 22.4 to 22.0 \$/tCO2 under the linear and from 56.9 to 51.0 \$/tCO2 under the quadratic hazard). The target atmospheric carbon stock is curbed from 1623 to 1557 GtC under the linear hazard function and from 1281 to 1279 GtC under the quadratic hazard function. Less intergenerational inequality aversion curbs the target rate of consumption, but only by a very modest amount.

4. Gradual and Catastrophic productivity damages of global warming

In most IAMs global mean temperature *Temp* decreases total factor productivity in gradual fashion. Nordhaus (2008) uses $A = (1 + 0.00284 Temp^2)^{-1}$ in his DICE-2007, which implies damages of 1.7% of GDP at 2.5° Celsius. Temperature is often described by $Temp = \chi \ln(P / P_{PI}) / \ln(2)$ with χ the climate sensitivity (i.e., the temperature rise from doubling the carbon stock *P*) and P_{PI} = 596.4 GtC the preindustrial carbon stock, so total factor productivity decreases in the carbon stock. Golosov et al. (2014) use χ = 3 (cf., IPPC, 2007) and show that damages are well approximated by $A(P) = e^{-\xi(P-\bar{P})}\bar{A},$ where ξ = 0.02379 US$/tC is the damage coefficient, $\bar{P}$ the 2010 stock of carbon, and $\bar{A}$ total factor productivity in 2010.

With these additional gradual damages, the social cost of carbon is (see Appendix 3):

$$\tau(t) = \underbrace{\frac{\psi}{U'\left(C^B(t)\right)}\int_t^\infty e^{-\int_t^s r^U(s')ds'} H'\left(P(s)\right)\left[V^B\left(K(s),P(s)\right) - V^A\left(K(s),P(s)\right)\right]ds}_{\text{risk-averting part of the price of carbon}}$$

(18)
$$+\underbrace{\xi\psi\int_t^\infty e^{-\int_t^s r^Y(s')ds'} AF(s)ds}_{\text{conventional part of the price of carbon}}$$

$$+\underbrace{\psi\int_t^\infty e^{-\int_t^s r^Y(s')ds'} H\left(P(s)\right)\left[\frac{-V_P^A\left(K(s),P(s)\right)}{U'\left(C(s)\right)}\right]ds}_{\text{raising-the stakes part of the price of carbon}}.$$

The first term is the present value of the expected loss of a future climate catastrophe. It curbs carbon emissions and thus the risk of a climate calamity. The second term is the usual present value of marginal climate damages with the discount rate being the rate of interest plus decay rate of atmospheric carbon augmented with the hazard of the catastrophe *plus* the precautionary return.[10] The third term is the expected present value of future increases in after-calamity welfare from curbing emissions by one unit.

4.1. Decomposing the social cost of carbon and the precautionary return on capital

The first three columns of table 2 give the after- and before-calamity steady states for both a linear and a quadratic hazard; compare with table 1 to show effects of adding marginal damages. As a result of the marginal damages, after the calamity there is a conventional carbon tax of 11 \$/tCO2 which curbs atmospheric carbon from 1502 to 1107 GtC. This slightly lowers capital and consumption after the calamity. Before the disaster strikes, there is a much bigger carbon tax for the linear hazard function (55 instead of 22 \$/tCO2) than for the quadratic hazard function (71 instead of 57 \$/tCO2), as compared with the situation that there are no marginal climate damages. This before-calamity price of carbon is in both cases dominated by the catastrophe components, especially the risk-averting component. As a consequence of introducing marginal climate damages, the carbon stock is much more reduced with the linear hazard function (from 1623 to 1287 GtC) than with the quadratic hazard function (from 1281 to 1161 GtC) where global warming has already been curbed without taking account of marginal damages as in table 1. The differences between the effects of the two hazard functions thus become smaller.

Table 2: Target steady states with catastrophic and marginal damages

	Naive solution	20% shock in TFP			10% shock in TFP		
		After shock	Linear	Quadratic	After shock	Linear	Quadratic
Capital stock (T $)	378	271	492	465 (426)	323	431	421
Consumption (T $)	57.1	40.8	58.3	58.2 (58.0)	48.7	57.8	57.8
Carbon stock (GtC)	1502	1107	1287	1161 (1119)	1303	1425	1320
Temperature (^{o} Celsius)	4.00	2.68	3.33	2.88 (2.72)	3.38	3.77	3.44
Precautionary return (%/year)	0	0	1.10	0.90 (0.56)	0	0.57	0.49
Price of carbon ($/GtCO2)	15.4	11.0	54.8	71.2 (73.1)	13.2	29.8	41.5
Marginal	15.4	11.0	4.3	5.7 (8.5)	13.2	3.8	4.7
risk averting	0	0	35.0	51.9 (54.8)	0	12.4	24.2
raising stakes	0	0	15.4	13.7 (9.8)	0	13.7	12.5

Note: In brackets are results for when the hazard rates are halved for each level of *P*.

Internalizing marginal as well as catastrophic damages curbs global warming and lessens the need for precautionary capital accumulation, especially for the linear hazard function (from 530 to 492 trillion dollars for the linear and from 486 to 465 trillion dollars for the quadratic hazard function). Hence, the precautionary return, largely driven by the hazard rate and carbon stock, drops by much more with the linear hazard (from 1.23 to 1.10 percent per annum) than with the quadratic hazard (from 0.99 to 0.90 percent per annum). The convex hazard function has a lower marginal hazard rate now at the target level of the carbon stock (4.1×10^{-5} for the quadratic and 5.0×10^{-5} for the linear hazard function) but the risk-averting component of the price of carbon is still higher as the hazard rate is lower. It is not so much higher now, since the hazard rates differ less between the linear and the quadratic hazard function.

The remaining columns of table 2 show the consequences of a catastrophic drop of 10 percent in total factor productivity. Since there is more economic activity and more

[10] With logarithmic utility, Cobb-Douglas production and 100% depreciation, this part of the SCC in a discrete-time Ramsey model is proportional to GDP, $\tau = \psi\xi AF / (\rho + \gamma)$ (Golosov et al., 2014).

carbon emissions after the calamity than with the 20 percent disaster, the after-calamity price of carbon increases from 11 to 13 \$/tCO2. Before the calamity strikes, both risk-averting components of the optimal price of carbon fall relative to the outcome with a 20 percent calamity but the biggest falls occur for the risk-averting components (from 35 to 12 \$/tCO2 for the linear and from 52 to 24 \$/tCO2 for the quadratic hazard function). Although there is more economic activity with a 10% shock after the calamity, the initial capital stock is lower because there is less precautionary capital accumulation. The precautionary return drops more or less in line with the drop in total factor productivity, so that less precautionary saving occurs before the calamity. Despite the lower initial capital stock after the calamity, the lower structural drop of only 10% in total factor productivity leads to bigger carbon stocks and more global warming.

In brackets we show how results change if the quadratic hazard rates are halved for each level of the carbon stock. This curbs precautionary saving, but increases the price of carbon by a small amount. The marginal hazard rates are lower but the hazard rates are lower also and the last effect dominates so the carbon price increases.[11]

4.2. Carbon and capital catastrophes

Instead of an economic catastrophe triggered by global warming we consider here a climate catastrophe which induces an abrupt increase in the climate sensitivity χ.[12] After the disaster the carbon stock and temperature increase and gradually erode productivity (see Appendix 4). Table 3 shows the effects of a catastrophic increase in the climate sensitivity from 3 to 4 with a quadratic hazard.[13] The after-calamity response requires a much higher carbon price than in the naive outcome (27 instead of 16 \$/tCO2). Before the calamity also a substantial target carbon price is needed (27 \$/tCO2); not different from the after-calamity carbon price, but there is a shift from

[11] With a linear hazard (not reported in table 2) there is still substantial precautionary capital accumulation after halving hazard rates (437 T\$). The carbon price is a little higher for the linear (77.2 \$/tCO2) than for the quadratic hazard, mainly due to the risk-averting component as the marginal hazard for the linear is higher than for the quadratic hazard ($2.52\times10^{-5} > 1.78\times10^{-5}$). The interplay between marginal hazard and the hazard rate itself can work out in different ways but it is clear that a big price of carbon is needed to curb the risk of catastrophe.

[12] An alternative is to have abrupt positive feedback by changing the system dynamics of the carbon cycle as discussed in Lemoine and Traeger (2014) and van der Ploeg (2014).

[13] Responses with a linear hazard hardly differ from those with a quadratic hazard function.

marginal to raising-the-stakes damages. Once the calamity has struck, damages are higher and after-calamity welfare is lower so that the raising-the-stakes component of the carbon price is big. This in itself curbs carbon emissions and lessens the need for correction of marginal global warming damages. Before the calamity a small precautionary return is required and thus the target capital stock is a little higher than in the naive outcome (despite the higher carbon price). The precautionary return is only small because the shock to total factor productivity induced by the shock to climate sensitivity is not so big. Just as with the total factor productivity shocks, the carbon stock drops after the calamity. But the after-calamity temperature is now higher than with the productivity shock presented in table 2.

Table 3: Target steady states for capital and carbon catastrophes

	Naive outcome	*CS* jumps from 3 to 4		20% jump in *P*	20% drop in *K*
		After calamity	Before calamity		
Capital stock (T $)	378	372	382	381	433
Consumption (T $)	57.1	56.3	57.3	57.1	57.6
Carbon stock (GtC)	1502	1374	1400	1490	1534
Temperature ($^\circ$ Celsius)	4.00	4.82	3.69	3.96	4.09
Precautionary return (%/year)	0	0	0.05	0.03	0.57
Price of carbon ($/GtCO2)	15.4	26.7	26.5	16.9	18.5
marginal	15.4	26.7	4.1	3.8	3.8
risk averting	0	0	2.2	1.4	2.5
raising stakes	0	0	20.2	11.7	12.2

Table 3 also shows the effects of a hazard of a 20% increase in the carbon stock. There is little precautionary saving with almost no effect on the rate of consumption and the carbon tax is small (17 $/tCO2) as it is a *recoverable* catastrophe rather than a permanent regime shift. The carbon stock and global warming are cut somewhat below the naive outcome (1490 instead of 1502 GtC). Both carbon and climate sensitivity catastrophes are less imminent than a productivity catastrophe resulting from, say, a reversal of the Gulf Stream and will typically be less abrupt as they may take centuries to have their full impact (e.g., Lenton and Ciscar, 2013; Lontzek et al., 2013).

Finally, table 3 shows the effects of a hazard of a 20% destruction of the capital stock. Now there is a sizeable precautionary return (0.57% per year) and substantial capital accumulation to prepare for sudden disaster (433 instead of 378 trillion dollars). Since this induces more fossil fuel use, there is despite a higher price of carbon (18.5 instead of 15.4 \$/tCO2) *more* global warming than in the naive outcome (1534 GtC).

Before the calamity strikes, the risk-averting components of the price of carbon are tiny as the damages done by the catastrophe are temporary. Comparing with the naive outcome, the raising-the-stakes component takes over most of the marginal damages of the price of carbon, as we have also seen for the shock in climate sensitivity. There is substantial accumulation of capital before the calamity strikes with an impending capital disaster but unsurprisingly hardly any for an impending carbon stock disaster.

5. Conclusion

Climate change will probably manifest itself in the future as a regime shift in the climate system resulting from a climate tipping point (Lenton and Ciscar, 2013). This means that the potential shock to the economy becomes an important driver of the social cost of carbon. It has also been argued that spending money now to slow down global warming should be conceptualized primarily as an issue about how much insurance to buy to offset the small chance of a ruinous catastrophe (Weitzman, 2007). We have taken up these challenges by analysing the optimal reactions to a tipping point which becomes more imminent with global warming.

The most striking result is that both precautionary capital accumulation and a price of carbon are needed. Precautionary saving may be picked up by the market but if not, a capital subsidy is needed. More capital is required to be prepared for the shock and to smooth consumption over time and less fossil fuel use is required to curb the risk of catastrophe. In combination with the standard Pigouvian price of carbon, this carbon price becomes even higher because the regime shift also increases the marginal cost of fossil fuel emissions after the shock. Higher intertemporal substitution or lower risk aversion and intergenerational inequality lower both the price of carbon and precautionary accumulation of capital.

Regime shifts are characterized in our analysis as structural shocks to total factor productivity or to climate sensitivity. We show that the effects on optimal policy are much larger than in case of shocks to the capital stock or the stock of atmospheric carbon which only set back growth temporarily and do not affect the steady state of the economy. Hence, there is less need for optimal policy to curb the risk of tipping.

Another implication is that a constant marginal hazard rate requires a balanced package of precautionary saving and pricing carbon. However, for an increasing marginal hazard rate a higher carbon price is needed so that the stock of atmospheric carbon is kept down and precautionary accumulation of capital can be more modest. Hence, if climate change is still far away but approaches more rapidly when the globe warms up, optimal climate policy requires a high price of carbon.

Our results highlight the importance of a price of carbon to curb the risk of catastrophe and precautionary capital accumulation to be better prepared when disasters strike. Our illustrative calculations indicate that the catastrophe parts and especially the risk-averting part of the price of carbon are substantial compared with the carbon taxes based on only marginal damages (e.g., Nordhaus, 2008; Golosov et al., 2014).

Our conclusion is that destruction of non-recoverable factors of production by climate catastrophes requires urgent action both to mitigate the risks of such actions occurring and to be better prepared for them.[14] Our illustrative calculations suggest that conventional marginal global warming damages necessitate a global price of carbon of 15 \$/tCO2.[15] Allowing for an impending negative shock of 20% to total factor productivity with an expected arrival in 40 years and falling to 15 years as the carbon stock doubles, boosts this figure to 55 or 71 \$/tCO2 for a linear or quadratic hazard function, respectively. In addition, a precautionary return of about 1% per annum induces capital accumulation of 30% and 23%, respectively. A -10% shock halves the precautionary returns and requires a global carbon price of 30 to 40 \$/tCO2. Halving the hazard rates for any given carbon stock hardly changes the required carbon price, but does diminish the precautionary return considerably to 0.16% per annum. Catastrophic shocks to recoverable factors of production such as the capital stock are only temporary and thus require much less action. On the other hand, catastrophes that

[14] Specific adaptation capital curbs the damage of catastrophe. This follows from setting the expected averted marginal damages equal to the marginal productivity of capital in production (Appendix 5).

[15] This is not too different from the latest ballpark measure of 12 \$/tCO2 (including uncertainty, equity weighting and risk aversion) given by Nordhaus (2014).

unleash hitherto dormant positive feedback loops in the carbon cycle may need much more risk-averting action to prevent runaway global warming from occurring.

Catastrophes might resonate more with policy makers to convince them of the need to implement ambitious climate policy. However, as argued by Lenton and Ciscar (2013), more research with climate scientists is needed to improve information on the different types of productivity, capital, carbon and climate sensitivity catastrophes that can occur and the different hazard and marginal hazard rates of such disasters. We also need to be more precise about how long it will take before the impact of the catastrophe is fully felt and adopt our analysis for this purpose (cf., Lontzek et al., 2014). For example, drops in total factor productivity resulting from reversal of the Atlantic Meriodonal Overturning Circulation may take a century, full disappearance of the Greenland or the West Antarctic Ice Sheet may take as long as three centuries, and release of permafrost may take close to a century. Desertification of agricultural land may take many decades before its full impact is felt. Future research is thus needed on the nature of catastrophes, their hazard rates and their dependence on global warming and the time it takes to have their effect, and how these affect the optimal policy prescriptions for dealing with climate tipping points.

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Appendix 1: After the climate shift

The Hamilton-Jacobi-Bellman (HJB) equation in the value function V^A is

(A1) $$\rho V^A(K,\pi) = \underset{C}{\text{Max}} \left[U(C) + V_K^A(K,\pi)\left\{ Y(K,\pi) - C \right\} \right].$$

Optimality demands $U'(C^A) = V_K^A(K,\pi),$ which gives the policy rule:

(A2) $$C^A = C^A(K,\pi), \qquad C_K^A = V_{KK}^A / U'' > 0, \quad C_\pi^A = V_{K\pi}^A / U'' < 0.$$

Consumption increases with the capital stock given that the value function is concave in capital as the utility and production functions are both concave. A bigger disaster π boosts the marginal value of capital V_K^A which requires a corresponding boost to the marginal utility of consumption $U'(C^A)$ and thus necessitates a drop in consumption.

Differentiating (A1) with respect to *K* and using (A2) yields a differential equation for consumption after the regime shift C^A as a function of *K*:

(A3) $$\left[Y(K,\pi) - C^A(K,\pi) \right] C_K^A(K,\pi) = \sigma \left[Y_K(K,\pi) - \rho \right] C^A(K,\pi),$$

where $\sigma \equiv -U'/CU'' > 0$ is the elasticity of intertemporal substitution. Both relative risk aversion and relative intergenerational inequality aversion equal $1/\sigma$. Equation (A3) can be written as a saddle-point system of differential equations in *C* and *K* as functions of time corresponding to the Keynes-Ramsey rule and the capital dynamics:

(A4) $$\begin{aligned} \dot{C}^A(t) &= \sigma \left[Y_K\left(K(t),\pi \right) - \rho \right] C^A(t), \\ \dot{K}(t) &= Y\left(K(t),\pi \right) - C^A(t), \quad K(T) = K_T. \end{aligned}$$

The steady-state capital stock $K^{A*}(\pi)$ follows from the modified golden rule of capital accumulation $Y_K(K^{A*},\pi) = \rho,$ and is low if the disaster is large and the discount rate high. As far as the transient phase is concerned, the capital stock is predetermined at time *T*, but the rate of consumption jumps down then to place the economy on the stable manifold, $C^A(T+) = C^A\left(K(T+), \pi\right)$. The optimal path along the stable manifold is $C^A(t) = C^A\left(K(t), \pi\right), t \geq T.$ Rearranging (A1) gives the value function:

$$\text{(A5)} \quad V^A(K,\pi) = \frac{U\left(C^A(K,\pi)\right) + U'\left(C^A(K,\pi)\right)\left[Y(K,\pi) - C^A(K,\pi)\right]}{\rho},$$

where in the sequel we use the approximation

$$\text{(A6)} \quad C^A = C^A(K,\pi) \cong Y\left(K^{A^*}(\pi),\pi\right)\left(\frac{K}{K^{A^*}(\pi)}\right)^{\phi}, \quad \phi \equiv z\frac{K^{A^*}(\pi)}{Y\left(K^{A^*}(\pi),\pi\right)} > 0,$$

where $z = \frac{\rho}{2} + \frac{1}{2}\sqrt{\rho^2 - 4\sigma Y_{KK}(K^{A^*},\pi)C^{A^*}} > \rho > 0.$ The direction of the stable manifold is derived from (A4) with l'Hôpital's rule $C_K^A(K^{A^*},\pi) =$ $\lim_{K \to K^{A^*}} \left\{\sigma\left[Y_K(K,\pi) - \rho\right]C_K^A(K,\pi) + \sigma Y_{KK}(K,\pi)C^A(K,\pi)\right\} / \left\{Y_K(K,\pi) - C_K^A(K,\pi)\right\}.$ This yields the quadratic $z^2 - \rho z + \sigma Y_{KK}(K^{A^*},\pi)C^{A^*} = 0$ in terms of $z \equiv C_K^A(K^{A^*},\pi).$ The positive solution is the slope of the stable manifold in the steady state. Although we could have solved the after-calamity problem by using value function iteration to solve the HJB equation, the approximation (A6) is very accurate and is very convenient.

Appendix 2: Calibration

We use a utility function U with constant elasticity of intertemporal substitution σ = 0.5 (and 0.8 in a sensitivity test) and a pure rate of time preference ρ = 0.014. Our parameters and initial values are calibrated to figures for the world economy for the year 2010. Data sources are the BP Statistical Review and the World Bank Development Indicators. Output AF before and $(1-\pi)AF$ after the disaster, the Cobb-Douglas production function $F(K,E,R) = K^{\alpha}(E^{\omega}R^{(1-\omega)})^{\beta}$ with share of capital in value added α = 0.3, share of fossil fuel in energy ω = 0.9614 and share of energy in value added β = 0.0651. The share of fossil fuel and labour in value added are thus $\beta\omega$ = 0.0626 and $1 - \alpha - \beta$ = 0.6349. We use a depreciation rate for manmade capital of δ = 0.05. Total factor productivity is set to A = 11.9762, so that initial world GDP equals Y_0 = 63 trillion US \$. The corresponding initial value of the aggregate capital stock is K_0 = 200 trillion US \$, initial fossil use is E_0 = 468.3 million GBTU or 8.3 GtC, and of initial renewable energy use is R_0 = 9.4 million GBTU. The calibrated production share parameters are compatible with a cost of fossil fuel of d = 9 US \$/million BTU or 504 US \$/tC and a cost of renewable energy of c = 18 US \$/million BTU. For the carbon cycle we have an initial stock of carbon of P_0 = 826 GtC or 338 ppm by volume CO2, a rate of decay γ = 0.003 and a fraction of carbon staying up of ψ = 0.5, and an equilibrium climate sensitivity χ = 3 (or 4 in a sensitivity run). The catastrophic shock to total factor productivity is π = 0.2 (or 0.1 in a sensitivity run).

Appendix 3: Gradual and catastrophic damages

Now $V^A(K,P,\pi)$ requires $\tau^A(K,P,\pi) \equiv -\psi V_P^A(K,P,\pi)/U'(C) > 0$ to internalize gradual damages, After-calamity consumption $C^A = C^A(K,P,\pi)$ gives

(A7)
$$V^A(K,P,\pi) = U\left(C^A(K,P,\pi)\right)/\rho + U'\left(C^A(K,P,\pi)\right)\left[Y^A(K,P,\pi) - C^A(K,P,\pi) + \gamma\tau^A(K,P,\pi)P/\psi\right]/\rho.$$

As a consequence, the second part of (11) and (12) become:

(A8) $$\dot{V}_P^B = \left[\rho + \gamma + H(P)\right]V_P^B + H'(P)(V^B - V^A) - A'FV_K^B - H(P)V_P^A,$$

(A9) $$\dot{\tau} = \left[Y_K^B(K,\tau) + \gamma + H(P) + \theta\right]\tau - \frac{\psi}{U'(C^B)}\left[H'(P)(V^B - V^A) - A'FV_K^B - H(P)V_P^A\right].$$

Integrating (A9), we get (18). The target before-calamity steady-state price of carbon is

$\tau^{B*} = \tau^{B*}_{\text{conventional}} + \tau^{B*}_{\text{risk-averting}} + \tau^{B*}_{\text{raising-the-stakes}}$, where $\tau^{B*}_{\text{conventional}} \equiv \dfrac{\xi\psi A(P^{B*})F(K^{B*},\tau^{B*})}{\rho + \gamma + H(P^{B*})}$,

$$\tau^{B*}_{\text{risk-averting}} \equiv \frac{\psi H'(P^{B*})\left[U(C^{B*}) - \rho V^A(K^{B*},P^{B*})\right]}{\left[\rho + \gamma + H(P^{B*})\right]\left[\rho + H(P^{B*})\right]U'(C^{B*})},$$

$\tau^{B*}_{\text{raising-the-stakes}} \equiv \dfrac{H(P^{B*})\tau^A(K^{B*},P^{B*})U'\left(C^A(K^{B*},P^{B*})\right)}{\left[\rho + \gamma + H(P^{B*})\right]U'(C^{B*})}$ and $A(P)F(K,\tau)$ is the maximum level of gross output.

Appendix 4: Carbon catastrophes

Our damages and χ = 3 imply $A(Temp) = \bar{A}\exp\left[-\xi\left(2^{Temp/3}P_{PI} - \bar{P}\right)\right]$. A catastrophic increase in the climate sensitivity to χ > 3 raises the temperature response and thus increases post-calamity damages as can be seen from substituting the temperature response: $A(P) = \bar{A}\exp\left[-\xi\left(\left(P/P_{PI}\right)^{\frac{\chi}{3}}P_{PI} - \bar{P}\right)\right]$. The catastrophic rise in χ pushes up damages ($A'(P) = -\xi(\chi/3)\left(P/P_{PI}\right)^{(\chi-3)/3}A(P) < -\xi A(P) < 0$).

Figure A.1: Effects of climate sensitivity on global warming damages

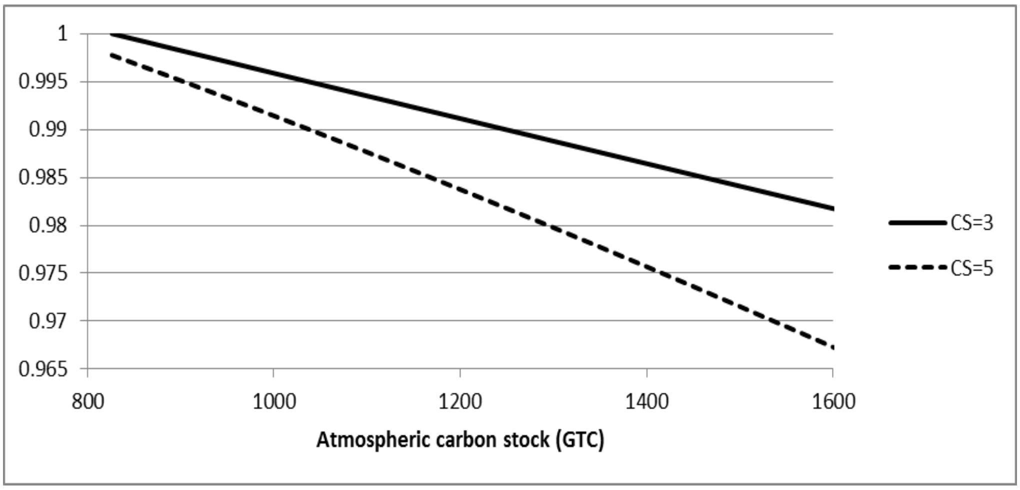


Figure A1 indicates that a bigger climate sensitivity of 4 (dashed lines) than our benchmark of 3 (solid lines) leads, for a given carbon stock, to bigger damages, with the hazard function unaltered. A sudden increase in climate sensitivity thus leads to a tougher challenge. Doubling the initial carbon stocks induces a 2.0% drop in total factor productivity if climate sensitivity is 3 and a 3.4% drop if climate sensitivity is 4.

Appendix 5: Adaptation capital

The risk of a tipping point induces precautionary capital accumulation so as to be prepared when the shock hits. Alternatively, one can invest in specific adaptation capital L (e.g., seawalls, storm surge barriers, dune reinforcement and creation of marshlands as protection to sea level rises, crop relocation, diversifying tourist attractions, adjusting rail and roads to cope with warming and drainage).[16] Adaptation capital L reduces the shock, so $\pi = \Pi(L)$ with $\Pi'(L) < 0$. Suppose both types of capital are *ex ante* perfect substitutes and have the same depreciation rate, δ. We assume a "putty-clay" technology: from the tip onwards, L is constant as there is no need to cover for depreciation and L cannot be turned *ex post* into productive capital.

[16] A third option is to invest in mitigation capital to curb the hazard of climate change (e.g., institutions, norms and networks that facilitate the implementation of optimal climate policy, but also CCS).

We ignore marginal damages, so after-calamity does not depend on the carbon stock. The right-hand side of the Hamilton-Jacobi-Bellman equation becomes

$$\text{(A10)}\qquad \begin{aligned}\Omega \equiv \underset{C,L}{\text{Max}} \Big[& U(C) + V_K^B(K,P)\left\{Y^B(K-L,P) - C\right\} \\ & - H(P)\left\{V^B(K,P) - V^A\left(K-L,\Pi(L)\right)\right\}\Big],\end{aligned}$$

with the condition for the optimal stock of adaptation capital,[17]

$$\text{(A11)}\quad Y_{K-L}^B(K-L,P) = \frac{H(P)\left[V_\pi^A\left(K-L,\Pi(L)\right)\Pi'(L) - V_{K-L}^A\left(K-L,\Pi(L)\right)\right]}{V_K^B(K,P)}.$$

We define $\hat{V}_L^A(K,L) \equiv V_\pi^A\left(K-L,\Pi(L)\right)\Pi'(L) - V_{K-L}^A\left(K-L,\Pi(L)\right) > 0$ and suppose $\hat{V}_L^A > 0,$ so the total marginal return on adaptation capital is decreasing in *L* and increasing in *K*, i.e. $V_{LL}^A < 0$ and $V_{LK}^A > 0.$ The share of adaptation capital *L* is zero if the net effect of a lower shock Π(*L*) and a lower productive capital *K* - *L* after the event cannot be positive. Else, (A11) requires that the expected marginal return on adaptation capital in final goods units must be equal to the marginal productivity of capital used in production. The optimal amount of adaptation capital then increases in both capital and carbon stocks:

(A12 $\quad L = L(P,K)$ with $L_P > 0$ and $L_K > 0.$

If the hazard rate is invariant to global warming, (A12) becomes $L = L(h,K)$ with $L_h > 0$ and $L_K > 0.$ A higher hazard rate *h* induces precautionary capital accumulation. Here we see that a bigger risk of catastrophe leads to more adaptation capital *L*. In general, the hazard rate increases with global warming in which case the optimal level of adaptation capital and the aggregate capital stock increase with the carbon stock.

[17] Condition (A11) follows from $\Omega_L = -V_K^B Y_{K-L}^B + H(P)\hat{V}_L^A = 0.$ The second-order condition is $\Omega_{LL} = V_K^B Y_{K-L,K-L}^B + H(P)\hat{V}_{LL}^A < 0.$ Since $\Omega_{LP} = H'(P)\hat{V}_L^A > 0,$ we have $L_P = -\Omega_{LP} / \Omega_{LL} > 0.$ Since $\Omega_{LK} = -V_K^B Y_{K-L,K-L}^B + H(P)\hat{V}_{LK}^A - V_{KK}^B Y_{K-L}^B > 0,$ so $L_K = -\Omega_{LK} / \Omega_{LL} > 0.$

SUPPLEMENTARY BACKGROUND MATERIAL

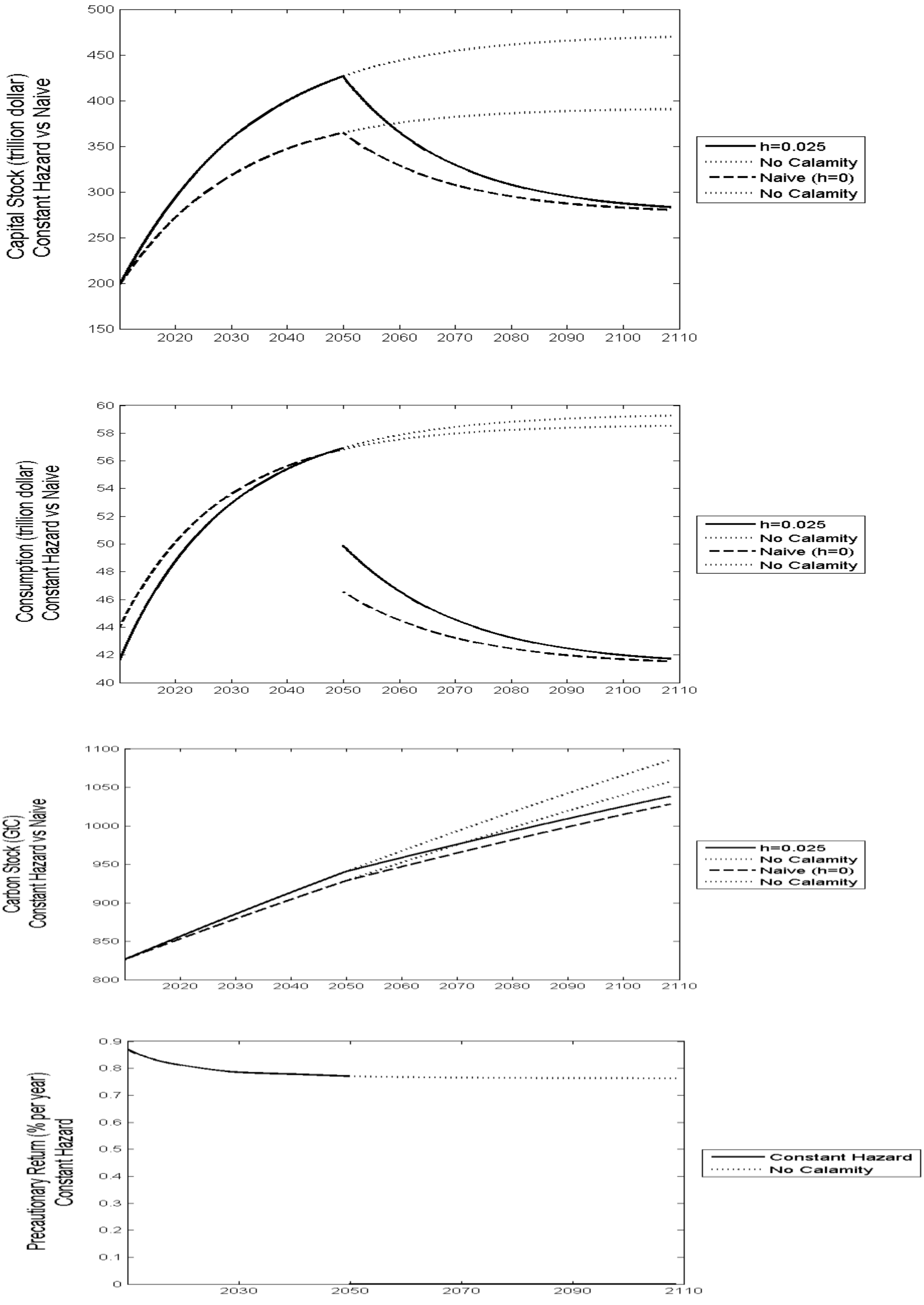
Appendix A: Constant hazard of a tipping point and the Green Paradox

We calibrate the parameters of our model to the world economy of 2010 using data from the BP Statistical Review and the World Bank Development Indicators (see Appendix 2). We consider a catastrophic shock of a 20 percent drop in total factor productivity ($\pi = 0.2$) with an expected arrival time of 40 years (h = 0.025).[18] Figure 1 offers two simulations for when the regime shift takes place in 2050: the naive outcome (solid lines) and the rational outcome (dashed lines). The dotted lines indicate that tipping has not taken place yet. The steady-state capital stocks for the naive, before-calamity and after-calamity outcomes are $K^* = 392, K^{B^*} = 473$ and $K^{A^*} = 276$ trillion dollar, respectively. The steady-state rates of consumption and carbon stocks for these three outcomes are $C^* = 58.6, C^{B^*} = 59.4$ and $C^{A^*} = 41.3$ trillion dollar, respectively, and $P^* = 1731,$ $P^{B^*} = 1838$ and $P^{A^*} = 1218$ GtC, respectively. The steady-state precautionary return is $\theta(K^{B^*}, \pi) = 0.76$ percent per annum.

Since the regime shift occurs sufficiently late, the capital stock starts to fall after the tip (as $K^B(T) > K^{A*}$). Precautionary capital accumulation in the rational outcome mitigates the drop in consumption compared with that in the naive outcome. This goesat the expense of less consumption especially in the short run when most precautionary saving takes place (witness the gradual decline in the precautionary return on capital from 0.87 to 0.76 percent per year). After the calamity, fossil fuel use is less and the carbon stock rises less rapidly than before. Due to precautionary capital accumulation, fossil fuel use rises before the calamity and thus the final stock of carbon in the atmosphere is higher than in the naive outcome. This worsens global warming and is akin to the Green Paradox (Sinn, 2008), but the mechanism differs. With a constant hazard rate h, this does not make the catastrophe more imminent.

[18] Barro (2013) has a sample of 158 rare macroeconomic disasters each with a drop of at least 10% of GDP. The mean disaster 20.7% of GDP, but note that no climate disasters have been recorded yet that exceed the 10% limit. Calibration of climate shocks is thus very speculative. Purely for illustrative purposes we focus at a shock of 20% in total factor productivity.

Figure 1: Naive and rational outcomes with constant hazard



Appendix B: Further calibration details

We set the 2010 capital stock to 200 trillion US$ which is below the steady-state level of 295 trillion US$ that is consistent with the 2010 figure for world GDP of 63 trillion US $ (i.e., 63/(ρ+δ)) to reflect that big parts of the global economy are still catching up. Bio-fuels production in 2010 was 1.45% of total oil production. Counting also nuclear and other renewable sources of energy might triple this figure. Hydro-electricity production is tiny. We thus take 2% for the share of renewable energy (9.4 million GBTU) in total fossil fuel energy output (468.3 million GBTU) in 2010.

Prices of oil, natural gas and coal are 14, 6.5 and 4 US$, respectively, per million BTU and we set the average cost of fossil fuel equal to 9 US$ per million BTU. We set the cost of the renewables twice as high. Hence, the budget share ω of fossil fuel in total energy is (9×468.3)/(9×468.3+18×9.4) = 0.9614. The share of fossil fuel in GDP is $\beta\omega/(1-\beta)$ = (9×468.3)/63000 = 0.0669, so that β = 0.0651 is the share of total energy in gross output and $\omega\beta$ = 0.0626 the share of fossil fuel in gross output. We use a share of capital α in gross output of 0.3 so that the share of labour in value added is 0.6349. We calibrate total factor productivity A to match 2010 world GDP, so that $(1-\beta)\times 63\times 200^{-\alpha}\times 468.3^{-\beta\omega}\times 9.4^{-\beta(1-\omega)} = 11.9762$.

To convert we use a conversion factor for fossil fuel of 1 GtC = 56 million GBTU. We measure fossil fuel in GtC, so the emission-input ratio equals one. Hence, the market price can be expressed as 504 US$ per ton of carbon. Other conversion factors are: 1 ppm by volume CO2 = 2.13 GtC and 1 kg carbon = 44/12 = 3.67 kg CO2. The fraction of carbon ψ that does not return quickly to the surface of the earth is 0.5.

Our production function gives the optimal choice of fossil-fuel use and renewable use

$$\text{(A15)}\quad E=\frac{\beta\omega\tilde{A}F}{d+\tau},\quad R=\frac{\beta(1-\omega)\tilde{A}F}{c},$$

and output *net* of total input costs and capital depreciation

$$\text{(A16)}\quad Y(K,\tau)=(1-\beta)\left[\tilde{A}\beta^{\beta}\left(\frac{\omega}{d+\tau}\right)^{\beta\omega}\left(\frac{1-\omega}{c}\right)^{\beta(1-\omega)}\right]^{\frac{1}{1-\beta}}K^{\frac{\alpha}{1-\beta}}-\delta K.$$

The modified golden rule of capital accumulation $Y_K(K^*,\tau^*)=\rho-\theta^*$ yields the before- and after-catastrophe steady-state capital stocks:

(A17a) $K^{B*} = \left[A\beta^{\beta} \left(\frac{\alpha}{\rho + \delta - \theta^{B*}} \right)^{1-\beta} \left(\frac{\omega}{d + \tau^{B*}} \right)^{\beta\omega} \left(\frac{1-\omega}{c} \right)^{\beta(1-\omega)} \right]^{\frac{1}{1-\alpha-\beta}}$,

(A17b) $K^{A*} = \left[(1-\pi)A\beta^{\beta} \left(\frac{\alpha}{\rho + \delta} \right)^{1-\beta} \left(\frac{\omega}{d + \tau^{A*}} \right)^{\beta\omega} \left(\frac{1-\omega}{c} \right)^{\beta(1-\omega)} \right]^{\frac{1}{1-\alpha-\beta}}$,

where $\tilde{A} = (1-\pi)A$ and θ^{*} = 0 after the catastrophe, and $\tilde{A} = A$ (and τ^{*} = 0 in case of a constant hazard rate) before the catastrophe.

Appendix C: Steady states, after-calamity rules and transient paths

C.1. No marginal climate damages:

With parameter values and functional forms discussed in Appendix 2 and a 20% TFP shock but no marginal climate damages, the after-event steady state has θ^{*} = τ^{*}= 0, K^{A*} = 276 T US$, C^{A*} = 41.3 T US$. Equation (A6) gives:

$$\phi = \left[\frac{1}{2}\rho + \frac{1}{2}\sqrt{\rho^{2} + 4\sigma(\rho+\delta)\left(\frac{1-\alpha-\beta}{1-\beta} \right)\frac{C^{A*}}{K^{A*}}} \right]\frac{K^{A*}}{C^{A*}} = 0.4310,$$

so $C^{A}(K) \cong 3.6589K^{0.4310}$. This policy rule gives explicit expressions for the marginal value function $V_{K}^{A}(K) = U'\left(3.6589K^{0.4310}\right)$ and for the value function $V^{A}(K)$. If the elasticity of intertemporal substitution is increased to σ = 0.8, the optimal path $C^{A(}K)$ can be approximated by $C^{A}(K) = 2.0808K^{0.5314}$.

The before-calamity steady-states follow from

$$\theta^{*} = H(P^{B*})\left[\left(\frac{C^{B*}}{C^{A}(K^{B*})} \right)^{1/\sigma} - 1 \right], \quad \tau^{*} = \frac{\psi H'(P^{B*})(C^{B*})^{1/\sigma}\left[\frac{(C^{B*})^{(1-1/\sigma)}}{1-1/\sigma} - \rho V^{A}(K^{B*}) \right]}{\left[\rho + H(P^{B*}) \right]\left[\rho + \gamma + H(P^{B*}) \right]},$$

$$C^{B^{*}} = \left[\left(1 - \beta + \frac{\beta\omega\tau^{*}}{d+\tau^{*}} \right)\left(\frac{\delta + \rho - \theta^{*}}{\alpha} \right) - \delta \right]K^{B*}, \quad P^{B*} = \frac{\psi}{\gamma}\left(\frac{\beta\omega}{d+\tau^{*}} \right)\left(\frac{\delta + \rho - \theta^{*}}{\alpha} \right)K^{B*},$$

where τ^{*} = 0 in case of a constant hazard rate *h*.

We solve the dynamic system used in the simulations for *C* and *K* backwards in time. In case the regime shift occurs at time *T*, the paths before the regime shift and after the shock are connected by $K^{B}(T) = K^{A}(T)$.

With a hazard rate that depends on the stock of atmospheric carbon, the saddle-point system (12)-(14) in Section 3 has K and P as predetermined variables and C and τ as non-predetermined variables with transversality condition that the system converges to the target steady state. Backward integration from the neighbourhood of the steady state or forward integration, guessing $C^B(0)$ and $\tau(0)$, is difficult due to the instability of small numerical mistakes. In the numerical simulations we reduce the dimension by setting the precautionary return θ and the price of carbon τ at their steady-state levels.

C.2. With marginal climate damages:

The after-calamity system for our functional forms (also allowing for general climate sensitivity χ, different from 3, and making use of the expression for $A(P)$) is:

$$\begin{aligned}\dot{K} &= Y^A(K,P,\tau)+\tau E - C,\\ \dot{P} &= \psi E - \gamma P,\\ \dot{C} &= \sigma\left[A^A(P)F_K(K,E,R)-\delta-\rho\right]C,\\ \dot{\tau} &= \left[A^A(P)F_K(K,E,R)+\gamma-\delta\right]\tau - \psi\xi^* A^A(P)F(K,E,R),\end{aligned}$$

where $\xi^* \equiv \xi\dfrac{\chi}{3}\left(P/P_{PI}\right)^{\frac{\chi-3}{3}}$, $A^A(P)=(1-\pi)\bar{A}\exp\left[-\xi\left(\left(P/P_{PI}\right)^{\frac{\chi}{3}}P_{PI}-\bar{P}\right)\right]$,

$E=\dfrac{\beta\omega A^A(P)F}{d+\tau}$ and $Y^A(K,P,\tau)$ with $Y_K^A=\rho$, $Y_P^A=-\xi^* A^A F$ and $Y_\tau^A=-E^A$, where all terms are evaluated at the after-calamity steady state. Define the vector $x\equiv\left(K,P,C,\tau\right)'$, $\Gamma_1\equiv\dfrac{1-\alpha-\beta}{1-\beta}$ and $\Gamma_2\equiv\xi^*\psi-\alpha\dfrac{\tau}{K}$. Using

$$E=\frac{\beta\omega}{d+\tau}\left[A^A(P)K^\alpha\beta^\beta\left(\frac{\omega}{d+\tau}\right)^{\beta\omega}\left(\frac{1-\omega}{c}\right)^{\beta(1-\omega)}\right]^{\frac{1}{1-\beta}},\ \ E_K=\frac{\alpha}{1-\beta}\frac{E}{K},$$

$$E_P=-\frac{\xi^*}{1-\beta}E,\ \ E_\tau=-\frac{1-\beta(1-\omega)}{1-\beta}\frac{E}{d+\tau},\ \ A^AF=\frac{Y^A(K,P,\tau)}{1-\beta}+\delta K$$

and $A^AF_K=\alpha\left[A^AK^{\alpha+\beta-1}\beta\left(\dfrac{\omega}{d+\tau}\right)^{\beta\omega}\left(\dfrac{1-\omega}{c}\right)^{\beta(1-\omega)}\right]^{\frac{1}{1-\beta}}$, we get the linearized after-calamity system $\dot{x}\cong\Upsilon(x-x^{A*})$ with state-transition matrix:

$$\Upsilon = \begin{pmatrix} \rho + \tau E_K & -\xi^*\left(A^A F + \dfrac{\tau E}{1-\beta}\right) & -1 & \tau E_\tau \\ \psi E_K & -\dfrac{\psi \xi^* E}{1-\beta} - \gamma & 0 & \psi E_\tau \\ -\sigma \dfrac{C}{K}(\rho+\delta)\Gamma_1 & -\dfrac{\sigma \xi^* C(\rho+\delta)}{1-\beta} & 0 & -\dfrac{\beta\omega\sigma C(\rho+\delta)}{(1-\beta)(d+\tau)} \\ -\left(\dfrac{\tau}{K} + \Gamma_2\right)(\rho+\delta) & \left[\dfrac{\Gamma_2}{1-\beta} - \dfrac{\psi}{P}\left(\dfrac{\chi-3}{3}\right)\right]\xi^* A^A F & 0 & \rho + \gamma + \dfrac{EF\Gamma_2}{1-\beta} \end{pmatrix}.$$

Log-linearization gives more accurate results so we use $\dot{\tilde{x}} = \tilde{\Upsilon}\tilde{x}$ where $\tilde{\Upsilon}_{ij} \equiv \Upsilon_{ij}\dfrac{x_j^{A*}}{x_i^{A*}}$, and $\tilde{x} \equiv (\tilde{x}_P{}', \tilde{x}_N{}')'$ with $\tilde{x}_p \equiv \left(\ln\left(\dfrac{K}{K^{A*}}\right), \ln\left(\dfrac{P}{P^{A*}}\right)\right)'$ and $\tilde{x}_n \equiv \left(\ln\left(\dfrac{C}{C^{A*}}\right), \ln\left(\dfrac{\tau}{\tau^{A*}}\right)\right)'$ being the vectors of predetermined and non-predetermined variables, respectively. Let $\tilde{\Upsilon}M = M\Lambda$ where the diagonal matrix Λ has the eigenvalues of $\tilde{\Upsilon}$ in decreasing order along the diagonal and the columns of the matrix M contain the eigenvectors corresponding to each of the eigenvalues. Suppose that the eigenvectors span the whole space so that each $\tilde{x}$ can be written as a linear combination of the eigenvectors, hence $\tilde{x} = M\tilde{y}$. It follows that the canonical form is $\dot{\tilde{y}} = \Lambda\tilde{y}$. If the system displays saddlepath stability, the first two eigenvalues have positive real parts and the last two have negative real parts. We rule out explosive trajectories, so the first two elements of $\tilde{y}$ must be zero and $\tilde{x} = M\tilde{y} = \begin{pmatrix} M_{pu} & M_{ps} \\ M_{nu} & M_{ns} \end{pmatrix}\begin{pmatrix} 0 \\ \tilde{y}_s \end{pmatrix} = \begin{pmatrix} M_{ps}\tilde{y}_s \\ M_{ns}\tilde{y}_s \end{pmatrix}$. The stable manifold is $\tilde{x}_n = M_{ns}\tilde{y}_s = M_{ns}M_{ps}{}^{-1}\tilde{x}_p$. Given initial values for the predetermined variables, the initial values for non-predetermined variables consumption and carbon price must be on the stable manifold: $\tilde{x}_n(0) = M_{ns}M_{ps}{}^{-1}\tilde{x}_p(0)$. From then on the economy stays on the stable manifold, so this manifold is an invariant subspace. We thus have:

$$\tilde{x}_p(t) = M_{ps}\begin{pmatrix} e^{\lambda_3 t} & 0 \\ 0 & e^{\lambda_4 t} \end{pmatrix} M_{ps}{}^{-1}\tilde{x}_p(0), \quad \tilde{x}_n(t) = M_{ns}\begin{pmatrix} e^{\lambda_3 t} & 0 \\ 0 & e^{\lambda_4 t} \end{pmatrix} M_{ps}{}^{-1}\tilde{x}_p(0).$$

For our benchmark parameter values, we get $\tilde{x}_n = \begin{pmatrix} 0.4313 & -0.02260 \\ 0.7258 & -0.01542 \end{pmatrix}\tilde{x}_p$ and

$$C^A(K,P) = C^{A*}\left(\frac{K}{K^{A*}}\right)^{0.4313}\left(\frac{P}{P^{A*}}\right)^{-0.02260}, \tau^A(K,P) = \tau^{A*}\left(\frac{K}{K^{A*}}\right)^{0.7258}\left(\frac{P}{P^{A*}}\right)^{-0.01542}.$$

The policy rule for after-calamity consumption reacts almost identically to changes in the capital stock as without marginal climate damages, but now also reacts negatively to the stock of atmospheric carbon to reflect the need to consume less when the stock of carbon is high. The after-calamity carbon price reacts positively to the capital stock, since the economy is then better able to cope with such a carbon price. Since in the damages proposed by Golosov et al. (2014) the concavity of the temperature response to the stock of carbon dominates the convexity of the function relating damages to temperature, climate damages are a concave function of the stock of atmospheric carbon. This explains why the after-calamity carbon tax reacts negatively to the stock of atmospheric carbon. However, with a climate sensitivity of 4 instead of 3, the after-calamity stable manifold is

$$C^A(K,P) = C^{A*}\left(\frac{K}{K^{A*}}\right)^{0.43113}\left(\frac{P}{P^{A*}}\right)^{-0.048597}, \tau^A(K,P) = \tau^{A*}\left(\frac{K}{K^{A*}}\right)^{0.72764}\left(\frac{P}{P^{A*}}\right)^{0.24346}.$$

The after-calamity carbon price now responds positively to the carbon stock.