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Just-noticeable difference as a behavioural foundation of the critical cost-efficiency index*

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Abstract

Critical cost-efficiency index (or CCEI), proposed in Afriat (1972, 1973) and Varian (1990), is the most commonly used measure of revealed preference violations. By representing consumer preference with *interval orders*, as in Fishburn (1970), we show that this index is equivalent to a particular notion of the just-noticeable difference, i.e., a measure of dissimilarity between alternatives that is sufficient for the agent to tell them apart. Therefore, CCEI can be interpreted as the consumer's cognitive inability to discriminate among options. This characterisation sheds new light on the existing empirical findings.

Keywords: utility maximisation, generalised axiom of revealed preference, critical cost-efficiency index, interval order, just-noticeable difference
JEL Classification: C14, C60, C61, D11, D12

1 Introduction

Suppose we observe a consumer making purchases from ℓ available goods at some prevailing prices. Formally, we consider the agent choosing a bundle of commodities $x_t \in \mathbb{R}^\ell_+$ at a price vector $p_t \in \mathbb{R}^\ell_{++}$. Given a finite number of such observations $t \in T$, under what condition it is possible to rationalise the observable choices with utility maximisation? Equivalently, when we can find a utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that

$$p_t \cdot y \leq p_t \cdot x_t \;\text{ implies }\; u(x_t) \geq u(y), \;\text{for all } t \in T;$$

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i.e., any bundle y affordable in observation t is inferior to x_t with respect to u. Afriat (1967), Diewert (1973), and Varian (1982) answered this question by characterising a necessary and sufficient condition on a finite set of observations, called the *generalised axiom of revealed preference* (or GARP, for short), under which it can be rationalised in the above sense with a locally non-satiated utility function.

One problematic feature of GARP lies in its deterministic nature. A dataset either satisfies this restriction, and thus can be rationalised with a locally non-satiated utility, or it does not. In practice, it is desirable to evaluate how severe is a violation once it occurs. Afriat (1972, 1973) and Varian (1990) addressed this issue by introducing the notion of efficiency indices. Formally, a dataset is rationalisable for efficiency parameters $e_t \in [0, 1]$, for all $t \in T$, if there is a locally non-satiated utility function u satisfying:

$$\text{if } \; p_t \cdot y \leq e_t(p_t \cdot x_t) \; \text{ then } \; u(x_t) \geq u(y), \text{ for all } \; t \in T.$$

Therefore, the observed choice x_t is maximising u over the perturbed budget set, for all $t \in T$. The farther the efficiency parameters are from 1, the more severe is the deviation from rationality. The *critical cost-efficiency index* (or CCEI, for short) is the supremum, with respect to some metric or aggregator function, of all efficiency parameters $(e_t)_{t \in T}$ that satisfy the above condition. Hence, it determines the minimal budget adjustments that are necessary for the dataset to be rationalisable.

Despite criticism, CCEI remains the most commonly used measure of revealed preference violations.[1] On one hand, it has an appealing economic interpretation in terms of the share of wealth wasted by the consumer relatively to a fully rational one. In fact, Halevy et al. (2017) show that it is closely related to the money metric. In addition, it is convenient for empirical applications, as it can be evaluated using computationally efficient methods — for particular specifications of the aggregator function.[2]

This paper presents a behavioural foundation of CCEI. Specifically, we argue that it is equivalent to a notion of the just-noticeable difference, i.e., a measure of dissimilarity between alternatives that is sufficient for the agent to tell them apart. The evidence from psychophysiology suggest that people can not discern between two physical stimuli unless their intensities are significantly (noticeably) different.[3] This idea was incorporated to

[1] For some critical analysis of this index see, e.g., Echenique et al. (2011), Apesteguia and Ballester (2015), Dean and Martin (2016), or Dziewulski (2017).

[2] See Smeulders et al. (2014) for details.

[3] See Laming (1997) or Algom (2001) for a comprehensive summary of this literature.

choice theory by Armstrong (1950) and Luce (1956), who claimed that due to imperfect powers of discrimination of the human mind, consumers are unable to distinguish between goods/bundles that are similar. In particular, the authors postulated that any form of such an imperfect discrimination would require for indifferences to violate transitivity.[4] Thus, this phenomenon can not be modelled using utility maximisation.

One way of approaching the problem of noticeable differences was proposed by Fishburn (1970), who characterised consumer preference with an *interval order*.[5] We postpone the formal definition of this notion till Section 2. However, an interval order can be though of as an asymmetric binary relation P for which there exists a utility u and a positive threshold function δ, defined over the space of alternatives, such that

$$xPy \text{ if and only if } u(x) > u(y) + \delta(x).$$

Thus, option x is strictly preferred to y if it yields a significantly higher utility, where the threshold is determined by $\delta(x)$. If neither xPy nor yPx, we say that x is indifferent to (or indistinguishable from) y and denote it by xIy, which is not transitive.

We capture consumers' insensitivity to differences among alternatives with a noticeable difference. An interval order P admits a *noticeable difference* $\lambda > 1$ if

$$\lambda' \geq \lambda \text{ implies } \big(\lambda' y\big) P y,$$

for any non-zero y. Roughly speaking, the parameter λ is the relative change in sizes of two bundles that is sufficient for the agent to correctly discern between the options.[6] This definition is inspired by the *Weber-Fechner law* in psychophysics. It states that people are unable to discriminate between two intensities of a physical stimulus unless the ratio of their magnitudes exceeds a particular value — a Weber's constant.[7] Ours is a natural extension of this notion to choices over multi-dimensional domains.

The just-noticeable difference is formally defined in Section 3. First, we introduce a model of consumer choice in which the sensitivity to dissimilarities across options varies

[4] Although it may be impossible to distinguish option x from y and y from z, alternative x may be noticeably different and strictly preferred to z, or vice versa.

[5] Alternatively, Luce (1956), Scott and Suppes (1958), and Beja and Gilboa (1992) address the problem of noticeable differences by characterising preferences with *semiorders*.

[6] This is similar to the notion of a noticeable difference defined in Dziewulski (2017). However, we require for P to be an interval order, rather than a semiorder.

[7] Note that *Stevens' power law* seems to better explain sensory discrimination than Weber-Fechner's law. Nevertheless, the latter is considered to be the best first approximation. See, e.g., Algom (2001).

from one decision problem to another. In each observation $t \in T$, preference of an agent is represented by an interval order P_t that admits a noticeable difference $\lambda_t > 1$. The choice is determined by maximisation of the relation over the affordable alternatives, i.e.,

$$p_t \cdot y \leq p_t \cdot x_t \ \text{ implies } \ \text{ not } yP_tx_t.$$

Moreover, in order to properly capture changes in the agent's sensitivity to differences among alternatives, rather than variations in the underlying preferences, we impose a particular consistency condition on the profile $(P_t)_{t\in T}$.

The *just-noticeable difference* is the infimum, with respect to some metric or aggregator function, of all noticeable differences $(\lambda_t)_{t\in T}$ for which the dataset is rationalisable in the above sense with some profile $(P_t)_{t\in T}$ of consistent interval orders. Therefore, it measures the least level of the consumer's cognitive inability to differentiate across options that is necessary to rationalise the observable choices.

In the Main Theorem, presented in Section 4, we show that CCEI coincides with the inverse of the just-noticeable difference. In particular, any efficiency parameters $(e_t)_{t\in T}$ that rationalise the set of observations are essentially equal to inverses of noticeable differences $(\lambda_t)_{t\in T}$ that support the data in the sense specified above. Therefore, instead of treating CCEI as a measure of budgetary adjustments, it can be interpreted in terms of imperfect sensory discrimination. This implies that the model of consumer choice introduced in this paper not only explains the observable choices but also determines how far away the set is from being rationalisable with utility maximisation — by reinterpreting the existing and well-established measure of revealed preference violations.

We conclude this paper in Section 5 where we revisit the experiment in Choi et al. (2014). The purpose of that large-scale study was to test for consistency with utility maximisation across various socio-economic groups. Since their analysis makes an extensive use of CCEI, our characterisation provides new insight to their results. We evaluate noticeable differences using the same dataset and compare our estimates of just-noticeable differences to those in the existing literature on sensory discrimination.

Our discussion pertains to a broad class of consumer choice problems that were introduced in Forges and Minelli (2009). In particular, none of the results depend on linearity of budget sets, as in the work of Afriat (1972, 1973) or Varian (1990).

2 Interval orders

In this section we introduce the notion of interval orders that is fundamental to our analysis. See Aleskerov et al. (2007) for a comprehensive treatment of this topic.[8]

Following Wiener (1914) and Fishburn (1970), an *interval order* over a set of alternatives X is an *irreflexive* binary relation P that satisfies the *interval order condition*:

$$\text{if} \quad xPy \text{ and } x'Py' \quad \text{then} \quad \text{either } xPy' \text{ or } x'Py,$$

for all x, y, x', and y' in X. It is easy to verify that any interval order is asymmetric and transitive. In the remainder of this paper, we associate P with the *strict preference*. If neither xPy nor yPx then x is *indistinguishable* from y, which we denote by xIy. Observe that the latter is reflexive and symmetric, but not transitive.

Any interval order is inherently related to a particular binary relation $\succeq$, with its asymmetric and symmetric components denoted by $\succ$ and $\sim$, respectively. Let $x \succ y$ whenever there is some $z \in X$ such that xIz and zPy. By symmetry of I, this guarantees that xPy implies $x \succ y$. If neither $x \succ y$ nor $y \succ x$, then $x \sim y$. In the remainder of the paper we say that such a relation $\succeq$ is *induced* by the interval order P.

Proposition 1. *Relation $\succeq$ induced by an interval order P is a weak order, i.e., a complete, transitive, and reflexive binary relation.*

A complete proof of this result can be found in Aleskerov et al. (2007, p. 60). However, to keep this paper self-contained, we show that the strict part $\succ$ of the relation is irreflexive and transitive, as it is required in the remainder of this paper.

Proof of Proposition 1. Suppose that $x \succ x$, for some $x \in X$. This implies that there is some z such that xIz and zPx, which is not possible by construction of I. Hence, $\succ$ is irreflexive. Next, let $x \succ x'$ and $x' \succ x''$. Thus, there is some z and z' such that xIz and zPx', as well as $x'Iz'$ and $z'Px''$. First, we claim that zPx''. By the interval order condition, if zPx' and $z'Px''$, then either zPx'' or $z'Px'$. However, since $x'Iz'$, it must be that zPx''. In particular, we obtain xIz and zPx'' which implies $x \succ x''$. □

An important characteristic of interval orders pertains to their utility representation. Following Fishburn (1970), for any interval order P defined over a countable set X there

[8] I would like to thank Ali Khan for pointing me to this publication.

is a utility $u : X \to \mathbb{R}$ and a positive threshold function $\delta : X \to \mathbb{R}_+$ such that

$$xPy \text{ if and only if } u(x) > u(y) + \delta(x). \tag{1}$$

Thus, option x is strictly preferred to y if and only if the utility of the former is sufficiently higher than that of the latter, where the threshold is determined by $\delta(x)$. Clearly, this implies that *not* yPx is equivalent to $u(x) + \delta(y) \geq u(y)$. Under some regularity conditions, the representation can be extended to interval orders defined over more general spaces. See Bridges (1985, 1986) and Chateauneuf (1987) for details.

Notice that, whenever $x \succ y$ then $u(x) > u(y)$. Therefore, the weak order $\succeq$ induced by P is consistent with the ranking generated by the utility function u. For this reason, we refer to the relation $\succeq$ as the *"true" consumer preference*, i.e., as if perfect discrimination were possible. In other words, if the consumer could hedonically distinguish between any two alternatives in X, or equivalently, if the threshold function δ were constantly equal to zero, the agent's preference would be characterised by the weak order $\succeq$.

Essentially, the above model describes a consumer with two binary relations. First, the interval order P represents preferences that the agent is using when making a choice. In particular, given our discussion, this captures the nature of decision making under imperfect discrimination among alternatives. Second, we assume that the induced relation $\succeq$ is the consumer's "true" preference that would be revealed if the agent could perfectly discern among all options. This distinction plays a crucial role in the model of consumer choice with variable noticeable differences, presented in Section 3.3.

3 Efficiency indices and noticeable differences

In this section we introduce the framework of our analysis and formally define the critical cost-efficiency index and the just-noticeable difference.

3.1 Setup

Suppose that a researcher monitors a finite number of observations $t \in T$, each consisting of a set of alternatives $B_t \subseteq \mathbb{R}^\ell_+$ available to the consumer and a bundle of ℓ goods $x_t \in B_t$ selected from it. The *set of observations* is given by $\mathcal{O} = \{(B_t, x_t) : t \in T\}$. With a slight abuse of the notation, we denote its cardinality by T.

Throughout this paper we consider choices over *generalised budget sets* as in Forges and Minelli (2009). In particular, we assume that set B_t is compact and downward comprehensive, for all $t \in T$.[9] Moreover, there is some $y \in B_t$ such that $y \gg 0$. With a slight abuse of our notation, let the *upper bound* of the set B_t be given by

$$\partial B_t := \{y \in B : \text{ if } z \gg y \text{ then } z \notin B\}$$

and suppose that, for any $y \in \partial B_t$ and scalar $\theta \in [0, 1)$, we have $\theta y \in B_t \setminus \partial B_t$. That is, for any non-zero $y \in \mathbb{R}^\ell_+$, ray $\{\theta y : \theta \geq 0\}$ intersects the boundary ∂B_t exactly once. Finally, we assume that in each observation, at least one commodity is chosen in a strictly positive amount, i.e., $x_t \neq 0$. This is not without loss of generality, but it simplifies our analysis and is insignificant from the empirical point of view.

It is straightforward to verify that our framework admits linear budget sets, as in the original work of Afriat (1967), Diewert (1973), and Varian (1982). In such a case, each set of alternatives is given by $B_t := \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\}$, for some $p_t \in \mathbb{R}^\ell_{++}$.

3.2 Critical cost-efficiency index

We proceed with a generalised definition of the critical cost-efficiency index introduced in Varian (1990).[10] A set of observations $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ is rationalisable with *efficiency parameters* $(e_t)_{t \in T}$, where $e_t \in [0, 1]$, for all $t \in T$, whenever there is a locally non-satiated utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfies

$$\text{if } \ y \in e_t B_t \ \text{ then } \ u(x_t) \geq u(y), \ \text{ for all } \ t \in T,$$

where $e_t B_t := \{e_t y : y \in B_t\}$. This condition requires that the observed choice x_t is maximising the function u over the perturbed set $e_t B_t$. In particular, whenever $(e_t)_{t \in T}$ is the unit vector, this restriction coincides with utility maximisation.

Let $F : \mathbb{R}^T_+ \to \mathbb{R}$ be a well-defined, continuous, and increasing *aggregator function* that maps efficiency parameters $(e_t)_{t \in T}$ to real numbers. The corresponding *critical cost-efficiency index* (or CCEI), denoted by e^*_F, is the *supremum* of $F\big((e_t)_{t \in T}\big)$ with respect to

[9] We endow space $\mathbb{R}^\ell_+$ with the natural product order $\geq$. That is, we have $x \geq y$ if and only if $x^i \geq y^i$, for all $i = 1, 2, \ldots, \ell$. Moreover, let $x \gg y$ whenever $x^i > y^i$, for all $i = 1, 2, \ldots, \ell$. A set $B \subseteq \mathbb{R}^\ell_+$ is *downward comprehensive* whenever $y \in B$ and $y \geq z$ implies $z \in B$, for all $z \in \mathbb{R}^\ell_+$.

[10] Some authors refer to this measure as *Varian's (in)efficiency index.*

all vectors $(e_t)_{t\in T}$ for which the dataset $\mathcal{O}$ is rationalisable.[11] Thus, this measure evaluates the minimal budget adjustments (with respect to the function F) that are required for the data to be supported with utility maximisation. Equivalently, it determines the least money waste that is necessary for the set of observations be rationalisable. In fact, Halevy et al. (2017) show that the index is equivalent to the normalised money metric.

In the remainder of this subsection we characterise the class of datasets $\mathcal{O}$ that are rationalisable with a locally non-satiated utility function for some efficiency parameters $(e_t)_{t\in T}$. We begin by introducing the following condition.

Axiom 1. *Given numbers $e_t \in [0,1]$, $t \in T$, for any cycle $\mathcal{C} = \big\{(a,b),(b,c),\dots,(z,a)\big\}$ in $T \times T$ such that $x_s \in e_tB_t$, for all $(t,s) \in \mathcal{C}$, we have $x_s \in \partial(e_tB_t)$, for all $(t,s) \in \mathcal{C}$.*

This restriction requires that for any cycle $\mathcal{C}$ induced by the set of observations such that bundle x_s belongs to the perturbed budget set e_tB_t, for all $(t,s) \in \mathcal{C}$, each bundle x_s must belong to the upper bound of the corresponding set e_tB_t. Notice that, whenever $(e_t)_{t\in T}$ is the unit vector, Axiom 1 is equivalent to the generalised version of GARP as in Forges and Minelli (2009) — or Afriat (1967), Diewert (1973), and Varian (1982) once we assume that budget sets B_t are linear, for all $t \in T$.

Proposition 2. *Set of observations $\mathcal{O}$ is rationalisable with efficiency parameters $(e_t)_{t\in T}$ if and only if it satisfies Axiom 1 for $(e_t)_{t\in T}$.*

To show that Axiom 1 is necessary for a set of observations $\mathcal{O}$ to be rationalisable in the above sense, suppose that there is a locally non-satiated function u such that $y \in e_tB_t$ implies $u(x_t) \geq u(y)$, for all $t \in T$. Clearly, for any t, $s \in T$ with $x_s \in e_tB_t$, we have $u(x_t) \geq u(x_s)$, while the inequality is strict whenever x_s is not in the upper bound of e_tB_t — by local non-satiation. Therefore, for any cycle $\mathcal{C}$ specified as in the axiom,

$$u(x_a) \;\geq\; u(x_b) \;\geq\; \dots \;\geq\; u(x_z) \;\geq\; u(x_a),$$

which can be satisfied only if all the inequalities are binding. This requires for the bundle x_s to be in the upper bound of the perturbed set e_tB_t, for all $t \in T$. Therefore, Axiom 1 excludes the possibility of any strict cycles in the revealed preference relation.

[11] In the original definition of Varian (1990), the aggregator function is given by the mean of squares. However, other methods of aggregation may be considered. For examples, see Tsur (1989), Cox (1997), Alcantud et al. (2010), or Smeulders et al. (2014).

The sufficiency part of the result can be supported with a simple modification of the argument in Forges and Minelli (2009, Section 1.2). Thus, we omit the proof.

Proposition 2 implies that CCEI corresponding to a set of observations $\mathcal{O}$ is equivalent to the supremum of $F\big((e_t)_{t\in T}\big)$ over all efficiency parameters $(e_t)_{t\in T}$ for which the set obeys Axiom 1. We exploit this characterisation in the main result, were we show that CCEI is equivalent to the just-noticeable difference.

The definition in Varian (1990) generalises the so-called *Afriat's efficiency index*, proposed originally in Afriat (1972, 1973). In the latter, budget set adjustments are restricted to be identical across all observations, i.e., we have $e_t = e_s$, for all t, $s \in T$. Equivalently, it is a special case of CCEI whenever the aggregator function is given by $F\big((e_t)_{t\in T}\big) := \min\big\{e_t : t \in T\big\}$ or any strictly monotone transformation of thereof. We discuss this measure in more detail in Section 4.2.

3.3 Just-noticeable difference

In this subsection we define our notion of the just-noticeable difference. We begin by introducing a model of consumer choice in which the agent is unable to perfectly discriminate among alternatives. We consider a case in which sensitivity to dissimilarities across options varies from one decision problem to another, but the underlying "true" preference, i.e., as if perfect discrimination were possible, remain constant.

Formally, suppose that in each decision problem $t \in T$ the consumer is maximising an interval order P_t over the set of available options. We capture the agent's sensitivity to differences among alternatives with a noticeable difference. Specifically, we say that the relation P_t admits a *noticeable difference* $\lambda_t > 1$ whenever

$$\lambda' \geq \lambda_t \ \text{ implies } \ (\lambda' y) P_t y,$$

for all non-zero $y \in \mathbb{R}^\ell_+$. That is, number λ_t determines by how much one should inflate a consumption bundle y in order to guarantee that the agent perceives the difference. Roughly speaking, it is the relative change in sizes of two bundles that is sufficient for the agent to discern between the alternatives. This is inspired by the aforementioned Weber-Fechner law in psychophysiology according to which people are unable to discriminate between two intensities of a physical stimulus unless their ratio exceeds a particular constant. We extend this notion to choices over multi-dimensional domains.

By allowing for the relation P_t and the noticeable difference λ_t to vary across decision problems, we assume that the agent's cognitive ability to discriminate among options changes from one choice to another. Since it is irrelevant to our results, we are agnostic about how the scalars $(\lambda_t)_{t\in T}$ are determined in each observation $t \in T$.

For every $t \in T$, let $\succeq_t$ denote the weak order induced by interval order P_t, constructed as in Section 2. Recall that we interpret this relation as the "true" preference, i.e., as if perfect discrimination were possible. In order to properly capture changes in the agent's sensitivity to differences among alternatives, rather than variations in the underlying preferences, we restrict our attention to consistent interval orders. A profile $(P_t)_{t\in T}$ of interval orders is *consistent* with respect to the induced weak order whenever $\succeq_t = \succeq_s$, for all t, $s \in T$. That is, even though each ordering P_t in the profile may be different, they correspond to the same underlying "true" preference. In fact, whenever relations $(P_t)_{t\in T}$ admit a representation as in (1), this implies that xP_ty if and only if $u(x) > u(y) + \delta_t(x)$, where the utility u is invariant with respect to $t \in T$ and the induced weak order $\succeq_t$ is consistent with the ranking generated by u, for all $t \in T$.

A set of observations $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ is rationalisable with *noticeable differences* $(\lambda_t)_{t\in T}$ whenever there exists a profile $(P_t)_{t\in T}$ of consistent interval orders such that, for each $t \in T$, relation P_t admits the noticeable difference λ_t and

$$y \in B_t \ \text{ implies } \ \text{not } yP_tx_t,$$

i.e., the interval order P_t is maximised in observation t. This notion of rationalisation captures the idea that, even though the underlying "true" preference of the consumer remain unchanged throughout the observations, in each decision problem $t \in T$ the agent is affected by a different level of cognitive ability to discriminate among options.

In some instances, we are interested in a stronger notion of consistency across the interval orders. The profile $(P_t)_{t\in T}$ is *monotone* with respect to the noticeable difference if: $\lambda_t \geq \lambda_s$ implies $P_t \subseteq P_s$, or equivalently, if xP_ty then xP_sy, for all t, $s \in T$. Hence, whenever option x is strictly chosen over y under a greater noticeable difference, then it has to be strictly preferable to the latter when the agent is more sensitive to changes. In particular, monotonicity guarantees that: if $\lambda_t = \lambda_s$, then $P_t = P_s$. Thus, the noticeable difference is the only source of the consumer's insensitivity.

As previously, let $F : \mathbb{R}^T_+ \to \mathbb{R}$ be a well-defined, continuous, and increasing aggregator function. The corresponding *aggregate just-noticeable difference*, denoted by λ^*_F, is the *infimum* of $F\big((\lambda_t)_{t\in T}\big)$ over all noticeable differences $(\lambda_t)_{t\in T}$ that rationalise $\mathcal{O}$. Therefore, conditional on the criterion F, it measures the least noticeable differences that are sufficient to explain the observations with the model presented above.

We proceed with a characterisation of observation sets that are rationalisable in the aforementioned sense. Consider the following axiom.

Axiom 2. *Given numbers $\lambda_t > 1$, $t \in T$, there is no cycle $\mathcal{C} = \big\{(a,b),(b,c),\dots,(z,a)\big\}$ in $T \times T$ such that $(\lambda_t x_s) \in B_t$, for all $(t,s) \in \mathcal{C}$.*

This condition requires that there is no cycle $\mathcal{C}$ in the set of observations such that $x_s \in (1/\lambda_t)B_t$, for all $(t,s) \in \mathcal{C}$. Roughly speaking, this is to say that whenever there is a sequence $\mathcal{C}$ with $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, at least one bundle x_s has to be sufficiently close to the upper bound of the corresponding set B_t. In particular, the above condition refers to one element cycles $\mathcal{C} = \big\{(t,t)\big\}$. Thus, we have $(\lambda_t x_t) \notin B_t$, for each $t \in T$.

Proposition 3. *Set $\mathcal{O}$ is rationalisable with noticeable differences $(\lambda_t)_{t\in T}$ if and only if it satisfies Axiom 2 for $(\lambda_t)_{t\in T}$. In addition, the corresponding profile $(P_t)_{t\in T}$ is monotone with respect to the noticeable difference, with no loss of generality.*

Since the proof is extensive, we show only the necessity part of the result and postpone the rest till the Appendix. Suppose that a dataset $\mathcal{O}$ is rationalisable with noticeable differences $(\lambda_t)_{t\in T}$ and a corresponding profile $(P_t)_{t\in T}$ of interval orders. Since the profile is consistent, we have $\succeq_t = \succeq_s = \succeq$, for all t, $s \in T$.

Take any cycle $\mathcal{C} = \big\{(a,b),(b,c),\dots,(z,a)\big\}$ in $T \times T$ such that $(\lambda_t x_s) \in B_t$, for all $(t,s) \in \mathcal{C}$. By our definition of rationalisation, if $(\lambda_t x_s) \in B_t$ then *not* $(\lambda_t x_s) P_t x_t$. Thus, either $x_t I_t (\lambda_t x_s)$ or $x_t P_t (\lambda_t x_s)$, where I_t denotes the indistinguishable relation corresponding to P_t. In addition, it must be that $(\lambda_t x_s) P_t x_s$. The two conditions imply $x_t \succ_t x_s$, or equivalently $x_t \succ x_s$. Since this is true for all $(t,s) \in \mathcal{C}$, we obtain

$$x_a \;\succ\; x_b \;\succ\; x_c \;\succ\; \dots \;\succ\; x_z \;\succ\; x_a,$$

which by transitivity of $\succ$ yields $x_a \succ x_a$. However, this contradicts that the relation $\succ$ is irreflexive (recall Proposition 1). Thus, no such cycle $\mathcal{C}$ is admissible.

The above argument highlights that the testable implications for interval order maximisation pertain to the induced weak order, rather than the interval order itself. The sole purpose of Axiom 2 is to guarantee that there are no cycles in $\succ$.

Finally, Proposition 3 implies that the aggregate just-noticeable difference is equal to the infimum of $F\big((\lambda_t)_{t\in T}\big)$ with respect to all noticeable differences for which set $\mathcal{O}$ satisfies Axiom 2. This observation is crucial for our main result.

4 The equivalence result

In this section we present the main theorem of our paper. First, we state the equivalence between the critical cost-efficiency index and the inverse of just-noticeable difference. Then we apply the result to the special case of Afriat's efficiency index.

4.1 The main result

Recall that $F : \mathbb{R}^T_+ \to \mathbb{R}$ denotes a well-defined, continuous, and increasing aggregator function. We proceed with the main result of the paper.

Main Theorem. *For any set $\mathcal{O}$, the critical cost-efficiency index e^*_F is the supremum of $F\big((1/\lambda_t)_{t\in T}\big)$ with respect to all noticeable differences $(\lambda_t)_{t\in T}$ that rationalise $\mathcal{O}$. Conversely, the aggregate just-noticeable difference λ^*_F is the infimum of $F\big((1/e_t)_{t\in T}\big)$ with respect to all efficiency parameters $(e_t)_{t\in T}$ for which the dataset is rationalisable.*

Our main result states that, for an arbitrary dataset $\mathcal{O}$, evaluating the critical cost-efficiency index is equivalent to determining the corresponding just-noticeable difference. In fact, as we show in the remainder of this subsection, any efficiency parameters $(e_t)_{t\in T}$ that rationalise the set of observations as in Section 3.2 are essentially equivalent to inverses of noticeable differences $(\lambda_t)_{t\in T}$ that support the data as specified in Section 3.3. Therefore, the critical cost-efficiency index can be interpreted as a measure of the consumer's cognitive inability to differentiate among alternatives.

Depending on the functional form of the aggregator function F, the equivalence between the critical cost-efficiency index and the aggregate just-noticeable difference may be even stronger. The following is implied directly by the Main Theorem.

Corollary 1. *Suppose that the aggregator function $F : \mathbb{R}^T_+ \to \mathbb{R}$ is given by the geometric mean, i.e., $F\big((y_t)_{t\in T}\big) := \sqrt[T]{\prod_{t\in T} y_t}$. For any set $\mathcal{O}$, the critical cost-efficiency index is equal to the inverse of the aggregate just-noticeable difference, i.e., we have $e^*_F = 1/\lambda^*_F$.*

Proof. By the Main Theorem, we obtain

$$e_F^* = \sup \sqrt[T]{\prod_{t \in T} \frac{1}{\lambda_t}} = \sup \frac{1}{\sqrt[T]{\prod_{t\in T} \lambda_t}} = \frac{1}{\inf \sqrt[T]{\prod_{t\in T} \lambda_t}} = \frac{1}{\lambda_F^*},$$

where the supremum and infimum are taken with respect to all noticeable differences $(\lambda_t)_{t\in T}$ under which the set of observations $\mathcal{O}$ is rationalisable. □

The Main Theorem shows that, instead of thinking about the critical cost-efficiency index as a measure of budgetary adjustments, it can be interpreted in terms of imperfect sensory discrimination. This means that the model of consumer choice introduced in Section 3.3 can both explain the observable choices as well as determine how far the set is from being rationalisable with utility maximisation, by evoking the existing and well-known measure of revealed preference violations.

We devote the reminder of this subsection to the proof of the Main Theorem. The result is implied by the following two lemmas. First, we claim that any noticeable differences rationalising the set of observations are equal to inverses of the corresponding efficiency parameters. Consider the following result.

Lemma 1. *If a set $\mathcal{O}$ is rationalisable for some noticeable differences $(\lambda_t)_{t\in T}$, then it is rationalisable with any efficiency parameters $(e_t)_{t\in T}$ such that $e_t \leq 1/\lambda_t$, for all $t \in T$.*

Proof. Suppose that set $\mathcal{O}$ satisfies Axiom 2 for some noticeable differences $(\lambda_t)_{t\in T}$. Thus, there is no cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ such that $(\lambda_t x_s) \in B_t$, or equivalently $x_s \in (1/\lambda_t)B_t$, for all $(t,s) \in \mathcal{C}$. This suffices for Axiom 1 to be satisfied for any efficiency parameters $(e_t)_{t\in T}$ such that $e_t \leq 1/\lambda_t$, for all $t \in T$. □

The second result is a converse to Lemma 1. We show that efficiency parameters are essentially equivalent to inverses of noticeable differences.

Lemma 2. *Whenever a set of observations $\mathcal{O}$ is rationalisable with some efficiency parameters $(e_t)_{t\in T}$, then it is rationalisable for any noticeable differences $(\lambda_t)_{t\in T}$ such that $\lambda_t \geq 1/e_t$, for all $t \in T$, and $\lambda_t > 1/e_t$, for some $t \in T$.*

Proof. We prove the lemma by contradiction. Suppose that set $\mathcal{O}$ obeys Axiom 1 for efficiency parameters $(e_t)_{t\in T}$, but violates Axiom 2 for some noticeable differences $(\lambda_t)_{t\in T}$ such that $\lambda_t \geq 1/e_t$, for all $t \in T$, and $\lambda_t > 1/e_t$, for some $t \in T$. Therefore, there is a cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ such that $(\lambda_t x_s) \in B_t$ or equivalently

$x_s \in (1/\lambda_t)B_t$, for all $(t, s) \in \mathcal{C}$. However, this implies that, for any $t \in T$ with $\lambda_t > 1/e_t$, we have $x_s \in (1/\lambda_t)B_t \subset e_tB_t \setminus \partial(e_tB_t)$, which contradicts that $x_s \in \partial(e_tB_t)$. □

Given the two lemmas, we proceed with the proof of the Main Theorem. First, we argue that the critical cost-efficiency index e^*_F is the supremum of $F\big((1/\lambda_t)_{t\in T}\big)$ with respect to all noticeable differences $(\lambda_t)_{t\in T}$ that rationalise the dataset. Denote the latter value by v^*_F. To show that $e^*_F = v^*_F$, take any noticeable differences $(\lambda_t)_{t\in T}$ that rationalise $\mathcal{O}$. Following Lemma 1, any efficiency parameters satisfying $e_t \geq 1/\lambda_t$, for all $t \in T$, rationalise the dataset in the respective sense. By monotonicity of F, we have

$$e^*_F \;\geq\; F\big((e_t)_{t\in T}\big) \;\geq\; F\big((1/\lambda_t)_{t\in T}\big).$$

Taking the supremum over the right hand side implies $e^*_F \geq v^*_F$. Whenever $e^*_F > v^*_F$, there are some efficiency parameters $(e_t)_{t\in T}$ rationalising the data such that $F\big((e_t)_{t\in T}\big) > v^*_F$. By Lemma 2 as well as continuity and monotonicity of F, there are some noticeable differences $(\lambda_t)_{t\in T}$ rationalising the dataset $\mathcal{O}$ that satisfy

$$F\big((e_t)_{t\in T}\big) \;\geq\; F\big((1/\lambda_t)_{t\in T}\big) \;>\; v^*_F,$$

which contradicts that v^*_F is the supremum of $F\big((1/\lambda_t)_{t\in T}\big)$ with respect to noticeable differences rationalising $\mathcal{O}$. We prove the second part of the theorem analogously.

4.2 Interpreting Afriat's efficiency index

As pointed out in Section 3.2, Afriat's efficiency index proposed in Afriat (1972, 1973) is a special case of CCEI. In this subsection we show that the former coincides with the inverse of the *static* just-noticeable difference.

A dataset $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$ is rationalisable for a *constant efficiency parameter* $e \in [0, 1]$, whenever there is a locally non-satiated utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that

$$y \in eB_t \;\text{ implies }\; u(x_t) \geq u(y), \;\text{ for all } t \in T.$$

The above notion is a special case of the one discussed in Section 3.2 with efficiency parameters e_t constant and equal to e, for all $t \in T$. It is straightforward to show that a dataset is rationalisable with a constant efficiency parameter e if and only if it obeys Axiom 1 for efficiency parameters $(e_t)_{t\in T}$ satisfying $e_t = e$, for all $t \in T$.

Afriat's efficiency index, denoted by e^*_A, is the supremum over all constant efficiency parameters e under which set $\mathcal{O}$ is rationalisable as above. Recall that it coincides with CCEI for the aggregator function F given by $F\big((y_t)_{t\in T}\big) = \min\big\{y_t : t \in T\big\}$.

Next, we turn to a spacial case of the model of consumer choice introduced in Section 3.3. A set of observations $\mathcal{O}$ is rationalisable with a *static noticeable difference* $\lambda > 1$, if there is an interval order P that admits a noticeable difference λ, i.e., we have $(\lambda' y)Py$, for all $\lambda' \geq \lambda$ and non-zero $y \in \mathbb{R}^\ell_+$, while

$$y \in B_t \ \text{ implies } \ \text{not } yPx_t, \text{ for all } t \in T.$$

This is a special case of rationalisation with noticeable differences in which $\lambda_t = \lambda$ and $P_t = P$, for all $t \in T$. In particular, set $\mathcal{O}$ is rationalisable in the above sense if and only if it satisfies Axiom 2 for $\lambda_t = \lambda$, for all $t \in T$. Indeed, since a constant profile $(P_t)_{t\in T}$ is consistent with respect to the induced weak order, any dataset rationalisable in the above sense satisfies Axiom 2 for $\lambda_t = \lambda$, for all $t \in T$. Conversely, by Proposition 3, whenever the axiom holds there is a profile $(P_t)_{t\in T}$ of consistent interval orders in which relation P_t admits the noticeable difference $\lambda_t = \lambda$ and rationalises observation t, for all $t \in T$. Since $(P_t)_{t\in T}$ is monotone with respect to the noticeable difference, with no loss of generality, we have $\lambda_t = \lambda_s = \lambda$ only if $P_t = P_s = P$, for all t, $s \in T$.

The *static just-noticeable difference*, denoted by λ^*_S, is the infimum over all static noticeable differences $\lambda > 1$ that rationalise the data. Equivalently, it is the aggregate just-noticeable difference for the aggregator function $F\big((y_t)_{t\in T}\big) := \max\big\{y_t : t \in T\big\}$.

Proposition 4. *For any set of observations $\mathcal{O}$, Afriat's efficiency index is equal to the inverse of the static just-noticeable difference, i.e., we have $e^*_A = 1/\lambda^*_S$.*

Proof. This result follows directly from Lemmas 1 and 2. To show $e^*_A \leq 1/\lambda^*_S$, take any static noticeable difference $\lambda > 1$ that rationalises $\mathcal{O}$. By Lemma 1 and our previous observation, there is a constant efficiency parameter e rationalising $\mathcal{O}$ that satisfies

$$e \ \leq \ 1/\lambda \ \leq \ 1/\lambda^*_S.$$

Taking the supremum over the left hand side proves our claim.

To show $e^*_A \geq 1/\lambda^*_S$, take any constant efficiency parameter e that rationalises $\mathcal{O}$. By Lemma 2, there is a static noticeable difference λ rationalising the data such that

$$e^*_A \ \geq \ e \ > \ 1/\lambda.$$

Taking the infimum over the right hand side concludes our proof. $\square$

The notion of Afriat's efficiency index is equivalent to the inverse of the static just-noticeable difference. Therefore, the former can be interpreted as a measure of the consumer's cognitive inability to differentiate among options. Similarly to the Main Theorem, this proposes an alternative model of consumer choice that is parametrised by Afriat's efficiency index and rationalises the observable choices.

4.3 Noticeable differences and revealed preference

An important feature of revealed preference analysis is that it allows to infer consumer preference form observable choices. In this subsection we investigate the properties of revealed relations induced by the two models discussed in Sections 3.2 and 3.3.

First, we focus on implications of the model with efficiency parameters that was introduced in Section 3.2. Take any set of observations $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$ and some efficiency parameters $e = (e_t)_{t\in T}$, where $e_t \in [0,1]$, for all $t \in T$. Conditional on the vector e, we define the *directly revealed preference* relation $\succeq_e^{**}$ as follows:

$$x \succeq_e^{**} y \quad \text{whenever} \quad x = x_t \text{ and } y \in e_t B_t, \text{ for some } t \in T.$$

The relation is *strict* and denoted by $x \succ_e^{**} y$ if $x = x_t$ and $y \in e_t B_t \setminus \partial(e_t B_t)$, for some $t \in T$, i.e., whenever bundle y is in the relative interior of $e_t B_t$. The *revealed preference* relation $\succeq_e^{*}$ is the transitive closure of $\succeq_e^{**}$. That is, we have $x \succeq_e^{*} y$ whenever there is a sequence $\{z_k\}_{k=1}^K$ such that $z_1 = x$, $z_K = y$, and $z_k \succeq_e^{**} z_{k+1}$, for all $k = 1, \ldots, K-1$. The relation is *strict*, and denoted by $x \succ_e^{*} y$, if $z_k \succ_e^{**} z_{k+1}$, for some k.

Proposition 5. *Set $\mathcal{O}$ is rationalisable for efficiency parameters $e = (e_t)_{t\in T}$ if and only if the strict revealed preference relation $\succ_e^{*}$ is irreflexive.*

It is straightforward to verify that the relation $\succeq_e^{*}$ is consistent with the ordering induced by *any* locally non-satiated utility function u that rationalises the set of observations for $e = (e_t)_{t\in T}$. This is to say that $x \succeq_e^{*} y$ implies $u(x) \geq u(y)$, while $x \succ_e^{*} y$ only if $u(x) > u(y)$. Therefore, the relation recovers preferences that the agent is using when making a choice. Clearly, set $\mathcal{O}$ is rationalisable in this sense only if $\succ_e^{*}$ is irreflexive.

To show the converse, notice that we have $x \succ_e^{*} x$, for some $x \in \mathbb{R}_+^\ell$, only if Axiom 1 is violated. Hence, by Proposition 2, it suffices for $\succ_e^{*}$ to be irreflexive for the set of observations to be rationalisable with efficiency indices $e = (e_t)_{t\in T}$.

Next, we turn to the revealed preference relation induced by the model of consumer choice with noticeable differences. Conditionally on noticeable differences $\lambda = (\lambda_t)_{t\in T}$, we define the *directly revealed strict preference* relation $\succ^{**}_\lambda$, as follows:

$$x \succ^{**}_\lambda y \quad \text{whenever} \quad x = x_t \text{ and } (\lambda_t y) \in B_t, \text{ for some } t \in T.$$

As previously, the *revealed strict preference* relation $\succ^{*}_\lambda$ is defined as the transitive closure of $\succ^{**}_\lambda$. That is, for any x and y in $\mathbb{R}^\ell_+$, we have $x \succ^{*}_\lambda y$ if there is a sequence $\{z_k\}_{k=1}^K$ such that $z_1 = x$, $z_K = y$, and $z_k \succ^{**}_\lambda z_{k+1}$, for all $k = 1, \ldots, K-1$. Notice that, we do not define the weak counterpart of neither $\succ^{**}_\lambda$ nor $\succ^{*}_\lambda$.

Proposition 6. *Set $\mathcal{O}$ is rationalisable with noticeable differences $\lambda = (\lambda_t)_{t\in T}$ if and only if the strict revealed preference relation $\succ^{*}_\lambda$ is irreflexive.*

Suppose that set $\mathcal{O}$ is rationalisable with noticeable differences $\lambda = (\lambda_t)_{t\in T}$. In particular, there is a profile $(P_t)_{t\in T}$ of consistent interval orders such that P_t admits the noticeable difference λ_t, for all $t \in T$. It can be shown that the revealed preference relation $\succ^{*}_\lambda$ is consistent with the strict part of the weak order $\succeq$ induced by the profile. That is, if $x \succ^{*}_\lambda y$ then $x \succ y$. Given our discussion in Section 3.3, this allows us to recover the "true" preferences of the consumer, i.e., as if perfect discrimination were possible. Once the interval orders admit a representation as in (1), i.e, there is a utility u and a threshold function δ_t such that xP_ty if and only if $u(x) > u(y) + \delta_t(y)$, for each $t \in T$, then $\succ^{*}_\lambda$ is consistent with the ordering induced by u, i.e., if $x \succ^{*}_\lambda y$ then $u(x) > u(y)$.

Below we summarise the relationship between the above notions of revealed preference.

Proposition 7. *For any efficiency coefficients $e = (e_t)_{t\in T}$ and noticeable differences $\lambda = (\lambda_t)_{t\in T}$ such that $e_t = 1/\lambda_t$, for all $t \in T$, we have $\succ^{*}_e \subseteq \succ^{*}_\lambda = \succeq^{*}_e$.*

This observation follows directly from the definition of the revealed relations, hence, we skip the proof. The proposition states that the revealed preference induced by the two models are essentially equivalent. The only distinction pertains to revealed indifferences. In particular, whenever a set of observations is rationalisable with efficiency parameters $e = (e_t)_{t\in T}$ but fails to satisfy Axiom 2 for $\lambda_t = 1/e_t$, for all $t \in T$, it must be that $x \succeq^{*}_e x$, for some bundle $x \in \mathbb{R}^\ell_+$, i.e., a bundle is revealed indifferent to itself. For this reason, efficiency parameters and noticeable differences coincide only in the limit.

5 Reinterpreting empirical findings

The critical cost-efficiency index is the most widely used measure of revealed preference violations, e.g., see the empirical studies in Sippel (1997), Harbaugh et al. (2001), Andreoni and Miller (2002), Choi et al. (2007), Fisman et al. (2007), Ahn et al. (2014), Choi et al. (2014), and Halevy et al. (2017). Our characterisation of efficiency indices allows us to reinterpret these empirical findings and relate violations of rationality to the subjects' cognitive inability to differentiate among alternatives.

5.1 Empirical just-noticeable differences

We revisit Choi et al. (2014). The purpose of this large-scale study was to test for consistency with utility maximisation across various socio-economic groups. The online experiment was performed using the CentERpanel on 1,182 Dutch adult individuals. Every subject was presented with 25 decision problems under risk, each being a choice from a two-dimensional budget set. The individual divided her budget between two Arrow-Debreu securities, each paying one token if the corresponding state was realized and zero otherwise. The states were determined randomly with equal probability. Income was normalized to 1 and the state prices were chosen stochastically and varied across subjects. Thus, for each individual and observation $t = 1, \ldots, 25$, the budget set was given by $B_t = \left\{y \in \mathbb{R}^2_+ : p \cdot y \leq 1\right\}$, where prices $p \in \mathbb{R}^2_{++}$ were determined randomly.

Out of the 1,182 subjects, choices of 951 (80%) could *not* be rationalised with a locally non-satiated utility maximisation. The quality of decision-making of each individual was evaluated using Afriat's efficiency index as in Section 4.2.[12] In the experiment the mean measure was 0.881, which implied that on average budget sets needed to be reduced by 12% to rationalise a subject's choices. In addition, Choi et al. (2014) determined how the values of the index depended on selected socio-economic categories. On average, high-income and high-education subjects displayed greater levels of consistency with utility maximisation than lower-income and lower-education individuals. In addition, men were more consistent than women and those who were young tended to be more consistent than older subjects. See the original paper for details.

Finally, Choi et al. (2014) found a statistically significant correlation between consis-

[12]Choi et al. (2014) refer to this measure as the *critical cost-efficiency index* (or CCEI). However, we reserve this name to the general specification of the index as in Section 3.2.

	Mean	St. dev.	Percentile				
			10th	25th	50th	75th	90th
Aggregate just-noticeable difference (λ_F^*)	1.021	0.036	1.000	1.001	1.005	1.026	1.059
Static just-noticeable difference (λ_S^*)	1.180	0.309	1.000	1.003	1.075	1.238	1.480

Table 1: Descriptive statistics of the empirical distributions of the just-noticeable difference aggregated with the arithmetic mean (top) and the static just-noticeable difference (bottom).

tency in the experiment and household wealth. An increase of one standard deviation in the consistency score of the person who was primarily responsible for the household financial matters lead to 15–19% more household wealth. Our characterisation ties this relationship to the consumers' cognitive ability to differentiate among alternatives.

In Table 1 we provide descriptive statistics of the empirical distribution of just-noticeable differences evaluated for the dataset from Choi et al. (2014). In the top row, we present the aggregated just-noticeable differences λ_F^*, specified as in Section 3.3, with the aggregator function F being the arithmetic mean

$$F\big((y_t)_{t\in T}\big) := \frac{1}{T}\sum_{t\in T} y_t.$$

By the Main theorem, this is equivalent to the infimum of the mean of inverses of efficiency parameters $(e_t)_{t\in T}$ for which a dataset is rationalisable in the sense specified in Section 3.2.[13] The median value of the measure was 1.005. This is to say that, in order to rationalise choices of the median subject with variable noticeable differences, we had to assume that on average a 0.5% increase in the size of a bundle was sufficient for the consumer to perceive the difference. The mean was 1.021.

In the bottom row we show statistics for the static just-noticeable difference. Following Proposition 4, this is equivalent to the inverse of the Afriat's efficiency index evaluated in Choi et al. (2014). Moreover, recall from Section 4.2 that it is equal to the aggregate just-noticeable difference whenever $F\big((y_t)_{t\in T}\big) := \max\big\{y_t : t \in T\big\}$. Therefore, this measure always dominates the previous one. The median value of the index was 1.075. Hence, in

[13] In general, evaluating CCEI, or equivalently the aggregate just-noticeable difference, is very demanding computationally. See Theorem 4.2 in Smeulders et al. (2014). We take the advantage of the two-dimensional commodity space. Following Rose (1958), in such a case, it suffices to verify conditions stated in Axioms 1 and 2 for two-element cycles $\mathcal{C}$ only.

order to rationalise the median dataset with the static just-noticeable difference, we had to assume that a 7.5% increase in the size of a bundle was sufficient for the subject to notice the difference. The mean was 1.18.

Next, we performed an analysis of the correlation between noticeable differences and socio-economic characteristics. Given that, by definition, values of our measure are bounded from below by 1, we estimated the censored tobit model. We present the results in Table 2. Since the evaluated aggregate and static just-noticeable differences were strongly correlated ($\rho = 0.901$), the qualitative results were similar. Conditional on the just-noticeable difference being greater than 1, female, older, lower-income, and retired subjects were on average exhibiting higher values of the just-noticeable difference than their counterparts. Higher education had a statistically significant and negative impact only on the aggregated just-noticeable difference. For obvious reasons, these results are very similar to the ones obtained in Choi et al. (2014).

5.2 Existing evidence on sensory discrimination

In this subsection we compare our estimates of just-noticeable differences to measures of sensory discrimination obtained in the psychophysics literature. It is arguable whether sensitivity to changes among alternatives in the above experiment is related to responsiveness to physical stimuli. Thus, we approach such comparisons with caution.

The existing literature on sensory discrimination provides a variety of estimates for the Weber's constant. Recall that, this value is defined as the ratio of intensities of a physical stimulus that allows a subject to correctly discriminate between the magnitudes. See Laming (1997) for a detailed discussion on this and alternative measures of sensory discrimination. Depending on the stimulus, the estimates of this ratio range from 1.017 for the perception of cold and temperature, to 1.250 when it concerns discriminating between bitter substances. See Tables 8.1–8.3 in Laming (1997) for a summary of estimates obtained in the literature. Clearly, our median and mean evaluations of the aggregate and static just-noticeable differences are included in this range.

The evidence regarding the impact of socio-economic characteristics on sensory discrimination is limited and focus mainly on age and gender. Cowart (1989), Stevens et al. (1995), and Doty et al. (2016) show a decline of smell and taste sensitivity with respect to age. In particular, Doty et al. (2016) argue that the age-related changes become present

	Aggregate just-noticeable difference (λ_F^*)	Static just-noticeable difference (λ_S^*)
Constant	1.015***	1.074***
	[0.007]	[0.060]
Female	0.006**	0.064***
	[0.003]	[0.024]
Age		
35–49	0.005	0.040
	[0.004]	[0.034]
50–64	0.016***	0.147***
	[0.004]	[0.033]
65+	0.015**	0.156***
	[0.006]	[0.051]
Education		
Medium	-0.004	-0.007
	[0.003]	[0.027]
High	-0.007**	-0.041
	[0.003]	[0.028]
Household monthly income		
€2,500–3,499	-0.010***	-0.077**
	[0.004]	[0.031]
€3,500–4,499	-0.007*	-0.062**
	[0.004]	[0.031]
€5,000+	-0.011**	-0.095***
	[0.004]	[0.035]
Occupation		
Paid work	-0.010**	-0.052
	[0.005]	[0.042]
House work	-0.014**	-0.101**
	[0.006]	[0.050]
Other	-0.015***	-0.076
	[0.006]	[0.048]
Household composition		
Partner	0.007**	0.073**
	[0.004]	[0.030]
Number of children	0.000	0.000
	[0.001]	[0.012]

Table 2: Correlations of just-noticeable differences and subjects' individual characteristics using tobit (censored) regression. Omitted categories include: male, age under 35, lower education (primary and secondary), household monthly income below €2,500, retired, and not having a partner. Standard errors are presented in the brackets.

in middle age, i.e., before the age of 50, but are most marked after that time. In addition, Stevens et al. (1995) claim that, since individuals age psychologically at different rates, sensory discrimination is generally more uniform across younger individuals than older subjects. In a wide meta-study, Lautenbacher et al. (2017) show a similar relationships between age and perception of low intensity pain.

The above observations are consistent with our results. According to the estimation in Table 2, on average, the value of both the aggregate and static just-noticeable difference was increasing with respect to age. However, the effect was statistically significant only for subjects aged 50 and above. Therefore, similarly to Doty et al. (2016), this suggests that age-related changes in perception were observable after the middle age. In order to verify the latter claim by Stevens et al. (1995), we compared variances of the just-noticeable differences across all age categories. We found that, in general, the variance was increasing with respect to seniority. However, for the two youngest groups of subjects, i.e., younger than 35 and 35–49, it was not significantly different. Thus, sensitivity to differences across alternatives was more uniform for younger subjects.

The influence of gender on sensory discrimination is much less conclusive. As discussed in Rollman et al. (2000), there is a strong consensus that women are more sensitive to experimentally induced pain. However, as shown in Doty et al. (2016), there are no overall sex differences for taste and smell discrimination. This is opposite to our findings in which women had significantly higher just-noticeable differences than men.

Unfortunately, we found no reliable studies regarding how sensory discrimination is related to education, income, wealth, or household composition.

A Appendix

In this appendix we prove the sufficiency part of Proposition 3. Suppose that a set of observations $\mathcal{O} = \big\{(B_t, x_t) : t \in T\big\}$ obeys Axiom 2 for some noticeable differences $(\lambda_t)_{t\in T}$. We claim that set $\mathcal{O}$ is rationalisable with a profile $(P_t)_{t\in T}$ of consistent interval orders, where relation P_t admits the noticeable difference λ_t, for all $t \in T$. In addition, the profile is monotone with respect to the noticeable difference.

We begin our argument with an auxiliary result. In the following lemma we construct a collection of functions that play a crucial role in the remainder of the proof.

Lemma A.1. *For each $t \in T$, there is a continuous function $h_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that*

(i) $h_t(\theta y) > h_t(y)$, *for all* $\theta > 1$ *and non-zero* $y \in \mathbb{R}^\ell_+$;

(ii) $h_t(\lambda y) \geq h_t(y) + 1$, *for all* $\lambda \geq \lambda_t$ *and non-zero* $y \in \mathbb{R}^\ell_+$;

(iii) $k_{ts} \geq h_t(x_s) > k_{ts} - 1$, *where* $k_{ts} := \inf\big\{k \in \mathbb{Z} : x_s \in \lambda_t^k B_t\big\}$, *for all* $s \in T$.

Proof. Define the gauge function $\gamma_t : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ by $\gamma_t(y) := \inf\big\{\theta > 0 : y \in \theta B_t\big\}$, for all $t \in T$. Following Lemma 1 in Forges and Minelli (2009), it is continuous, homogeneous of degree one, and satisfies $\gamma_t(y) \leq 1$ if and only if $y \in B_t$.

Take any continuous and strictly increasing function $f_t : [1, \lambda_t] \to [0, 1]$ that satisfies $f_t(1) = 0$ and $f_t(\lambda_t) = 1$. Let $g_t : \mathbb{R}_+ \to \mathbb{R}$ be an extension of f_t to $\mathbb{R}_+$ given by

$$g_t(y) := \sum_{k\in\mathbb{Z}} \Big[f\big(y/\lambda_t^k\big) + k - 1\Big]\chi_{A_k}(y),$$

where χ_{A_k} is the indicator function and $A_k := \big(\lambda_t^k, \lambda_t^{k+1}\big]$, for all $k \in \mathbb{Z}$. Clearly, the function is continuous and strictly increasing. We argue that if $\lambda' \geq \lambda_t$ and $y > 0$ then $g_t(\lambda' y) \geq g_t(y) + 1$. Since $y \in A_k$ implies $(\lambda_t y) \in A_{k+1}$, we have

$$g_t(\lambda' y) - g_t(y) \;\geq\; g_t(\lambda_t y) - g_t(y) \;=\; f_t\big(\lambda_t y/\lambda_t^{k+1}\big) - f_t\big(y/\lambda_t^k\big) + 1 \;=\; 1,$$

where the first inequality follows from monotonicity of g_t. In addition, by construction of the function g_t, we know that $y \in A_k$ implies $k \geq g_t(y) > k - 1$.

Define $h_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ by $h_t(y) := (g_t \circ \gamma_t)(y)$, which is a continuous function. To show that it satisfies property (i), take any $\theta > 1$ and a non-zero $y \in \mathbb{R}^\ell_+$. By homogeneity of the gauge function γ_t and strict monotonicity of g_t, we obtain

$$h_t(\theta y) \;=\; g_t\big(\gamma_t(\theta y)\big) \;=\; g_t\big(\theta\gamma_t(y)\big) \;>\; g_t\big(\gamma_t(y)\big) \;=\; h_t(y).$$

In order to prove (ii), take any $\lambda' \geq \lambda_t$ and a non-zero $y \in \mathbb{R}^\ell_+$. By homogeneity of function γ_t and our previous observation regarding function g_t, we conclude that

$$h_t(\lambda' y) \;=\; g_t\big(\gamma_t(\lambda' y)\big) \;=\; g_t\big(\lambda'\gamma_t(y)\big) \;\geq\; g_t\big(\gamma_t(y)\big) + 1 \;=\; h_t(y) + 1.$$

Finally, take any $t,\, s \in T$. By construction of k_{ts}, we have $\lambda_t^{k_{ts}} \geq \gamma_t(x_s) > \lambda_t^{k_{ts}-1}$, or equivalently $\gamma_t(x_s) \in A_{k_{ts}}$. This implies that $k_{ts} \geq g_t\big(\gamma_t(x_s)\big) > k_{ts} - 1$, proving (iii). □

The proof of Lemma A.1 does not require for the set of observations $\mathcal{O}$ to satisfy Axiom 2. In fact, given our framework, existence of such functions is independent of that restriction. In our next result, we show that the axiom is sufficient for existence of a solution to a particular system of linear inequalities.

Lemma A.2. *Suppose that a set of observations $\mathcal{O}$ obeys Axiom 2 for some $(\lambda_t)_{t\in T}$. For any functions $(h_t)_{t\in T}$, specified as in Lemma A.1, there are numbers $(\phi_t)_{t\in T}$ and strictly positive numbers $(\mu_t)_{t\in T}$ such that $\phi_s < \phi_t + \mu_t\big[h_t(x_s)+1\big]$, for all t, s in T.*

The system of inequalities specified in the above lemma is very similar to the well-known *Afriat's inequalities*. However, unlike in the standard case, we require that each of the inequalities is strict. Nevertheless, the result can be proven in a relatively standard fashion. Below, we modify the approach by Fostel et al. (2004).

Proof of Lemma A.2. Denote $q_{ts} := \big[h_t(x_s)+1\big]$, for all t, $s \in T$. First, we claim that whenever set $\mathcal{O}$ obeys Axiom 2 there is no cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T\times T$ such that $q_{ts}\le 0$, for all $(t,s)\in\mathcal{C}$. Indeed, by condition (iii) in Lemma A.1, if $(\lambda_t x_s)\in B_t$ then $0\ge k_{ts}+1\ge h_t(x_s)+1 = q_{ts}$, which suffices for the property to hold.

Next, we argue that there is some $t\in T$ satisfying $q_{ts}>0$, for all $s\in T$. Otherwise, it would be possible to find indices a, b in T such that $q_{ab}\le 0$. Similarly, there would be some $c\in T$ such that $q_{bc}\le 0$, and so on. Eventually, we would construct a cycle $\mathcal{C} := \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ with $q_{ts}\le 0$, for all $(t,s)\in\mathcal{C}$, violating Axiom 2.

We conduct the remainder of the proof by induction on the size of the set of observations. Whenever set $\mathcal{O} = \big\{(B_t,x_t)\big\}$ is a singleton, it must be that $q_{tt}>0$. Clearly, this guarantees that for any numbers ϕ_t and $\mu_t>0$, we have $\phi_t<\phi_t+\mu_t q_{tt}$.

To show the inductive step, suppose that the claim in Lemma A.2 holds for any set of size $T-1$. Take any $t\in T$ such that $q_{ts}>0$, for all $s\in T$. By our earlier claim, such an index exists. Denote $T' := T\setminus\{t\}$. Clearly, the premise of the proposition must hold for numbers q_{ts}, where $(t,s)\in T'\times T'$. Therefore, there exist numbers $(\phi_s)_{s\in T'}$ and strictly positive numbers $(\mu_s)_{s\in T'}$ such that $\phi_s<\phi_r+\mu_r q_{rs}$, for all s, $r\in T'$. Take any ϕ_t satisfying $\phi_t<\phi_s+\mu_s q_{st}$, for all $s\in T'$. Finally, choose $\mu_t>0$ such that $\phi_s<\phi_t+\mu_t q_{ts}$, for all $s\in T$. Since $q_{ts}>0$, for all $s\in T$, it is always possible. □

In the next step, we show that whenever there is a solution to the above system of inequalities there exist particular functions u and v_t, for $t\in T$. These are instrumental for constructing interval orders that rationalise the data.

Lemma A.3. *Suppose that the system of inequalities in Lemma A.2 admits a solution. There exist continuous functions $u:\mathbb{R}^\ell_+\to\mathbb{R}$ and $v_t:\mathbb{R}^\ell_+\to\mathbb{R}$, for all $t\in T$, such that*

(i) $u(y)\ge v_t(y)$, for all $y\in\mathbb{R}^\ell_+$;

(ii) $\lambda' \geq \lambda_t$ *implies* $v_t(\lambda' y) > u(y)$*, for all non-zero* $y \in \mathbb{R}^{\ell}_{+}$*;*

(iii) $y \in B_t$ *implies* $u(x_t) \geq v_t(y)$*;*

(iv) $v_t(y) \geq v_s(y)$*, for all* $y \in \mathbb{R}^{\ell}_{+}$ *and* t*,* $s \in T$ *such that* $\lambda_t \leq \lambda_s$*.*

Proof. Take any functions $(h_t)_{t\in T}$ specified as in Lemma A.1 and numbers $(\phi_t)_{t\in T}$, $(\mu_t)_{t\in T}$ that solve the inequalities in Lemma A.2. Define function $u : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ by

$$u(y) := \min\Big\{\phi_s + \mu_s\big[h_s(y) + 1\big] : s \in T\Big\},$$

which is continuous and satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero $y \in \mathbb{R}^{\ell}_{+}$. In addition, following Lemma A.2, it must be that $\phi_t < u(x_t)$, for all $t \in T$.

For any $t \in T$, define a continuous function $w_t : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ by

$$w_t(y) := \min\Big\{u(x_s) + \mu_s h_s(y) : s \in T \text{ such that } \lambda_s \leq \lambda_t\Big\}.$$

Observe that, for any t, $s \in T$ such that $\lambda_s \leq \lambda_t$, we have $w_s(y) \geq w_t(y)$, for all $y \in \mathbb{R}^{\ell}_{+}$. Moreover, by Lemma A.1(ii), for any $t \in T$, number $\lambda' \geq \lambda_t$, and a non-zero $y \in \mathbb{R}^{\ell}_{+}$,

$$\begin{aligned} u(y) &:= \min\Big\{\phi_s + \mu_s\big[h_s(y) + 1\big] : s \in T\Big\} \\ &< \min\Big\{u(x_s) + \mu_s h_s(\lambda' y) : s \in T \text{ such that } \lambda_s \leq \lambda_t\Big\} \\ &=: w_t(\lambda' y), \end{aligned}$$

where the inequality is implied by $\phi_s < u(x_s)$, for all $s \in T$, and $h_s(\lambda' y) \geq h_s(y) + 1$, for all $s \in T$ such that $\lambda' \geq \lambda_t \geq \lambda_s$. Finally, following Lemma A.1(iii), whenever $y \in B_t$ then $h_t(y) \leq 0$. In particular, this implies

$$w_t(y) \;\leq\; u(x_t) + \mu_t h_t(y) \;\leq\; u(x_t).$$

For each $t \in T$, let $v_t : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ be given by $v_t(y) := \min\big\{u(y), w_t(y)\big\}$, which is continuous and satisfies $u(y) \geq v_t(y)$, for all $y \in \mathbb{R}^{\ell}_{+}$. Thus, condition (i) holds. Next, take any $\lambda' \geq \lambda_t$ and a non-zero $y \in \mathbb{R}^{\ell}_{+}$. By our previous observations,

$$u(y) \;<\; \min\big\{u(\lambda' y), w_t(\lambda' y)\big\} \;=\; v_t(\lambda' y),$$

since $\lambda' > 1$. This proves property (ii). In addition, we have $v_t(y) \leq w_t(y) \leq u(x_t)$, for any $y \in B_t$, which implies (iii). Finally, notice that if $\lambda_t \leq \lambda_s$ then $v_t(y) \geq v_s(y)$, for any $y \in \mathbb{R}^{\ell}_{+}$ and t, $s \in T$. Therefore, condition (iv) holds as well. □

In order to complete the proof, take any functions u and v_t, specified as in Lemma A.3, and define a continuous function $\delta_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ by $\delta_t(y) := u(y) - v_t(y)$, for all $t \in T$. By Lemma A.3(i), it is positive. Moreover, Lemma A.3(ii) implies

$$u(y) + \delta_t(\lambda' y) \;<\; v_t(\lambda' y) + \delta_t(\lambda' y) \;=\; u(\lambda' y),$$

for any $\lambda' \geq \lambda_t$ and non-zero $y \in \mathbb{R}^\ell_+$, while Lemma A.3(iii) guarantees that

$$u(x_t) \;\geq\; v_t(y) \;=\; u(y) - \delta_t(y),$$

for all $y \in B_t$. Finally, by property (iv) in Lemma A.3, we obtain $\delta_t(y) \leq \delta_s(y)$, for all vectors $y \in \mathbb{R}^\ell_+$ and indices $t,\, s \in T$ that satisfy $\lambda_t \leq \lambda_s$.

For each $t \in T$, construct a binary relation P_t by

$$xP_t y \;\text{ if and only if }\; u(x) \geq u(y) + \delta_t(x),$$

with $yP_t 0$, for all non-zero $y \in \mathbb{R}^\ell_+$. Clearly, relation P_t is an interval order satisfying $(\lambda' y)P_t y$, for all $\lambda' \geq \lambda_t$ and non-zero $y \in \mathbb{R}^\ell_+$. Moreover, if $y \in B_t$ then *not* $yP_t x_t$.

Next, we show that the profile $(P_t)_{t\in T}$ is consistent with respect to the induced weak order. Take any $t,\, s \in T$. We need to show that $x \succ_t y$ if and only if $x \succ_s y$. By construction, we have $y \succ_t 0$, for all non-zero $y \in \mathbb{R}^\ell_+$ and $t \in T$. Otherwise, if $x \succ_t y$ then $u(x) > u(y)$. Given that $u(\lambda_s x) - \delta_s(\lambda_s x) > u(x)$ and $u(y) \geq u(y) - \delta_s(y)$, by continuity of u and δ_s there is some z such that $u(x) \geq u(z) - \delta_s(z) > u(y)$. By definition of P_s, this implies *not* $zP_s x$ and $zP_s y$, which is equivalent to $x \succ_s y$.

Finally, to show that the profile is monotone with respect to the noticeable difference, take any $t,\, s \in T$ such that $\lambda_t \leq \lambda_s$ and suppose that $xP_s y$. By definition of P_s,

$$u(x) \;>\; u(y) + \delta_s(x) \;\geq\; u(y) + \delta_t(x),$$

since $\delta_s(x) \geq \delta_t(x)$. Clearly, this implies $xP_t y$, which completes our proof.

References

Afriat, S. N. (1967). The construction of a utility function from expenditure data. *International Economic Review 8*(1), 67–77.

Afriat, S. N. (1972). Efficiency estimation of production functions. *International Economic Review 13*(3), 568–598.

Afriat, S. N. (1973). On a system of inequalities in demand analysis: An extension of the classical method. *International Economic Review 14*(2), 460–472.

Ahn, D., S. Choi, D. Gale, and S. Kariv (2014). Estimating ambiguity aversion in a portfolio choice experiment. *Quantitative Economics 5*(2), 195–223.

Alcantud, J. C. R., D. L. Matos, and C. P. Palmero (2010). Goodness-of-fit in optimizing a consumer model. *Mathematical and Computer Modelling 52*(7), 1088–1094.

Aleskerov, F., D. Bouyssou, and B. Monjardet (2007). *Utility maximization, choice and preference* (2nd ed.). Springer Berlin Heidelberg.

Algom, D. (2001). Psychophysics. In *Encyclopedia of Cognitive Science*, pp. 800–805. Nature Publishing Group (Macmillan).

Andreoni, J. and J. Miller (2002). Giving according to GARP: An experimental test of the consistency of preferences for altruism. *Econometrica 70*(2), 737–753.

Apesteguia, J. and M. A. Ballester (2015). A measure of rationality and welfare. *Journal of Political Economy 123*(6), 1278–1310.

Armstrong, W. E. (1950). A note on the theory of consumer's behaviour. *Oxford Economic Papers 2*, 119–122.

Beja, A. and I. Gilboa (1992). Numerical representations of imperfectly ordered preferences (a unified geometric exposition). *Journal of Mathematical Psychology 36*, 426–449.

Bridges, D. S. (1985). Representing interval orders by a single real-valued function. *Journal of Economic Theory 36*, 149–155.

Bridges, D. S. (1986). Numerical representation of interval orders on a topological space. *Journal of Economic Theory 38*, 160–166.

Chateauneuf, A. (1987). Continuous representation of a preference relation on a connected topological space. *Journal of Mathematical Economics 16*, 139–146.

Choi, S., R. Fisman, D. Gale, and S. Kariv (2007). Consistency and heterogeneity of individual behavior under uncertainty. *American Economic Review 97*(5), 1921–1938.

Choi, S., S. Kariv, W. Müller, and D. Silverman (2014). Who is (more) rational? *American Economic Review 104*(6), 1518–1550.

Cowart, B. J. (1989). Relationships between taste and smell across the adult life span. *Annals of the New York Academy of Sciences 561*(1), 39–55.

Cox, J. C. (1997). On testing the utility hypothesis. *The Economic Journal 107*(443), 1054–1078.

Dean, M. and D. Martin (2016). Measuring rationality with the minimum cost of revealed preference violations. *The Review of Economics and Statistics 98*(3), 524–534.

Diewert, W. E. (1973). Afriat and revealed preference theory. *Review of Economic Studies 40*(3), 419–425.

Doty, R. L., J. M. Heidt, M. R. MacGillivray, M. Dsouza, E. H. Tracey, N. Mirza, and D. Bigelow (2016). Influences of age, tongue region, and chorda tympani nerve sectioning on signal detection measures of lingual taste sensitivity. *Physiology & Behavior 155*, 202–207.

Dziewulski, P. (2017). Eliciting the just-noticeable difference. Economics Series Working Papers 798, University of Oxford, Department of Economics.

Echenique, F., S. Lee, and M. Shum (2011). The money pump as a measure of revealed preference violations. *Journal of Political Economy 119*(6), 1201–1223.

Fishburn, P. C. (1970). Intransitive indifference with unequal indifference intervals. *Journal of Mathematical Psychology 7*, 144–149.

Fisman, R., S. Kariv, and D. Markovits (2007). Individual preferences for giving. *American Economic Review 97*(5), 1858–1876.

Forges, F. and E. Minelli (2009). Afriat's theorem for general budget sets. *Journal of Economic Theory 144*(1), 135–145.

Fostel, A., H. Scarf, and M. Todd (2004). Two new proofs of Afriat's theorem. *Economic Theory 24*(1), 211–219.

Halevy, Y., D. Persitz, and L. Zrill (2017). Parametric recoverability of preferences. *Journal of Political Economy*. forthcoming.

Harbaugh, W. T., K. Krause, and T. R. Berry (2001). GARP for kids: On the development of rational choice behavior. *American Economic Review 91*(5), 1539–1545.

Laming, D. (1997). *The Measurement of Sensation.* Oxford Psychology Series. Oxford University Press.

Lautenbacher, S., J. H. Peters, M. Heesen, J. Scheel, and M. Kunz (2017). Age changes in pain perception: A systematic-review and meta-analysis of age effects on pain and tolerance thresholds. *Neuroscience & Biobehavioral Reviews 75*, 104–113.

Luce, R. D. (1956). Semiorders and a theory of utility discrimination. *Econometrica 24*(2), 178–191.

Rollman, G. B., S. Lautenbacher, and K. S. Jones (2000). Sex and gender differences in responses to experimentally induced pain in humans. In R. B. Fillingim (Ed.), *Progress in pain research and management*, pp. 165–190. IASP Press.

Rose, H. (1958). Consistency of preference: The two-commodity case. *The Review of Economic Studies 25*(2), 124–125.

Scott, D. and P. Suppes (1958). Foundation aspects of theories of measurement. *Journal of Symbolic Logic 23*(2), 113–128.

Sippel, R. (1997). An experiment of the pure theory of consumer behaviour. *The Economic Journal 107*(444), 1431–1444.

Smeulders, B., L. Cherchye, B. De Rock, and F. C. R. Spieksma (2014). Goodness-of-fit measures for revealed preference tests: Complexity results and algorithms. *ACM Transactions on Economics and Computation 2*(1).

Stevens, J. C., L. A. Cruz, J. M. Hoffman, and M. Q. Patterson (1995). Test sensitivity and aging: High incidence of decline revealed by repeated threshold measures. *Chemical Senses 20*(4), 451–459.

Tsur, Y. (1989). On testing for revealed preference conditions. *Economics Letters 31*(4), 359–362.

Varian, H. R. (1982). The nonparametric approach to demand analysis. *Econometrica 50*(4), 945–974.

Varian, H. R. (1990). Goodness-of-fit in optimizing models. *Journal of Econometrics 46*(1–2), 125–140.

Wiener, N. (1914). A contribution to the theory of relative position. *Proceedings of the Cambridge Philosophical Society 17*, 441–449.