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Observable Consequences of Mental Accounting

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OBSERVABLE CONSEQUENCES OF MENTAL ACCOUNTING

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Abstract. We investigate necessary and sufficient nonparametric conditions for mental accounting.

1. Introduction

Mental accounting seems to be a broad notion which might encompass almost any kind of mechanism used by individuals and households to organise, evaluate, and keep track of financial activities (see Thaler (1999)). In this paper we investigate one narrow aspect of it: the assignment of particular consumption expenditures to specific "accounts". This is the kind of consumer budgeting defined by Strotz (1957, p.271) as follows:

> "A decision is first made as to how income should be allocated among the budget branches (given all prices). Each budget allotment is then spent optimally on the commodities in its branch, with no further references to purchases in other branches."

Note that Strotz only reserves optimality for the allocation of expenditures between items within an account (budget allotment), the way in which income is allocated amongst accounts is not so described. This aspect of mental accounting is intrinsically a boundedly rational activity.

The question we consider is the following. Suppose we observe, for an individual agent, a sequence of consumption decisions and the prices at which they were transacted $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$, can we model the data generating process for these data as if the consumer is operating a system of mental accounts? We investigate three variants. The first is what we call "pure mental accounting" in which the consumer has non-separable preferences over items and operates a system whereby items are allocated to certain accounts - this is the model described by Strotz above. The second concerns the situation in which these accounts correspond to groupings of items over which the consumer has separable preferences. We call this "separable accounts". The third concerns the situation in which the consumer has non-separable preferences and certain sources of income are labeled and then used exclusively for the labeled purpose. We call this "labeling". We also characterize mental accounting which is fully rational: the multi-stage budgeting procedure as described by Gorman (1959). In each case we derive necessary and sufficient nonparametric conditions for each model.

2. Boundedly Rational Mental Accounting

2.1. **Pure mental accounting.** The individual has standard preferences over a $K-$vector of goods described by $u(\mathbf{q})$. These preferences may be separable but there is no presumption that this is the case. To begin we suppose that the unit

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of time over which the accounts are operated correspond to the definition of a period/observation in our data. This might not be the case if food budgets (for instance) are balanced weekly, but clothing budgets are balance monthly. Nonetheless given data observed at high frequency (for example) it is possible to aggregate up so that budgets are defined over common periods (budgets which are balanced weekly also balance monthly). The individual has an income within the budgeting period t of x_t. This is divided into several (M) accounts $\{x_t^m\}_{t=1,...,T}$ which must sum to the total available budget. We take it that the "last" budget is a residual (petty cash) account: $x_t^M = x_t - \sum_{m=1}^{M-1} x_t^m$. The balance of each account may vary both between accounts and over time. Each item in the commodity vector is allocated to a specific account; thus we denote the vector of items in the m'th account by $\mathbf{q}^m$. The number of items in each account may differ across accounts. These groups of items form a *partition* of the commodity vector: $\mathbf{q} = [\mathbf{q}^1, ..., \mathbf{q}^M]$. The allocation of items to accounts does not vary over time (e.g. food items are always assigned to the food account).

The optimizing model is

$$\max_{\mathbf{q}} u(\mathbf{q}) \text{ subject to } \mathbf{p}_t^{1\prime}\mathbf{q}^1 \leq x_t^1, ..., \mathbf{p}_t^{M\prime}\mathbf{q}^M \leq x_t - \sum_{m=1}^{m=M-1} x_t^m$$

We suppose that only the prices and the commodity vectors are observed. In particular, neither the allocation of items to accounts nor the account balances themselves ($\{x_t^m\}_{t=1,...,T}$) are observed.

Definition 1. A mental accounting model with M accounts rationalizes the data $\{\mathbf{p}_t, \mathbf{q}_t\}_{t=1,...,T}$ if there exists a concave, continuous, monotonic, function u and a set of accounts $\{x_t^1, ..., x_t^M\}_{t=1,...,T}$ such that $u(\mathbf{q}_t) \geq u(\mathbf{q})$ for all $\mathbf{q}$ such that $\mathbf{p}_t^{m\prime}\mathbf{q}^m \leq x_t^m$ and $\sum_{m=1}^{m=M} x_t^m = x_t$ for all t and m.

This a statement of the principle of revealed preference: that the utility function must assign a higher value to the selected bundle than any other which satisfies the accounts and is affordable. The conditions for this model are given in the following Proposition (the proof of this and subsequent results are in the Appendix).

Proposition 1. *(Pure mental accounting). The following statements are equivalent.*
1. There exists a concave, continuous, monotonic utility function and a set of M accounts which rationalize the data.
2. There exist a partition of items into M groups such that there exists set of real numbers $\{U_t, \lambda_t^1, ..., \lambda_t^M\}_{t=1,...,T}$ *such that*

$$\begin{aligned} U_s &\leq U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m) \;\; \forall s,t \\ \lambda_t^m &> 0 \;\; \forall m,t \end{aligned}$$

This result provides necessary and sufficient conditions for the model: if a suitable partition exists then the model can provide a rationalization for the data and is a potential data-generating process; if not then the data are incompatible with mental accounting as described by the optimizing model above.

The empirical problem of determining whether the accounting model is data-consistent is a straightforward finite problem which contains both combinatorial and linear programming elements. For a given partition the question of determining whether or not a suitable set of Afriat numbers exists is easily determined using phase one of the simplex algorithm. The combinatorial element is a question of enumerating the partitions of the items. In principle this is a very simple and easily-parallelised problem. In practice, however, there may be a serious computational burden if the number of items in the commodity vector is large. The number of partitions of a n-element set is given by the Bell number B_n. The sequence of Bell numbers begins $B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$, $B_6 = 203$ and so on. This sequence satisfies the recursion $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$. The number of partitions grows very rapidly with the number of items in the commodity vector. There are, for example, 51,724,158,235,372 partitions of just 20 items.

However a blind search of over 51bn sets in not necessary thanks to the "Pigeonhole Principle" which states that when n items are partitioned into k groups, there exists at least 1 group containing not less than $\lceil \frac{n}{k} \rceil$ items and at least one group containing not more than $\lfloor \frac{n}{k} \rfloor$ items.[1] The pigeonhole principle can be used to reduce the combinatorial problem to more manageable proportions even when the number of items is large.

A second more powerful, and ultimately more sensible method of cutting down the computational burden is to simply rule out certain grouping of goods. By ruling out the idea that, for example, fly-paper and watches occupy the same account we can greatly reduce the number of combinations which need to be checked.

We have the following corollaries which relate to the number of accounts.

Corollary 1. *Proposition 1 with $M = 1$ is equivalent to GARP.*

If it is the case that $M = K$ then the data must satisfy the conditions in the Proposition 1.

If all of the data are in a single account then the model is the standard utility maximization model and the conditions in Proposition 1 reduce to GARP. Thus the standard rational choice model is a special case of the mental accounting model. At the other end of the spectrum if there are as many accounts as goods then the model is unfalsifiable.

2.2. **Separable mental accounting.** An interesting variant of the mental accounting model assumes that the accounts are defined over groups of goods which are separable and hence that the marginal rates of substitution between items within an account are independent of the consumption of items in other accounts. Then the consumer solves the following problem.

$$\max_{\mathbf{q}} u(v^1(\mathbf{q}^1), ..., v^M(\mathbf{q}^M)) \text{ subject to } \mathbf{p}_t^{1\prime}\mathbf{q}^1 \leq x_t^1, ..., \mathbf{p}_t^{M\prime}\mathbf{q}^M \leq x_t - \sum_{m=1}^{m=M-1} x_t^m$$

At first sight this seems odd - why should it so happen that the accounting structure exactly mirrors the separability of preferences in this way? Yet a collective household model in which each member was individually responsible for the purchase of certain classes of items would exactly reflect this (this is akin to Cherchye, De Rock and Vermeulen's (2009) "situation-dependent dictatorship" but with the partition of responsibility being between goods rather than periods). This is also the case considered by Strotz (1957). In this case the only restriction imposed by maximizing behaviour is that the data within each account satisfies GARP.

Proposition 2. *(Separable accounting). The following statements are equivalent.*
1. There exists a concave, continuous, monotonic utility function and a set of M separable accounts which rationalize the data.
2. There exist a partition of items into M groups such that there exists a set of real numbers $\{U_t^1 ..., U_t^M, \lambda_t^1, ..., \lambda_t^M\}_{t=1,...,T}$ such that

$$\begin{aligned} U_s^m &\leq U_t^m + \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m) \ \ \forall m, s, t \\ \lambda_t^m &> 0 \ \ \forall m, t \end{aligned}$$

[1] $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer not less than x and the greatest integer not greater than x respectively: e.g. $\lceil \pi \rceil = 4$ and $\lfloor \pi \rfloor = 3$.

3. There exist a partition of items into M groups such that the data $\{\mathbf{p}_t^m, \mathbf{q}_t^m\}_{t=1,...,T}$ *satisfies GARP for each partition* $m = 1, ..., M$.

Proposition 2 implies Proposition 1. Thus we have

Corollary 2. *If the data satisfy the condition for separable accounting they necessarily satisfy the conditions for pure mental accounts.*

2.3. **Labeling.** Suppose now that one part of the consumer's budget is labeled. This might be something like child benefit. Call this part of the budget x^A. Suppose the consumer then feels it necessary to spend this only on goods denoted by the group A (for example children's clothes or food). The rest of the goods we will denote by B. Non-satiation implies that the consumer will always exhaust this labeled part of his budget and may also wish to exceed it. Hence

$$\mathbf{p}^{A\prime}\mathbf{q}^A \geq x^A$$

Their optimization problem is therefore:

$$\max u\left(\mathbf{q}^A, \mathbf{q}^B\right) \text{ subject to } \mathbf{p}_t^{A\prime}\mathbf{q}_t^A \geq x_t^A \text{ and } \mathbf{p}_t^{A\prime}\mathbf{q}_t^A + \mathbf{p}_t^{B\prime}\mathbf{q}_t^B = x_t - x_t^A$$

If a consumer operates such a model then when the labeled income is *infra-marginal* (that is, less than the consumer would in any case wish to spend on the A goods given the prices they face and a budget of $(x_t + x_t^A)$) then the $\mathbf{p}_t^{A\prime}\mathbf{q}_t^A \geq x_t^A$ constraint would not bind and, as far as observables are concerned, they would operate as a standard utility maximizer. But if the labeled income is *extra-marginal* (that is, more than they would otherwise spend) then because they would have to increase their expenditure on the A goods (due to being non-satiated) they could violate GARP. The formal result is as follows.

Proposition 3. *(Labeling). The following statements are equivalent.*
1. There exists a concave, continuous, monotonic utility function which rationalize the data $\left\{\mathbf{p}_t, \mathbf{q}_t, x_t^A\right\}_{t=1,...,T}$.
2. $\mathbf{p}_t^{A\prime}\mathbf{q}_t^A \geq x_t^A$ *and there a set of real numbers* $\{U_t, \lambda_t\}_{t=1,...,T}$ *such that*

$$\begin{aligned} U_s &\leq U_t + (\lambda_t - \mu_t)\mathbf{p}_t^{A\prime}(\mathbf{q}_s^A - \mathbf{q}_t^A) + \lambda_t\mathbf{p}_t^{B\prime}(\mathbf{q}_s^B - \mathbf{q}_t^B) \quad \forall\, s,t \\ \lambda_t &> 0 \quad \forall\, t \\ \mu_t &\geq 0 \text{ with equality when } \mathbf{p}_t^{A\prime}\mathbf{q}_t^A = x_t^A \quad \forall\, t \end{aligned}$$

3. Rational Mental Accounting: Two stage budgeting

Two-stage, or multi-stage budgeting is a rational form of mental accounts. This is very close in spirit to some boundedly rational mental accounting but multi-stage budgeting requires that the overall allocation of expenditures is consistent with maximization - i.e. not just that it is optimal, conditional on the initial allocation of income to accounts. Strotz (1957) argued that a sufficient condition for two-stage budgeting is that the household's utility function be separable. Gorman (1959) showed that, while necessary, separability is not sufficient. In addition, it is required that the sub-utility functions enter utility either additively or through an intermediate function which is homogeneous of degree one. These restrictions on the consumer's preferences imply empirically refutable restrictions on the system of demand functions.

Proposition 4. *(Weakly separable two-stage budgeting). The following statements are equivalent:*
1. There exists a weakly separable utility function with homothetic sub-utilities

which provide a two-stage budgeting rationalization for the data.
2. There exist real numbers $\{U_t^m, \lambda_t^m\}_{t=,\ldots,T}^{m=1,\ldots,M}$ *such that*

$$U_t^m \leq U_s^m + \lambda_t^m \mathbf{p}_t^{m\prime} (\mathbf{q}_s^m - \mathbf{q}_t^m) \quad \forall\, s, t, m$$

$$U_t^m = \lambda_t^m \mathbf{p}_t^{m\prime} \mathbf{q}_m^m \quad \forall\, t, m$$

$$\lambda_t^m > 0 \quad \forall\, t, m$$

and $\{U_t^m, {}^1/\lambda_t^m\}_{t=1,\ldots,T}^{m=1,\ldots,M}$ *satisfies GARP.*

The following result covers the additive case.

Proposition 5. *(Additive two-stage budgeting). The following statements are equivalent:*
1. There exists an additively separable utility function with sub-utilities of the Gorman Polar Form which provide a two-stage budgeting rationalization for the data
2. There exist real numbers $\{U_t^m, \lambda_t\}_{t=,\ldots,T}^{m=1,\ldots,M}$ *such that*

$$\begin{aligned} &U_t^m \leq U_s^m + \lambda_t \mathbf{p}_t^{m\prime} (\mathbf{q}_s^m - \mathbf{q}_t^m) \quad \forall\, s, t, m \\ &\lambda_t > 0 \quad \forall\, t \end{aligned}$$

Both of these results link directly to the previous proposition regarding boundedly rational separable mental accounting.

Corollary 3. *If the data satisfy the condition for two-stage budgeting they necessarily satisfy the conditions for separable accounts.*

4. Conclusions

We characterise three alternative forms of mental accounting and, for comparison, two rational mental accounting models. It appears that all of the models considered have nonparametrically refutable observable consequences.

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Appendix - Proofs

Proof of Proposition 1.

Necessity. Assuming differentiability (alternatively the following arguments can be established using super-gradients) the first order conditions are

$$\nabla u(\mathbf{q}_t^m) \leq \lambda_t^m \mathbf{p}_t^m \quad \forall m, t$$

Concavity implies

$$u(\mathbf{q}_s^1, ..., \mathbf{q}_s^m) \leq u(\mathbf{q}_t^1, ..., \mathbf{q}_t^m) + \sum_{m=1}^{m=M} \nabla u(\mathbf{q}_t^m)'(\mathbf{q}_s^m - \mathbf{q}_t^m)$$

$$u(\mathbf{q}_s^1, ..., \mathbf{q}_s^m) \leq u(\mathbf{q}_t^1, ..., \mathbf{q}_t^m) + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m)$$

Since u real-valued there exist real numbers $\{U_t, \lambda_t^1, ..., \lambda_t^M\}_{t=1,...,T}$ and a partition $\mathbf{q}_t = [\mathbf{q}_t^1, ..., \mathbf{q}_t^M]$ such that

$$\begin{aligned} U_s &\leq U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m) \;\; \forall s, t \\ \lambda_t^m &> 0 \;\; \forall m, t \end{aligned}$$

Sufficiency. Now suppose that there exist numbers and a partition which satisfy these inequalities. Let

$$U(\mathbf{q}) = \min_t \{U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m)\}$$

Note that this is piece-wise linear, monotonic, continuous and concave. Consider $\mathbf{q}_t$ and the partition $\mathbf{q}_t = [\mathbf{q}_t^1, ..., \mathbf{q}_t^M]$

$$U(\mathbf{q}_t) = \min_s \{U_s + \sum_{m=1}^{m=M} \lambda_s^m \mathbf{p}_s^{m\prime}(\mathbf{q}_t^m - \mathbf{q}_s^m)\}$$

For $t = s$ we have $U(\mathbf{q}_t) = U_t$ and we have (by construction)

$$U_t \leq U_s + \sum_{m=1}^{m=M} \lambda_s^m \mathbf{p}_s^{m\prime}(\mathbf{q}_t^m - \mathbf{q}_s^m) \;\; \forall s$$

Hence $U(\mathbf{q}_t) = U_t$. Now consider some arbitrary $\mathbf{q}$ and the partition $\mathbf{q} = [\mathbf{q}^1, ..., \mathbf{q}^M]$ such that $\mathbf{p}_t^{m\prime}\mathbf{q}^m \leq x_t^m$ for all m. We have

$$U(\mathbf{q}) = \min_t \{U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m)\}$$

We have that

$$\mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m) \leq 0 \;\; \forall m$$

and hence since $\lambda_t^m > 0$ we have

$$\lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m) \leq 0 \;\; \forall m$$

and

$$\sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m) \leq 0$$

Thus

$$U_t \geq U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m)$$

and hence

$$U_t \geq U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m) \geq \min_t\{U_t + \sum_{m=1}^{m=M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}^m - \mathbf{q}_t^m)\} = U(\mathbf{q}_t)$$

Thus $u(\mathbf{q}_t) \geq u(\mathbf{q})$.■

Proof of Corollary 1

The condition in Proposition 1 reduces to

$$U_t \geq U_t + \lambda_t^1 \mathbf{p}_t^{1\prime}(\mathbf{q}_s^1 - \mathbf{q}_t^1)$$

with $\mathbf{p}_t^1 = \mathbf{p}_t$and $\mathbf{q}_t^1 = \mathbf{q}_t$for all t. By Afriat's Theorem this is equivalent to GARP.
■

Proof of Corollary 2

With a single good, each in its own account, the condition in Proposition 1

$$U_s \leq U_t + \sum_{m=1}^{m=M} \lambda_t^m p_t^{m\prime}(q_s^m - q_t^m)$$

can be written as

$$U_s \leq U_t + \mu_t \pi_t^{\prime}(\mathbf{q}_s - \mathbf{q}_t)$$

where

$$\lambda_t^m p_t^m = \pi_t^m$$

This is is equivalent to the requirement that there exist some virtual prices $\{\pi_t\}_{t=1,\ldots,T}$ such that the data $\{\pi_t, \mathbf{q}_t\}_{t=1,\ldots,T}$ satisfies GARP. Varian (Theorem 1, 1988) shows that such prices always exist.■

Proof of Proposition 2

Weak separability is necessary and sufficient for the second (lower) stage of two-stage budgeting. This implies GARP within each separable group (Varian (1984, Theorem 3)) and hence the existence of sub-utility functions which rationalize the model

$$max_{\mathbf{q}} v^m(\mathbf{q}^m) \text{ subject to. } \mathbf{p}_t^{m\prime}\mathbf{q}^m \leq x_t^m$$

Summing the corresponding Afriat Inequalities

$$U_s^m \leq U_t^m + \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m)$$

over the groups gives

$$\sum_{m=1}^{M} U_s^m \leq \sum_{m=1}^{M} U_t^m + \sum_{m=1}^{M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m)$$

or, setting $U_t = \sum_{m=1}^{M} U_t^m$, we can write this as

$$U_s \leq U_t + \sum_{m=1}^{M} \lambda_t^m \mathbf{p}_t^{m\prime}(\mathbf{q}_s^m - \mathbf{q}_t^m)$$

The rest of the proof parallels Proposition 1.■

Proof of Corollary 3

Immediate from inspection of th conditions in Propositions 1 and 2.■

Proof of Proposition 3.

Necessity. Assuming differentiability (alternatively the following arguments can be established using super-gradients) the first order conditions are

$$\begin{bmatrix} \nabla u(\mathbf{q}_t^A) \\ \nabla u(\mathbf{q}_t^B) \end{bmatrix} \leq \begin{bmatrix} \lambda_t \mathbf{p}_t^A - \mu_t \mathbf{p}_t^A \\ \lambda_t \mathbf{p}_t^B \end{bmatrix} \forall t \text{ in which the } \mathbf{p}_t^{A\prime}\mathbf{q}_t^A \geq x_t^A \text{constraint binds}$$

$$\begin{bmatrix} \nabla u(\mathbf{q}_t^A) \\ \nabla u(\mathbf{q}_t^B) \end{bmatrix} \leq \lambda_t \begin{bmatrix} \mathbf{p}_t^A \\ \mathbf{p}_t^B \end{bmatrix} \text{ otherwise}$$

Concavity implies

$$u(\mathbf{q}_s) \leq u(\mathbf{q}_t) + \nabla u(\mathbf{q}_t^A)'(\mathbf{q}_s^A - \mathbf{q}_t^A) + \nabla u(\mathbf{q}_t^B)'(\mathbf{q}_s^B - \mathbf{q}_t^B)$$

Using the first order conditions we have

$$u(\mathbf{q}_s) \leq u(\mathbf{q}_t) + (\lambda_t - \mu_t)\mathbf{p}_t^{A\prime}(\mathbf{q}_s^A - \mathbf{q}_t^A) + \lambda_t \mathbf{p}_t^{B\prime}(\mathbf{q}_s^B - \mathbf{q}_t^B)$$

where $\lambda_t > 0$ due to non-satiation, and $\mu_t > 0$ if the$\mathbf{p}_t^{A\prime}\mathbf{q}_t^A \geq x_t^A$ constraint binds and $\mu_t = 0$ otherwise. This implies that there exist real numbers $\{U_t, \lambda_t > 0, \mu_t \geq 0\}_{t=1,\ldots,T}$ each period such that

$$\begin{aligned} U_s &\leq U_t + (\lambda_t - \mu_t)\mathbf{p}_t^{A\prime}(\mathbf{q}_s^A - \mathbf{q}_t^A) + \lambda_t \mathbf{p}_t^{B\prime}(\mathbf{q}_s^B - \mathbf{q}_t^B) \\ \lambda_t &> 0 \\ \mu_t &\geq 0 \text{ with equality when } \mathbf{p}_t^{A\prime}\mathbf{q}_t^A = x_t^A \end{aligned}$$

Sufficiency: Suppose we have data which satisfies the conditions . Using similar construction to Proposition 1 define the utility function

$$u(\mathbf{q}) = \min_t \left\{ U_t + (\lambda_t - \mu_t)\mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A) + \lambda_t \mathbf{p}_t^{B\prime}(\mathbf{q}^B - \mathbf{q}_t^B) \right\}$$

We need to show that this assigns a higher number to $\mathbf{q}_t$ than it does to any arbitrary bundle $\mathbf{q}$ which is affordable ($\mathbf{p}_t'\mathbf{q} \leq \mathbf{p}_t'\mathbf{q}_t$) and satisfies the accounting constraint $\mathbf{p}_t^{A\prime}\mathbf{q}^A \geq x_t^A$. Firstly note that the condition in the Proposition ensures that

$$\min_t \left\{ U_t + (\lambda_t - \mu_t)\mathbf{p}_t^{A\prime}(\mathbf{q}_s^A - \mathbf{q}_t^A) + \lambda_t \mathbf{p}_t^{B\prime}(\mathbf{q}_s^B - \mathbf{q}_t^B) \right\} = U_t$$

Now consider

$$u(\mathbf{q}) = \min_t \left\{ U_t + (\lambda_t - \mu_t)\mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A) + \lambda_t \mathbf{p}_t^{B\prime}(\mathbf{q}^B - \mathbf{q}_t^B) \right\}$$

Expanding this out gives

$$u(\mathbf{q}) = \min_t \left\{ U_t - \mu_t \mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A) + \lambda_t \mathbf{p}_t'(\mathbf{q} - \mathbf{q}_t) \right\}$$

Since $\lambda_t > 0$ and $\mathbf{p}_t'(\mathbf{q} - \mathbf{q}_t) \leq 0$ we know that the final term is non-positive. We need to sign $\mu_t \mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A)$. There are two cases to consider. Firstly suppose that$\mathbf{p}_t^{A\prime}\mathbf{q}_t^A = x_t^A$. Then the accounting constraint did not bind at this observation and $\mu_t = 0$ and so $\mu_t \mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A) = 0$. Consequently we know that $u(\mathbf{q}) \leq U_t$. Now suppose that $\mathbf{p_t^{A\prime}q^A} = x_t^A$ so the constraint binds and $\mu_t > 0$. Then since $\mathbf{p}_t^{A\prime}\mathbf{q}^A \geq x_t^A$ we have $\mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A) \geq 0$ and hence $\mu_t \mathbf{p}_t^{A\prime}(\mathbf{q}^A - \mathbf{q}_t^A) \geq 0$ with the result that $u(\mathbf{q}) \leq U_t$.■

Proof of Proposition 4

See Diewert and Parkan (1978) and Varian (1983, Theorem 5). ■

Proof of Proposition 5

Varian (1983, Theorem 6), Cherchye *et al* (2013, Theorem 2).■