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A Model of Technological Unemployment

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A MODEL OF TECHNOLOGICAL UNEMPLOYMENT

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Abstract

The traditional 'task-based' literature supports an optimistic view about the threat of automation in the labor market. In this paper, I use updated reasoning about how new machines operate. This supports a pessimistic view. I introduce a new distinction between two types of capital – 'traditional' and 'advanced'. The former is a q-complement to labor in performing tasks, but the latter displaces labor from those q-complemented tasks. In a dynamic model, this 'task encroachment' drives labor out the economy at an endogenously determined rate and wages decline to zero. In the limit, labor is fully immiserated and 'technological unemployment' follows. (JEL: J20, J21, J23)

The traditional 'task-based' literature that explores the consequences of technological change on the labor market supports an optimistic view about the threat of automation.[1] This optimism relies on the claim that there exists a large set of types of tasks that cannot be automated and, in turn,

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[1] I refer to 'optimism' and 'pessimism' throughout this paper. I do so as a form of short-hand. Put simply, 'pessimistic outcomes' are those that make people feel pessimistic about labor's future and vice-versa for 'optimistic outcomes'. More formally, in this paper 'pessimism' is captured by a fall in relative, and absolute, wages as the stock and productivity of 'advanced' capital increases.

that those "tasks that cannot be substituted by automation are generally complemented by it" (Autor 2015). This paper explores the consequences for earnings and employment if the set of types of tasks in which labor has the comparative advantage is eroded – beyond that imagined by the traditional literature. As defenders of this traditional view, Autor and Salomons (2017a, b) nevertheless explicitly recognise the importance of my argument. The model in this paper, they write, captures a new process of "task encroachment".

There are important reasons to take this argument seriously. Though forecasting the future capabilities of machines is very difficult, the traditional 'task-based' literature has often underestimated them.[2] For instance, Autor, Levy, and Murnane (2003) noted that the task of driving a car could not be readily automated, but a type of driverless car appeared two years later;[3] Autor and Dorn (2013) noted that order-taking and table-waiting could not be readily automated, but later that year the US restaurants Chili's and Applebee's announced they were installing 100,000 tablets to allow customers to order and pay without a human waiter;[4] Autor (2015) noted that the task of identifying a species of bird based on a fleeting glimpse could not be readily automated, but later that year an app was released to do that as well.[5]

This suggests that the traditional literature's conception of how machines operate and the capabilities that this implies – known as the 'ALM hypothesis' – may be incorrect.[6] These tasks were believed to be out of reach of automation because they were 'non-routine' rather than 'routine'. But in practice this has proven not to be the case. It is important to note that the traditional literature recognised that this constraint might not hold indefinitely. Autor, Levy, and Murnane (2003), for instance, refers to "*present* technologies" and only claimed that the ALM hypothesis is "at *present*" a binding constraint (emphasis added). In turn, Autor (2015) con-

[2]In this paper, I use the terms 'machine' and 'capital' interchangeably.

[3]The Society of Automative Engineers defines five levels of vehicle 'autonomy'. These early cars were at a low level. Since 2005, further progress has been made. I thank Frank Levy for this point.

[4]See Pudzer (2016).

[5]See http://merlin.allaboutbirds.org/photo-id/.

[6]'ALM', after 'Autor, Levy and Murnane' – the authors of Autor, Levy, and Murnane (2003).

siders whether this will continue to hold in the future. In Susskind (2017) I set out two distinct explanations for why this constraint no longer binds and propose a new hypothesis about the capabilities of machines, based on these explanations, that encompasses the ALM hypothesis as a special case. In this paper I explore the consequences for earnings and employment if the set of types of tasks in which human beings have the comparative advantage continues to be eroded in this way. If many 'non-routine' tasks can also be automated, then the set of types of tasks that offer a 'refuge' for labor will be far smaller than the traditional literature assumed.

The model in this paper uses a new distinction between two types of capital – 'traditional' capital and 'advanced' capital. Whilst traditional capital cannot perform the same type of tasks as labor, advanced capital can. Traditional capital is a q-complement to a distinct set of types of tasks. This means that an increase in the quantity or productivity of traditional capital raises the value of those complemented tasks.[7] This captures the traditional channel for optimism. But in the new model in this paper, these complemented tasks are either performed by labor – *or advanced capital*. An increase in the quantity or productivity of advanced capital erodes the comparative advantage of labor in performing those complemented tasks. Labour is forced to specialise in a shrinking set of types of complemented tasks. This captures the new channel for pessimism. In a static version of the model, an increase in the quantity or productivity of advanced capital drives down relative wages and the labor share of income and forces labor to specialise in a shrinking set of tasks. In a dynamic version, the endogenous accumulation of advanced capital drives labor out the economy at an endogenously determined rate, and absolute wages fall towards zero. In the limit, labor is fully immiserated and 'technological unemployment' follows. As Autor and Salomons (2017a, b) describe, labor has "no place left to hide" in this new model.

The central argument for optimism in the task-based literature is that

[7] The definition of 'q-complementarity' is more nuanced in a many-good setting. This is because in the set-up in which q-complementarity was originally defined – Hicks (1970), and Sato and Koizumi (1973) – the models had only a unique final good. When I use the term 'q-complement' I mean that an increase in the allocation of a factor's task input to the production of a *given* good, *ceteris paribus*, causes the marginal product of the other task input involved in the production of that good to increase.

people tend to "overstate the extent of machine substitution for labor" and "ignore the strong complementarities" (Autor 2015). This paper does not challenge the existence of strong complementarities between certain types of tasks performed by capital and other types of tasks. But it does challenge the idea that labor is uniquely placed to perform those other tasks – on the contrary, labor's comparative advantage over capital in performing those complemented tasks appears to be diminishing. These new models show that the ALM hypothesis, and the boundary it has imposed between what capital can and cannot do, may have created a misleading sense of optimism about the prospects for labor. This new process of 'task encroachment' is an important argument for pessimism that requires further theoretical and empirical research.

I. A Static Model

In the new model that follows there are two sets of types of tasks. The first is a set of tasks that are performed only by 'traditional' capital. This type of capital cannot perform the same type of tasks as labor. The second is a set of tasks that are either performed by labor or 'advanced' capital. These tasks that are performed by either labor or advanced capital are ordered in a line from left to right, going from 'simple' to 'complex', and the relative productivity of labor with respect to advanced capital increases as tasks become more 'complex'. This feature is similar to that in Acemoglu and Autor (2011).[8] To produce any good in the economy requires a combination of a task performed by traditional capital and a task performed by either labor or advanced capital.

This distinction between two different types of capital is new and important. In this model, traditional capital is a q-complement to a distinct set of types of tasks. But an increase in the quantity or productivity of advanced capital erodes the comparative advantage of labor in performing those complemented tasks. Though both these effects are present in Ace-

[8] And in turn it shares features with Dornbusch, Fischer, and Samuelson (1977), from which Acemoglu and Autor (2011) also draw. Dornbusch, Fischer, and Samuelson (1977) is a two-country trade model where a continuum of goods are traded and the result is a Ricardian pattern of specialisation.

moglu and Autor (2011), they are entangled – because of the structure of production, and the use of one type of capital, both these effects occur simultaneously. This new model disentangles them in a revealing way. Each type of capital has only one effect and, as a result, it is possible to explore what happens if the set of tasks in which labor is q-complemented by capital shrinks – holding constant the 'intensity' of that q-complementarity.

The purpose of this new model is to study the effect of technological progress in the use of the different types of capital. In equilibrium there is a single cut-off and all of the tasks to the left of the cut-off are performed by advanced capital with traditional capital and all of the tasks to the right are performed by labor with traditional capital. When capital is 'traditional', with a fixed and distinct role from labor in production, improvements in its capability have a neutral effect on labor. Relative wages are unaffected. This is optimism at work. But when capital is 'advanced', increasingly capable advanced capital erodes the set of types of tasks in which traditional capital q-complements labor. The relative return to labor falls and labor is forced to specialise in a shrinking set of types of tasks. This is the new pessimism at work.

I.A. Consumers

There is a spectrum of consumers $j \in [0,1]$ who are either high-skilled workers or capitalists. If consumer j is a high-skilled worker he sells his labor l_j^H for a wage $w^H \geq 0$. If he is a capitalist with advanced capital k_j^A or traditional capital k_j he rents them to earn $r^A \geq 0$ and $r \geq 0$ respectively. There is a spectrum of types of goods $x(i)$ where $i \in [0,1]$ and each consumer j has Cobb-Douglas preferences over those goods:

$$\ln u(x) = \int_0^1 \theta(i) \ln x_j(i)\, di \tag{1}$$

Note that, given the Cobb-Douglas utility function, this economy can be captured by a representative consumer who owns all the factors. For simplicity, I assume that all goods have the same expenditure density:

ASSUMPTION 1: $\theta(i) = 1 \; \forall i$.

I.B. Production and Firms

Goods are produced by combining two different types of tasks, $z_1(i)$ and $z_2(i)$, where again $i \in [0, 1]$. The first set of types of task, $z_1(i)$, are those that can performed by labor *and* advanced capital. The second set of types of tasks, $z_2(i)$, can only be performed by traditional capital. Perfectly competitive firms must hire factors to perform these tasks. The total stock of available factors is equal to the sum of l_j^H, k_j^A, k_j, owned by the consumers – L^H, K^A, and K respectively. The task-based production functions for the goods are:

$$x(i) = z_1(i)^{\psi} z_2(i)^{1-\psi} \tag{2}$$

where $\psi \in (0, 1)$. The factor-based production functions for the tasks are:

$$\begin{aligned} z_1(i) &= a^A(i)K^A(i) + a^H(i)L^H(i) \\ z_2(i) &= a^K(i)K(i) \end{aligned} \tag{3}$$

where $L^H(i)$, $K^A(i)$, and $K(i)$ are the allocations of high-skilled labor, advanced capital, and traditional capital to each type of task, and $a^H(i)$, $a^A(i)$ and $a^K(i)$ are their respective productivities. Again, these factor-based production functions for tasks reflect the fact that traditional capital performs its own distinct set of tasks, but advanced capital and labor perform the same tasks. The productivities of advanced capital and labor combine to form a 'relative productivity schedule' over the $z_1(i)$ task-spectrum:

$$A^H(i) = \frac{a^H(i)}{a^A(i)} \tag{4}$$

A second important assumption follows:

ASSUMPTION 2: *$A^H(i)$ is continuous, $A^H(0) > 0$, $A^{H\prime}(i) > 0$, and $A^{H\prime\prime}(i) = 0$.*

The assumption that $A^{H\prime}(i) > 0$ is a 'comparative advantage' assumption.[9] It reflects two principles. First, as i increases, the task that is performed by labor or advanced capital, $z_1(i)$, becomes more 'complex'. And secondly, that high-skilled labor has an increasing comparative advantage over advanced capital at performing more complex tasks. This is the sense in which labor is 'high' skilled. This reflects the fact that the most complex tasks draw on creative, problem-solving, and interpersonal faculties of human beings that, as yet, are hardest to automate. Following Susskind (2017), the most complex tasks are relatively hard to *routinise*.

The concept of 'routinisability' used here is fundamentally different to the traditional concept of 'routineness' that is used in the literature. Under the ALM hypothesis, the criteria that is used to determine whether a task can be automated is whether it is 'routine' or 'non-routine' – a task is 'routine' if a human being finds it easy to articulate his or her thinking process for it and 'non-routine' if not. This is because the ALM hypothesis uses traditional reasoning about how machines work – machines must follow explicit instructions or rules that reflect human reasoning, and so if people cannot explain the rules that they follow then it is difficult to write a set of rules for a machine to follow. However, new technologies – advances in processing-power, data retrieval and storage capabilities, and algorithm design – now make it feasible to perform tasks with machines that follow rules which do not reflect the rules that human beings follow (for example, the cancer diagnostic system developed by Esteva et al. 2017). As a result, the inability of human-beings to articulate their thought processes is no longer necessarily a binding constraint on automation. It follows that the appropriate criteria is not the 'routineness' of a task from the standpoint of a human being but whether it has features that make it more or less *routinisable* from the standpoint of a machine. If a task is routinisable, a routine can be composed that allows a machine to perform it – but that routine may not necessarily reflect the way in which a human being performs the task. This argument is set out in detail in Susskind (2017).

[9] Acemoglu and Autor (2011) also use this approach.

I.C. Equilibrium

The Supply-Side

The firms must decide which factors to hire to perform the tasks that will produce each type of good. It is clear that to perform the tasks $z_2(i)$ the firms will need to rent traditional capital, $K(i)$ – it is the only factor capable of performing those types of task. Less obviously, the firms will hire either labor *or* advanced capital to carry out the tasks $z_1(i)$ – but never both factors together. This is Lemma 1:

LEMMA 1: *In equilibrium, there exists some cut-off $\tilde{i}$ such that advanced capital works with traditional capital to produce goods of type $i \in [0, \tilde{i}]$, and labor works with traditional capital to produce the goods of type $i \in [\tilde{i}, 1]$.*

The proof is intuitive.[10] For any given w^H and r^A in equilibrium, a perfectly competitive firm will hire labor rather than advanced capital to perform $z_1(i)$ if:

$$\frac{w^H}{a^H(i)} \leq \frac{r^A}{a^A(i)}$$
$$A^H(i) \geq \frac{w^H}{r^A} \tag{5}$$

which takes place to the right of $\tilde{i}$, given the properties of $A^H(i)$ in Assumption 1. The opposite argument applies for the choice of advanced capital over labor. At $\tilde{i}$, the cost of performing a unit of $z_1(\tilde{i})$ to produce that good $\tilde{i}$ is the same for labor and advanced capital. This is shown on the left-hand side of Figure I.

– FIGURE I HERE –

For any $\frac{w^H}{r^A} \in [a, b]$ there is a single 'cut-off' type of good $\tilde{i}$. Given that factor-price ratio, firms would want to hire advanced capital (to work with

[10]Lemma 1 and the proof are similar to Lemma 1 in Acemoglu and Autor (2011).

traditional capital) to produce all goods to the left of $\tilde{i}$ and hire labor (to work with traditional capital) to produce all goods to the right of $\tilde{i}$. This is intuitive. $A^H(i)$ is a factor demand schedule, its shape determined by Assumption 1, and it describes the factor-price ratio for each type of good i that would make a firm indifferent between using either labor or advanced capital for a given $z_1(i)$. Moving left to right along the task-spectrum, if a firm is to remain indifferent, the relative price of labor must rise. This is because the relative advantage of labor over advanced capital at performing $z_1(i)$ increases.

Given this reasoning and Lemma 1, (2) and (3) combine to form the following factor-based production functions for goods:

$$
\begin{aligned}
x(i) &= \left[a^A(i)K^A(i)\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \quad \forall i \in [0,\tilde{i}] \\
x(i) &= \left[a^H(i)L^H(i))\right]^{\psi}\left[a^K(i)K(i)\right]^{1-\psi} \quad \forall i \in [\tilde{i},1]
\end{aligned} \tag{6}
$$

The Demand-Side

Call $\gamma(\tilde{i})$ the share of total consumer expenditure on all goods that are produced by advanced capital and traditional capital i.e. type $i \in [0,\tilde{i}]$. Assumption 1 implies that:

$$
\begin{aligned}
\gamma(\tilde{i}) &= \int_0^{\tilde{i}} \theta(i)\, di \\
&= \tilde{i}
\end{aligned} \tag{7}
$$

In turn, it follows from (2) that the share of total consumer expenditure that is spent on all the *tasks* performed by advanced capital is equal to $\psi \cdot \tilde{i}$.[11] The same argument applies to the share of total consumer expenditure on tasks performed by labor and traditional capital – these are equal to $\psi \cdot (1-\tilde{i})$ and $(1-\psi)$ respectively.

[11] This is because the task-based production function for goods is Cobb-Douglas. To see this formally, call the implicit 'price' of $z_1(i)$, $p_{z_1}(i)$. Perfectly competitive firms will set this 'price' equal to the marginal revenue product of $z_1(i)$ in producing $x(i)$, which is $p(i)\cdot\psi z_1(i)^{\psi-1}z_2(i)^{1-\psi}$. As a result the $z_1(i)$ share of the expenditure on a particular $x(i)$ is $\frac{p(i)\cdot\psi\cdot z_1(i)^{\psi-1}z_2(i)^{1-\psi}\cdot z_1(i)}{p(i)\cdot x(i)} = \psi$.

Equilibrium Factor Prices and Specialisation

As firms are perfectly competitive, the zero-profit condition requires that the total consumer expenditure on the types of tasks that are performed by each factor is equal to the income that the factor receives in return for carrying out those tasks. As a result, given (7) and the accompanying discussion, the following three conditions must hold:

$$\begin{aligned} r^A K^A &= \psi \cdot \tilde{i} \cdot Y \\ w^H L^H &= \psi \cdot (1-\tilde{i}) \cdot Y \\ rK &= (1-\psi) \cdot Y \end{aligned} \tag{8}$$

where $Y = r^A K^A + w^H L^H + rK$. The first two expressions in (8) implies:

$$\begin{aligned} \frac{w^H}{r^A} &= \frac{1-\tilde{i}}{\tilde{i}} \cdot \frac{K^A}{L^H} \\ &= C(\tilde{i}) \end{aligned} \tag{9}$$

$C(i)$ generates a market equilibrium schedule. This is shown in Figure II.

– FIGURE II HERE –

From inspection, it is clear that for any $\frac{w^H}{r^A}$ there is a unique cut-off good of type $\tilde{i}$ that ensures market equilibrium holds. (9) and Assumption 1 imply: $C'(i) < 0$; $C''(i) > 0$; $\lim_{i \to 0} C(i) = \infty$, and $C(1) = 0$.

The $A^H(i)$ schedule describes the pattern of specialisation $\tilde{i}$ for each factor-price ratio $\frac{w^H}{r^A}$. This is a factor-demand schedule. $C(i)$ describes the wage ratio $\frac{w^H}{r^A}$ that ensures market equilibrium for each pattern of specialisation $\tilde{i}$. This is a zero profit schedule. To derive the equilibrium cut-off $\bar{i}$, rather than a hypothetical cut-off $\tilde{i}$, these two schedules are combined. This is shown in Figure III and Proposition 1 follows.

PROPOSITION 1: *Given the properties of $A^H(i)$ and $C(i)$, there is a* *unique equilibrium wage ratio, f, and a unique equilibrium cut-off type of good, $\bar{i}$.*

The uniqueness of the equilibrium can be seen from an inspection of Figure III.

– FIGURE III HERE –

Since $A^H(0) > 0$ and $C(1) = 1$, the $A^H(i)$ schedule must start above the $C(i)$ schedule. As $A'(i) > 0$, $C'(i) < 0$, and $\lim_{i \to 0} C(i) = \infty$ the schedules must cross only once. In equilibrium there is a clear pattern of specialisation. It is 'Ricardian'.[12] Advanced capital specialises in producing the types of goods $i \in [0, \bar{i}]$ with traditional capital, and labor specialises in producing the goods $i \in [\bar{i}, 1]$ – i.e. $\tilde{i}$ is replaced by $\bar{i}$ in (6).

The full set of relative factor prices follow from (8) and (9) – again, recognising that in equilibrium the actual cut-off $\bar{i}$ replaces the hypothetical cut-off $\tilde{i}$:

$$\begin{aligned} \frac{w^H}{r^A} &= \frac{1-\bar{i}}{\bar{i}} \cdot \frac{K^A}{L^H} \\ \frac{w^H}{r} &= \frac{\psi \cdot (1-\bar{i})}{1-\psi} \cdot \frac{K}{L^H} \\ \frac{r^A}{r} &= \frac{\psi \cdot \bar{i}}{1-\psi} \cdot \frac{K}{K^A} \end{aligned} \tag{10}$$

I.D. Comparative Statics

This model can be used to compare the effect of technological progress in the use of the two different types of capital. First consider traditional capital, and a rise in $a^K(i)$ across the $z_2(i)$ task-spectrum. This has no effect on the wage relative to advanced capital, $\frac{w^H}{r^A}$, nor on the pattern of labor specialisation, $\bar{i}$. This follows the definition of the $A^H(i)$ and $C(i)$

[12]Equilibrium here is similar to Dornbusch, Fischer, and Samuelson (1977), though theirs is an equilibrium in international trade. Acemoglu and Autor (2011) note that their model is 'isomorphic' to Dornbusch, Fischer, and Samuelson (1977).

schedules and (10). Figure III remains unchanged. Nor does it affect the relative wage with respect to traditional capital, $\frac{w^H}{r}$. However, advanced capital does have an effect on both relative wages. A uniform rise in the relative productivity of advanced capital is shown in Figure IV.

– FIGURE IV HERE –

The result is a fall in $\frac{w^H}{r^A}$ and $\frac{w^H}{r}$ and a rise in $\bar{i}$ – labor is forced to specialise in a shrinking set of types of tasks, and is relatively worse off.[13]

I.E. Review of the Static Model

The two types of capital have very different consequences for labor. As traditional capital becomes more productive, the relative wages $\frac{w^H}{r^A}$ and $\frac{w^H}{r}$ do not change. This is because traditional capital is a q-complement to labor and advanced capital in the production of all types of goods and, because the task-based production functions for goods are Cobb-Douglas, this implies a rise in the marginal productivity of traditional capital in producing a given good causes an equiproportionate rise in the marginal productivity of either the labor or advanced capital producing that good.[14] This is the traditional channel of optimism – there exists a large set of types of tasks out of reach of automation, in which labor is complemented by capital. However, as advanced capital becomes more productive, $\frac{w^H}{r^A}$ and $\frac{w^H}{r}$ fall and labor is instead relatively worse off. This is because advanced capital is a perfect substitute for labor in performing those tasks that are q-complemented by traditional capital. Labour's comparative advantage diminishes, and is forced to specialise in a shrinking set of q-complemented tasks. This is the new pessimism at work.

[13] Exploring absolute, rather than relative, prices is more complex are requires the simulations that follow in the dynamic setting.

[14] Note again in this many-good setting the traditional definition of 'q-complementarity' does not apply straightforwardly. This is because in those settings there is only a unique final good.

II. A Dynamic Model

In the static model, as advanced capital becomes more productive, consumers do not respond to any changes in the rate of return to capital. Now I place that new model in a dynamic setting and introduce an endogenous process of advanced capital accumulation. The outcome is remorselessly pessimistic – labor is displaced at an endogenously determined rate, is forced into specialising in a shrinking set of tasks, and absolute wages are driven to zero. No steady-state is possible until labor has been entirely driven out the economy by advanced capital i.e. the economy must approach a steady-state where $\bar{i}(t) = 1$. Labour is fully immiserated, and technological unemployment follows.

To solve this dynamic model, I nest the static analysis in the previous section in a Ramsey-type setting. In any given t, the factor prices and pattern of specialisation are determined instantaneously by that static analysis – the $A^H(\cdot)$ and $C(\cdot)$ schedules in (5) and (9) are now time dependent such that $A^H(i,t)$ depends upon $a^A(i,t)$ and $a^H(i,t)$, and $C(i,t)$ on $\bar{i}(t)$, $K^A(t)$, $K(t)$, and ψ. A further important feature of this model is the innovative use of a numeraire good. This numeraire good allows me to do two things. First it allows me to reduce the dimensionality of the many-good model and make it tractable in the dynamic setting. As I will show, with Cobb-Douglas preferences across the range of goods, the law of motion for the numeraire good is the same as the law of motion for aggregate consumption across all goods – deriving this particular law of motion allows me to focus on aggregate consumption alone, rather than track the full set of laws of motion for each good. Secondly, with Cobb-Douglas production across the range of goods, the numeraire good allows me again to derive analytically tractable expressions for the absolute factor prices, r and r^A.

II.A. Consumers

Consumers have the same preferences as before. Again, the economy can be captured by a representative consumer. Only advanced capital $K^A(t)$ is accumulated, and the consumer faces the dynamic maximisation problem:

$$\max_{\boldsymbol{x(t)}} \int_0^\infty e^{-\rho t} \int_0^1 \ln x(i,t)\, di\, dt$$

s.t.

$$\dot{K}^A(t) = r^A(t) \cdot K^A(t) + r(t) \cdot K + w^H(t) \cdot L^H - c(t) \tag{11}$$

$$K^A(0) = K_0^A$$

$$K^A(t) \geq 0$$

I assume there is exogenous growth in the productivity of advanced capital. Any growth process must satisfy the following:

ASSUMPTION 3: *For any exogenous growth process used, it must be the case that* $a^A(\tilde{\tilde{i}}, t) \leq a^A(\tilde{i}, t) \cdot \frac{a^H(\tilde{\tilde{i}},t)}{a^H(\tilde{i},t)}$ *for* $\tilde{\tilde{i}} > \tilde{i}\ \forall t$.

This is the dynamic version of Assumption 2. It ensures that, as technological progress takes place, advanced capital does not become so productive in more complex tasks as to overturn the general principle that labor has the comparative advantage in these more complex tasks i.e. that $A_1^H(i,t) \geq 0\ \forall t$. Initially, I assume that the particular growth process is:

$$\frac{\dot{a}^A(i,t)}{a^A(i,t)} = g \quad \forall i, t \tag{12}$$

In the Appendix I show that this satisfies Assumption 3.

II.B. Production and Firms

Production is the same as in the static setting – the task-based production function for goods, and the factor-based production functions for tasks, are as in (2) and (3). For simplicity, I assume in the dynamic setting that there is no depreciation. In closing this section, I explain why depreciation does not change the central results.

The traditional approach to finding the steady-state in a Ramsey model with technological progress is to re-define the variables in 'effective' terms, dividing each variable by the prevailing level of labor-augmenting technology. The result is that a steady-state is reached not in the actual variables, but instead in these 'effective' variables, the variable 'per efficiency unit of labor'. In exactly the same way, solving this model requires that the advanced capital augmenting technological progress I am considering is instead exactly reflected in a process of traditional capital-augmenting technological progress. Consider again a good $x(\tilde{i},t)$ that is produced by advanced capital and traditional capital. The transformation of the production function is as follows:

$$\begin{aligned} x(\tilde{i},t) &= \left[a^A(\tilde{i},t)\cdot K^A(\tilde{i},t)\right]^{\psi}\left[a^K(\tilde{i},t)\cdot K(\tilde{i},t)\right]^{1-\psi} \\ &= \left[K^A(\tilde{i},t)\right]^{\psi}\left[a^A(\tilde{i},t)^{\frac{\psi}{1-\psi}}\cdot a^K(\tilde{i},t)\cdot K(\tilde{i},t)\right]^{1-\psi} \end{aligned} \tag{13}$$

(13) implies that a traditional-capital augmenting process of technological change in $a^A(\tilde{i},t)^{\frac{\psi}{1-\psi}}$ is identical to the advanced-capital augmenting process of technological change in $a^A(\tilde{i},t)$ in Assumption 3. This transformation has an important role in solving the dynamic model.

II.C. Dynamic Equilibrium

From the maximisation problem in (11) a current-value Hamiltonian follows:

(14)

$$\begin{aligned} H = &\int_0^1 \ln x(i,t)\,di \\ &+\ \mu(t)\left[r(t)\cdot K + r^A(t)\cdot K^A(t) + w^H(t)\cdot L^H - \int_0^1 x(i,t)\cdot p(i,t)\,di\right] \end{aligned}$$

It is possible to solve this Hamiltonian – with one co-state variable $\mu(t)$ and a spectrum of control variables, $x(i,t)$ where $i\in[0,1]$ – in the traditional way. A set of first order conditions follow for each good $x(i,t)$:

$$H_{x(i)} = \frac{1}{x(i,t)} - \mu(t) \cdot p(i,t) \qquad (15)$$
$$= 0$$

And for $K^A(t)$, the state variable:

$$H_{K^A} = \mu(t) \cdot r^A(t) \qquad (16)$$
$$= \rho \cdot \mu(t) - \dot{\mu}(t)$$

Together, (15) and (16) imply that for good $x(i,t)$:

$$\frac{\dot{x}(i,t)}{x(i,t)} = -\frac{\dot{p}(i,t)}{p(i,t)} + r^A(t) - \rho \qquad (17)$$

This is derived in the Appendix. In a traditional Ramsey-growth model, there is only a unique final output and so there is no need to consider how the price of the good changes over time. But (17) shows that in this many-good setting the rate of growth of demand for $x(i,t)$ will depend upon how its price changes over time. To maintain tractability, I now use a numeraire price normalisation. In particular, I assume that:

ASSUMPTION 4: $\tilde{i} = 0$ *such that the numeraire good is* $x(0,t)$*. Factor productivities and factor stocks are finite such that* $\bar{i}(t) \neq 0 \; \forall t$*.*

Given Assumption 4 it follows from (17) that for $x(0,t)$:

$$\frac{\dot{x}(0,t)}{x(0,t)} = r^A(t) - \rho \qquad (18)$$

Given Assumption 1 it follows that the law of motion for the numeraire good $x(0,t)$ is the same as the law of motion for total consumption $c(t)$.[15] And so:

[15] Where $c(t) = \int_0^1 x(i,t) \cdot p(i,t)\, di$. To see this note that since $p(\tilde{i},t) = 1$:

$$\frac{\dot{c}(t)}{c(t)} = r^A(t) - \rho \tag{19}$$

It follows that if $x(\tilde{i}, t)$ reaches a steady-state then total consumption $c(t)$ will also be in steady-state. From now, I use the law of motion in (19) and focus on $c(t)$. The law of motion for $K^A(t)$ follows from (11):

$$\dot{K}^A(t) = Y(t) - c(t) \tag{20}$$

At this point, the dynamic system is expressed in terms of $c(t)$ and $K^A(t)$. In order to find the steady-state in this model it is necessary to transform these variables into 'effective' terms. For any variable $v(t)$, I use the following transformations:

$$\hat{v}(t) = \frac{v(t)}{a^A(0,t)^{\frac{\psi}{1-\psi}}} \qquad \hat{\hat{v}}(t) = \frac{v(t)}{\bar{i}(t) \cdot a^A(0,t)^{\frac{\psi}{1-\psi}}} \tag{21}$$

The intuition for the form of these effective variable is revealed once the dynamic equilibrium is derived. But from inspection it is clear that a '$\hat{...}$' term is 'effective' with respect to a term that captures the productivity of advanced capital at time t, whereas the '$\hat{\hat{...}}$' term is 'effective' with respect to a term that captures the productivity of advanced capital *and* the value of the cut-off at time t. In the analysis that follows I look for an equilibrium in $\hat{c}(t) - \hat{\hat{K}}^A(t)$ space. (19) and (21) imply a law of motion for $\hat{c}(t)$:

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = r^A(t) - \rho - \frac{\psi}{1-\psi} \cdot g \tag{22}$$

$$c(t) = \frac{x(\tilde{i},t) \cdot p(\tilde{i},t)}{\theta(\tilde{i})} = \frac{x(\tilde{i},t)}{\theta(\tilde{i})}$$

and so $\frac{\dot{c}(t)}{c(t)} = \frac{\dot{x}(\tilde{i},t)}{x(\tilde{i},t)}$ since $\theta(\tilde{i})$ is constant and equal to 1.

and (20) and (21) imply a law of motion for $\hat{\hat{K}}^A(t)$:

$$\text{(23)} \qquad \frac{\dot{\hat{\hat{K}}}^A(t)}{\hat{\hat{K}}^A(t)} = \frac{\hat{Y}(t) - \hat{c}(t)}{\hat{K}^A(t)} - \left[g^{\gamma}(t) + \frac{\psi}{1-\psi} \cdot g \right]$$

where $g^{\gamma}(t)$ is the growth rate in $\gamma(t)$ and, given (7), in $\bar{i}(t)$ as well. Both these laws of motion can be expressed in terms of $\hat{c}(t)$ and $\hat{\hat{K}}^A(t)$ alone. The former requires an expression for $r^A(t)$ in terms of $\hat{\hat{K}}^A(t)$. The latter requires an expression for $\hat{Y}(t)$ and $g^{\gamma}(t)$ in terms of $\hat{\hat{K}}^A(t)$. First, consider $r^A(t)$. The static model implies that $p(0,t)$ is equal to:

$$\text{(24)} \qquad p(0,t) = \left[\frac{r^A(t)}{\psi \cdot a^A(0,t)} \right]^{\psi} \left[\frac{r(t)}{(1-\psi) \cdot a^K(0,t)} \right]^{1-\psi}$$

This is shown in the Appendix. Substituting the expression for $r(t)$ in terms of $r^A(t)$ that follows from (10), and using the price normalisation that $p(0,t) = 1$, (24) implies:

$$\text{(25)} \qquad \begin{aligned} r^A(t) &= \left[a^A(0,t)\right]^{\psi} \left[a^K(0)\right]^{1-\psi} \cdot [\gamma(\bar{i}(t))]^{1-\psi} \cdot \left[\frac{1}{K^A(t)} \right]^{1-\psi} \cdot K^{1-\psi} \cdot \psi \\ &= \hat{\hat{K}}^A(t)^{\psi-1} \cdot D \end{aligned}$$

where D is a positive constant equal to $\left(a^K(0) \cdot K\right)^{1-\psi} \cdot \psi$.[16] To find the

[16] (25) implies that the *level* of $r^A(t)$ depends on the choice of the numeraire good, $\tilde{i}$ – if a different $\tilde{i}$ is chosen, the level of $a^A(\tilde{i},t)$ and $a^K(\tilde{i},t)$ will differ from those in (25) where $\tilde{i} = 0$. However, this does not affect the important features of absolute factor prices in equilibrium. In the case where there is a one-off change in the productivity of advanced capital, so long as that change is uniform – as in Dornbusch, Fischer, and Samuelson (1977) – (25) implies that, *ceteris paribus*, $r^A(t)$ will always move in the same direction regardless of the choice of numeraire. In the case where there is an increase in the growth rate of the productivity of advanced capital, so long as the growth rates remain the same across different tasks, *ceteris paribus*, (25) implies that $r^A(t)$ will always increase at the same rate regardless of the choice of numeraire. To explore non-uniform changes in productivity, or to make the *level* of $r^A(t)$ invariant to the normalisation, it is necessary to use a simplex price normalisation. But this leads to an implicit, rather than explicit and tractable, solution to the model. This need for

expression for $\hat{Y}(t)$ in terms of $\hat{\hat{K}}^A(t)$, note that the structure of production in (2) and (3) implies that total advanced capital income is equal to:

$$r^A(t) \cdot K^A(t) = \gamma(\bar{i}(t)) \cdot \psi \cdot Y(t) \tag{26}$$

and so substituting in $r^A(t)$ from (25) it follows that:

$$\begin{aligned} \hat{Y}(t) &= \hat{\hat{K}}^A(t)^{\psi-1} \cdot D \cdot \hat{K}^A(t) \cdot \frac{1}{\bar{i}(t)} \cdot \frac{1}{\psi} \\ &= \hat{\hat{K}}^A(t)^{\psi} \cdot \frac{D}{\psi} \end{aligned} \tag{27}$$

Finally, to find the expression for $g^\gamma(t)$ note that by definition:

$$g^\gamma(t) = \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \dot{\hat{\hat{K}}}^A(t) \cdot \frac{1}{\bar{i}(t)} \tag{28}$$

Using the expression for $r^A(t)$ in (25), $\hat{Y}(t)$ in (27), and $g^\gamma(t)$ in (28), the laws of motion in (22) and (23) can now be re-expressed in terms of $\hat{c}(t)$ and $\hat{\hat{K}}^A(t)$. The former follows straightforwardly from (22) and (25):

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \hat{\hat{K}}^A(t)^{\psi-1} \cdot D - \rho - \frac{\psi}{1-\psi} \cdot g \tag{29}$$

The latter is more complex to derive. The full derivation is shown in the Appendix. Using (27) and (28), (23) can be written as:

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uniformity is an interesting limitation of Dornbusch, Fischer, and Samuelson (1977) that was not explored. Complications involving price normalisations are discussed elsewhere in the literature – Dierker and Grodal (1999), for example, on models of imperfect competition.

$$\frac{\dot{\hat{\hat{K}}}^A(t)}{\hat{\hat{K}}^A(t)} = \frac{\hat{\hat{K}}^A(t)^{\psi-1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{K}}^A(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \hat{\hat{K}}^A(t)} \tag{30}$$

At this stage, the traditional approach with a Ramsey-type growth model is to consider these schedules in $\hat{c}(t) - \hat{\hat{K}}^A(t)$ space. (29) implies that the $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$ schedule is:

$$\hat{\hat{K}}^A(t) = \left[\left[\rho + \frac{\psi}{1-\psi} \cdot g\right] \cdot \frac{1}{D}\right]^{\frac{1}{\psi-1}} \tag{31}$$

This is identical to a traditional Ramsey model with a labor-augmenting growth process at rate $\frac{\psi}{1-\psi} \cdot g$ – the stable arm for $\hat{c}(t)$ is simply a vertical schedule at some $\hat{\hat{K}}^{A*}$ that ensures $r^A(t) = \rho + \frac{\psi}{1-\psi} \cdot g$ and $\hat{c}(t)$ is constant. (30) implies that the $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule is equal to:

$$\hat{c}(t) = \hat{\hat{K}}^A(t)^{\psi} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g \cdot \hat{\hat{K}}^A(t) \tag{32}$$

This is almost identical to a traditional Ramsey model, except in one important respect – the presence of $\bar{i}(t)$. The reason that $\bar{i}(t)$ appears in the $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule in this model is critical. This is because in any period t, advanced capital is only used to produce $\bar{i}(t)$ of the goods in the economy. The remaining $1 - \bar{i}(t)$ goods are produced by labor whose productivity is unaffected by technological progress. However as $\bar{i}(t)$ rises, and more goods are produced by advanced capital rather than labor, the production of more goods in the economy is affected by the technological progress. As $\bar{i}(t)$ rises it is as if the 'effective' rate of technological progress – this is $\bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g$ in (32) – rises. Indeed, the increase in $\bar{i}(t)$ in this new model with many goods has the same consequence as an increase in technological progress in a traditional Ramsey model with a unique final good.[17] Intuitively, in a

[17] If the rate of technological progress were to increase a traditional Ramsey model, the

traditional Ramsey model with a unique final good, the economy 'feels the full force' of the technological progress, whereas in this many-good model only $\bar{i}(t)$ of the economy does.

This role for $\bar{i}(t)$ first appears because of its role in determining $r^A(t)$ in (25). That expression implies that $r^A(t)$ is increasing, at a diminishing rate, in $\hat{\hat{K}}^A(t)$. Given the definition of $\hat{\hat{K}}^A(t)$ implied by (21), this means that $r^A(t)$ is decreasing in $K^A(t)$ and increasing in $a^A(0,t)$ – as in a traditional Ramsey model – but that it is *also* increasing in $\bar{i}(t)$. This is because while there are diminishing returns to the use of $K^A(t)$ in the production of any given good, increasing the range of goods $\bar{i}(t)$ that $K^A(t)$ is used to produce 'spreads out' the stock of capital across the economy, and offsets those diminishing returns. It is this new role for $\bar{i}(t)$ that means the transformations in (21) must be used to find the dynamic equilibrium.

Because of the $\bar{i}(t)$ term in the stable arm for $\hat{\hat{K}}^A(t)$ in (32), dynamic equilibrium is not analytically tractable. It requires a non-linear simulation of the three equation system:[18]

$$\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = \hat{\hat{K}}^A(t)^{\psi-1} \cdot D - \rho - \frac{\psi}{1-\psi} \cdot g$$

(33)
$$\frac{\dot{\hat{\hat{K}}}^A(t)}{\hat{\hat{K}}^A(t)} = \frac{\hat{\hat{K}}^A(t)^{\psi-1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{K}}^A(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \hat{\hat{K}}^A(t)}$$

$$\bar{i}(t) = f(K^A(t), A^H(\bar{i}(t), t), L^H)$$

The first two are the familiar differential equations for $\hat{c}(t)$ and $\hat{\hat{K}}^A(t)$. The third is a static equation that determines $\bar{i}(t)$ in any given t. I will demonstrate a simulation later in this paper. Nevertheless, it is still possible to identify analytically the unique steady-state that this model approaches and also to provide an intuitive conjecture for transition to this steady-

corresponding 'effective' capital schedule would fall, requiring a lower rate of effective consumption to ensure that the effective capital stock is constant.

[18] $\hat{c}(t)$ is a forward-looking state variable, $\hat{\hat{K}}^A(t)$ is a backward-looking state variable, and $\bar{i}(t)$ is a simultaneously endogenous non-state variable (i.e. it has no associated differential equation).

state, without performing the simulation.

Steady-state exists where the stable arms for $\hat{c}(t)$ and $\hat{\hat{K}}^A(t)$ intersect. The steady-state is unique and must take place when $\bar{i}(t) = 1$ i.e. when labor is entirely driven out of the economy by advanced capital. To see why, first note that the stable arm for $\hat{c}(t)$ in $\hat{\hat{K}}^A(t)$–$\hat{c}(t)$ space is a vertical schedule at the level of $\hat{\hat{K}}^{A*}$ in (31). Now consider the following proof by contradiction. Suppose that the stable arm for $\hat{\hat{K}}^A(t)$ crosses the stable arm for $\hat{c}(t)$ schedule where $\bar{i}(t) \neq 1$. In this potential steady-state the productivity of advanced capital grows at a rate $\frac{\psi}{1-\psi} \cdot g$. If this is to be a steady-state, then $\bar{i}(t)$ must remain constant. If $\bar{i}(t)$ does not remain constant, then the stable arm for $\hat{\hat{K}}^A(t)$ will shift. This is implied by (31). The analysis of the static equilibrium in section I implies that $\bar{i}(t)$ will only stay constant if $K^A(t)$ decreases to offset this increase in productivity. But if $K^A(t)$ decreases such that $\bar{i}(t)$ remains constant then $\hat{\hat{K}}^A(t)$ will fall, violating the condition that $\hat{\hat{K}}^{A*}$ is constant at the steady-state, as required by (31). $K^A(t)$ must therefore rise to ensure that $\hat{\hat{K}}^{A*}$ remains constant. However, this growth in the stock of advanced capital increases $\bar{i}(t)$ further. From the definition of the stable arm for $\hat{\hat{K}}^A(t)$ in (32) this will drive down the intersection point of the stable arms for $\hat{c}(t)$ and $\hat{\hat{K}}^A(t)$. The steady-state value of $\hat{c}^*$ will be driven down. Repeating this argument implies that if steady-state is to exist in $\hat{\hat{K}}^A(t)$–$\hat{c}(t)$ space, it must take place when $\bar{i}(t) = 1$.[19]

It is possible to provide an intuitive conjecture for this transition path to this steady-state, where $\bar{i}(t) = 1$. This is shown in Figure V, when the steady-state is approached from the left i.e. $\hat{\hat{K}}^A(0) < \hat{\hat{K}}^{A*}$.

– FIGURE V HERE –

This conjecture for the transition to this steady-state is based on three observations. The first is that the stable arm for $\hat{c}(t)$ is known, given in (31). The second is that as $\hat{\hat{K}}^A(t)$ increases, $\bar{i}(t)$ also increases – this is intuitive, but proven in the Appendix. This gives the stable arm for $\hat{\hat{K}}^A(t)$

[19] Note that it is not possible for a steady-state where $\bar{i}(t) = 0$ – although $\bar{i}(t)$ is constant, this would imply there is no advanced capital and so the steady-state condition in (31) is violated.

its hump-shape in Figure V, given (32).[20] The third is that, supposing the economy reaches the steady-state where $\bar{i}(t) = 1$, the new model must collapse to have the same general form as a traditional Ramsey model in which there is labor and capital, and a labor-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$ – but rather than there being labor and capital, there is instead only two types of capital, with an advanced capital-augmenting process of technological change taking place at rate g that is equivalent to a traditional capital-augmenting process of technological change taking place at rate $\frac{\psi}{1-\psi} \cdot g$. As a result, locally to the steady-state, the new model has the same approximated saddle-path as this traditional Ramsey model. This is what is shown in Figure V.[21]

The implication of this conjectured transition path is intuitive – during transition $\hat{\hat{K}}^A(t)$ increases, driving up $\bar{i}(t)$ as the stock and productivity of $K^A(t)$ increases. However this is only a conjecture and for an important reason – the approximated saddle-path in Figure V is only correct locally to the steady-state. It is based on the claim that $\bar{i}(t) = 1$ in steady-state, but as soon as $\hat{\hat{K}}^A(t)$ deviates from $\hat{\hat{K}}^{A*}$ along the saddle-path, $\bar{i}(t) \neq 1$, and this claim no longer holds. In order to find the actual saddle-path to this steady-state, rather than the conjectured one, it is necessary to perform the non-linear simulation.

To make the simulation tractable, I need to explicitly define the absolute productivity schedules for the factors – the $a^K(i,t)$, $a^H(i,t)$, and $a^A(i,t)$ schedules. To do this, I make the following assumptions:

$$\begin{aligned} a^K(i,t) &= a^K \\ a^H(i,t) &= a^H \\ a^A(i,t) &= a^A(0,t) - b \cdot i(t) \\ a^A(0,t) &= a^A(0,0) \cdot e^{gt} \end{aligned} \tag{34}$$

[20] In effect, the $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule is bounded below by a hypothetical $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule where $\bar{i}(t)$ is fixed at 1 and above by a hypothetical $\dot{\hat{\hat{K}}}^A(t) = 0$ schedule where $\bar{i}(t)$ is fixed at 0.

[21] By 'collapse', I mean the static three-factor analysis is replaced by a two-factor analysis where there is no labor, and advanced and traditional capital combine to produce all goods.

where a^K, a^H, b, and $a^A(0,0)$ are positive constants. The first three expressions in (34) are absolute productivity schedules. They are the simplest possible schedules that generate a relative productivity schedule, $A^H(i,t)$, with the properties set out in Assumption 2. Intuitively, they imply that the absolute productivity of advanced capital is diminishing in i, at a constant rate b, and the productivity of the other factors is constant across the task-spectrum. The final expression is a growth process. It is the simplest possible process that ensures Assumption 3 is maintained. This is shown in the Appendix. It implies that, again, the productivity of advanced capital in producing the numeraire good grows at rate g. (34) allows for an explicit solution to be derived for $\bar{i}(t)$. From the $A^H(i,t)$ and $C(i,t)$ schedules it follows that:[22]

(35)

$$\left[b \cdot a^A(0,t)^{\frac{\psi}{1-\psi}}\right] \cdot \bar{i}(t)^2 - \left[a^A(0,t)^{\frac{1}{1-\psi}} + b \cdot a^A(0,t)^{\frac{\psi}{1-\psi}}\right] \cdot \bar{i}(t) + \left[a^A(0,t)^{\frac{1}{1-\psi}} - \frac{a^H(i,t) \cdot L^H}{\hat{\hat{K}}^A(t)}\right] = 0$$

and so given the quadratic form of (35), $\bar{i}(t)$ can be found in a straightforward way. Figure VI shows the results for one particular parameterisation of the dynamic model. This is solved using a relaxation algorithm, adapting Trimborn et al. (2008).[23]

– FIGURE VI HERE –

Effective advanced capital, $\hat{\hat{K}}^A(t)$, effective consumption, $\hat{c}(t)$, and the cut-off $\bar{i}(t)$ rise during transition to the steady state. Intuitively, the accumulation of advanced capital drives up consumption but also drives out labor. The outcome is remorselessly pessimistic as wages are driven to zero. The process is driven by the fact that advanced capital is accumulated not only to offset the fact that it is becoming more productive, but also that it is

[22]This is in full in the Appendix.

[23]$a^K = 1$, $a^H = 40$, $a^A(0,0) = 10$, $g = 0.05$, $\rho = 0.02$, $\psi = 0.5$, $b = 1$, $D = 1$, $L^H = 50$. Note that the transition path of $r^A(t)$ now does depend upon the normalisation – although the steady-state r^{A*} does not depend upon it. This simulation approach is not robust to all parameterisations.

becoming more prevalent (i.e. $\bar{i}(t)$ is rising, offsetting the diminishing returns in the production of any particular good). It is as if the economy is chasing a steady-state that is continually slipping out of its grasp – until $\bar{i}(t) = 1$ and labor is fully driven out. This is the remorselessly pessimistic conclusion that was conjectured previously. Proposition 2 follows.

PROPOSITION 2: *When the static model with advanced capital is placed in a dynamic setting with endogenous advanced capital accumulation, the economy approaches a unique steady-state at $\hat{c}^*$, $\hat{\hat{K}}^{A*}$, where $\bar{i}^* = 1$ – in this steady-state, labor is entirely driven out by advanced capital. During transition, $w^H(t)$ is driven to zero. The capital share of income rises steadily to 1.*

Note that Proposition 2 is derived without depreciation. This may seem like a significant omission, since the depreciation of the existing stock of advanced capital may appear to act as a counterbalance to the accumulation of advanced capital and the displacement of labor. But this intuition is incorrect. Suppose depreciation takes place at rate δ. As in a traditional Ramsey model, the introduction of depreciation changes the stable arm for $\hat{c}(t)$, $\frac{\dot{\hat{c}}(t)}{\hat{c}(t)} = 0$:

$$\hat{\hat{K}}^A(t) = \left[\left[\rho + \delta + \frac{\psi}{1-\psi} \cdot g\right] \cdot \frac{1}{D}\right]^{\frac{1}{\psi-1}} \tag{36}$$

and also the stable arm for $\hat{\hat{K}}^A(t)$, $\dot{\hat{\hat{K}}}^A(t) = 0$:

$$\hat{c}(t) = \hat{\hat{K}}^A(t)^\psi \cdot D - \bar{i}(t) \cdot \left[\delta + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{K}}^A(t) \tag{37}$$

The implication of (36) and (37) is that while the introduction of depreciation will change the level of steady-state $\hat{\hat{K}}^{A*}$ – implied by (36) – it is still the case that the model must approach a steady-state in $\hat{c}(t) - \hat{\hat{K}}^A(t)$ space when $\bar{i}(t) = 1$. This is again implied by (37). The outcome again is remorselessly pessimistic for labor.

As a final observation note that, with or without depreciation, the outcome for owners of capital is remorselessly *optimistic*. The advanced capital share of income, $\bar{i}(t) \cdot \psi$, rises over time. And the return to the fixed stock of traditional capital, $r(t)$, which (10) and (25) imply is equal to:

$$r(t) = \hat{\hat{K}}^A(t)^{\psi} \cdot a^A(0,t)^{\frac{\psi}{1-\psi}} \cdot \frac{a^K(0)^{1-\psi} \cdot (1-\psi)}{K^{\psi}} \tag{38}$$

This implies $r(t)$ rises during transition and in steady-state when $\bar{i}(t) = 1$.

III. Conclusion

Autor (2015) captures the case for optimism in the task-based literature:

> "These questions underline an economic reality that is as fundamental as it is overlooked: tasks that cannot be substituted by automation are generally complemented by it." (p. 6)

This is repeated in Autor (2014):

> "The fact that a task cannot be computerized does not imply that computerization has no effect on that task. On the contrary: tasks that cannot be substituted by computerization are generally complemented by it." (p. 8)

In an important sense, the analysis in this paper is in agreement with this claim. Those tasks that cannot be automated are indeed complemented by traditional capital. The value of those tasks increases as the quantity or productivity of traditional capital increases. However, the new model does challenge the assumption that *necessarily* labor will indefinitely be best placed to perform those complemented tasks. This is the new role for advanced capital. As it increases in quantity or productivity, it erodes the set of types of q-complemented tasks in which labor retains the comparative advantage. Labour is forced to specialise in a diminishing set of types of tasks. Under the ALM hypothesis, the set of types of tasks performed by labor that are q-complemented is protected from this erosion. But for the reasons set out in Susskind (2017), this no longer seems appropriate. It is as if capital in these new models is now able to 'reach over' the production function into that set of types of tasks that was previously thought to be for human beings alone. That the consequences of this 'task encroachment' are so pessimistic for labor – a remorseless displacement of labor, a continual fall in absolute wages, and technological unemployment – suggests the traditional literature may have already created a false sense of optimism about the prospects for labor. Note again, however, that for the owners of *capital* this conclusion is far from pessimistic. All the returns to technological progress flow to them.[24] From an equity standpoint, it follows that who owns and controls capital in this model becomes an increasingly important question over time.

If the new supply-side argument for pessimism in this paper is right, then understanding how we might offset it is an increasingly important task. One response to this process of 'task encroachment' is described in Acemoglu and Restrepo (2017) – the falling labor scarcity that takes place leads to an incentive for producers to create new tasks in which labor has the comparative advantage. Labour is displaced from old tasks, but can take refuge in these new tasks instead. However, the problem with approach is that it conflicts with the original argument used to motivate it. The authors present Leontief (1952) and his mistaken claim that labor in the 21st century would share the fate of horses in the 20th – that they "will become less and less important ... More and more workers will be replaced by machines". And the authors argue their model offers a response – "the difference between human labor and horses is that humans have a comparative advantage in new and more complex tasks. Horses did not." But this raises the fundamental question – why does this model not apply equally well to horses, too? If technological change also caused falling horse scarcity, why was there not a surge in the creation of new types of tasks in the 20th century in which horses had the comparative advantage, as the model would suggest? An important part of the theoretical account appears to be missing.

Nevertheless, the general idea of a new 'race' in task-space is an important area for further work. Acemoglu and Restrepo (2017) is a supply-side version of the race – *producers* create new types of tasks to produce

[24] $r(t)$, the return to traditional capital, rises over time. See (38).

output. A new alternative is a demand-side version of the race – where *consumers* demand different goods and services that in turn require new types of tasks to produce them. This claim is often made informally – for instance, in making a case for optimism, Mokyr et al. (2015) argues "the future will surely bring new products that are currently barely imagined", Autor and Dorn (2013) discusses "new products and services that raise national income", and Autor (2014) appeals to the unforeseen rise of "health care, finance, informational technology, consumer electronics, hospitality, leisure, and entertainment" to explain why those displaced from farming in the 20th century would not lack for work in the 21st. Only a model like the one in this paper, with a variety of goods, is capable of exploring this demand-side version. Acemoglu and Restrepo (2017), with a unique final good, cannot. Optimism will only follow, though, if the new goods and services that have to be produced require tasks in which labor, and not advanced capital, has the comparative advantage. This paper lays the necessary foundations for this further work.

IV. Appendix: For Online Publication

IV.A. Assumption 3

To derive the condition in Assumption 3, take two arbitrary levels of i such that $\tilde{i} < \tilde{\tilde{i}}$. Suppose Assumption 2 initially holds so that at $t = 0$:

$$A^H(\tilde{i}, 0) \leq A^H(\tilde{\tilde{i}}, 0) \tag{39}$$

Given the definition of $A^H(i, t)$ this implies:

$$a^A(\tilde{\tilde{i}}, 0) \leq a^A(\tilde{i}, 0) \cdot \frac{a^H(\tilde{\tilde{i}}, 0)}{a^H(\tilde{i}, 0)} \tag{40}$$

For Assumption 2 to continue hold over time, over time it must be that $\forall t$:

$$a^A(\tilde{\tilde{i}},t) \leq a^A(\tilde{i},t) \cdot \frac{a^H(\tilde{\tilde{i}},t)}{a^H(\tilde{i},t)} \tag{41}$$

This is Assumption 3. There are two growth processes used in the dynamic model. The first is used when the model is solved analytically. This is (12) and implies:

$$a^A(i,t) = a^A(i,0) \cdot e^{gt} \tag{42}$$

To see that this maintains Assumption 3, note that (40) and (42) imply:

$$\begin{aligned} a^A(\tilde{\tilde{i}},0) \cdot e^{gt} &\leq a^A(\tilde{i},0) \cdot e^{gt} \cdot \frac{a^H(\tilde{\tilde{i}},0)}{a^H(\tilde{i},0)} \\ a^A(\tilde{\tilde{i}},t) &\leq a^A(\tilde{i},t) \cdot \frac{a^H(\tilde{\tilde{i}},0)}{a^H(\tilde{i},0)} \end{aligned} \tag{43}$$

and so Assumption 3 holds, since the productivities of labor do not change over time. The second growth process is used when the model is solved computationally, and is implied by (34):

$$a^A(i,t) = a^A(0,0) \cdot e^{gt} - b \cdot i \tag{44}$$

To see that this maintains Assumption 3, note that (44) and (40) imply:

(45)

$$\begin{aligned} a^A(\tilde{\tilde{i}},t) = a^A(0,0) \cdot e^{gt} - b \cdot \tilde{\tilde{i}} &\leq a^A(0,0) \cdot e^{gt} - b \cdot \tilde{\tilde{i}} \cdot e^{gt} \\ &\leq \left(a^A(0,0) \cdot e^{gt} - b \cdot \tilde{i} \cdot e^{gt}\right) \cdot \frac{a^H(\tilde{\tilde{i}},0)}{a^H(\tilde{i},0)} \\ &\leq \left(a^A(0,0) \cdot e^{gt} - b \cdot \tilde{i}\right) \cdot \frac{a^H(\tilde{\tilde{i}},0)}{a^H(\tilde{i},0)} = a^A(\tilde{i},t) \cdot \frac{a^H(\tilde{\tilde{i}},0)}{a^H(\tilde{i},0)} \end{aligned}$$

Again, since (39) will hold $\forall t$ so long as:

$$\dot{a}^A(\tilde{\tilde{i}}, t) \leq \dot{a}^A(\tilde{i}, t) \cdot \frac{a^H(\tilde{\tilde{i}}, t)}{a^H(\tilde{i}, t)} \tag{46}$$

it follows from (45) that Assumption 3 again holds, since the productivities of labor do not change over time.

IV.B. Law of Motion for $x(i, t)$

From (15) it follows that:

$$x(i, t) = \frac{1}{\mu(t) \cdot p(i, t)} \tag{47}$$

And from (16) that:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -r^A(t) + \rho \tag{48}$$

It follows from (47) that:

$$\frac{\dot{x}(i, t)}{x(i, t)} = -\frac{\dot{p}(i, t)}{p(i, t)} - \frac{\dot{\mu}(t)}{\mu(t)} \tag{49}$$

(48) and (49) therefore imply that:

$$\frac{\dot{x}(i, t)}{x(i, t)} = -\frac{\dot{p}(i, t)}{p(i, t)} + r^A(t) - \rho \tag{50}$$

IV.C. Expression for $p(0, t)$

To find any $p(i, t)$, suppose that $i(t) \in [0, \bar{i}(t)]$. This implies that $x(i, t)$ is produced by advanced capital and traditional capital:

$$x(i,t) = \left[a^A(i,t)K^A(i,t)\right]^{\psi}\left[a^K(i,t)K(i,t)\right]^{1-\psi} \;\; \forall i(t) \in [0, \tilde{i}(t)] \tag{51}$$

This implies that the marginal product of advanced capital in producing $x(0,t)$ is:

$$\begin{aligned} MPK^A(i,t) &= a^A(i,t) \cdot \psi \left[a^A(i,t)K^A(i,t)\right]^{\psi-1}\left[a^K(i,t)K(i,t)\right]^{1-\psi} \\ &= \psi \cdot \frac{x(i,t)}{K^A(i,t)} \end{aligned} \tag{52}$$

and the marginal product of traditional capital in producing $x(i,t)$ is:

$$\begin{aligned} MPK(i,t) &= a^K(i,t) \cdot (1-\psi) \cdot \left[a^L(i,t)L^L(i,t)\right]^{\psi}\left[a^K(i,t)K(i,t)\right]^{-\psi} \\ &= (1-\psi) \cdot \frac{x(i,t)}{K(i,t)} \end{aligned} \tag{53}$$

Given perfectly competitive profit-maximising firms, the price of each of these factors – $r^A(t)$ and $r(t)$ – must be equal to their respective marginal revenue products:

$$\begin{aligned} r^A(t) &= p(i,t) \cdot \psi \cdot \frac{x(i,t)}{K^A(i,t)} \qquad \forall i(t) \in [0, \bar{i}(t)] \\ r(t) &= p(i,t) \cdot (1-\psi) \cdot \frac{x(i,t)}{K(i,t)} \qquad \forall i(t) \in [0, 1] \end{aligned} \tag{54}$$

These can be re-arranged:

$$\begin{aligned} K^A(i,t) &= p(i,t) \cdot \psi \cdot \frac{x(i,t)}{r^A(t)} \qquad \forall i(t) \in [0, \bar{i}(t)] \\ K(i,t) &= p(i,t) \cdot (1-\psi) \cdot \frac{x(i,t)}{r(t)} \qquad \forall i(t) \in [0, 1] \end{aligned} \tag{55}$$

and substituting these expressions for $K^A(i,t)$ and $K(i,t)$ into (51) implies:
(56)

$$x(i,t) = \left[a^A(i,t) \cdot p(i,t) \cdot \psi \cdot \frac{x(i,t)}{r^A}\right]^{\psi} \left[a^K(i,t) \cdot p(i,t) \cdot (1-\psi) \cdot \frac{x(i,t)}{r(t)}\right]^{1-\psi}$$

$$p(i) = \left[\frac{r^A}{\psi \cdot a^A(i)}\right]^{\psi} \left[\frac{r}{(1-\psi) \cdot a^K(i)}\right]^{1-\psi}$$

(56) is therefore the $p(i,t)$ for any good $i(t) \in [0, \bar{i}(t)]$. A similar exercise to derive $L^H(i,t)$ provides $p(i,t)$ for those remaining goods $i(t) \in [\bar{i}(t), 1]$ produced by labor with traditional capital.

IV.D. Law of Motion for $\hat{\hat{K}}^A(i,t)$

Using (27) and (28), (23) can be written as:

$$\text{(57)} \quad \frac{\dot{\hat{\hat{K}}}^A(t)}{\hat{\hat{K}}^A(t)} = \frac{\hat{K}^A(t)^{\psi} \cdot \frac{D}{\psi} - \hat{c}(t)}{\hat{K}^A(t)} - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \dot{\hat{\hat{K}}}^A(t) \cdot \frac{1}{\bar{i}(t)} + \frac{\psi}{1-\psi} \cdot g\right]$$

And so:

(58)

$$\dot{\hat{\hat{K}}}^A(t) = \frac{1}{\bar{i}(t)} \cdot \left[\hat{\hat{K}}^A(t)^{\psi} \cdot \frac{D}{\psi} - \hat{c}(t)\right] - \left[\frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \dot{\hat{\hat{K}}}^A(t) \cdot \frac{1}{\bar{i}(t)} + \frac{\psi}{1-\psi} \cdot g\right] \cdot \hat{\hat{K}}^A(t)$$

$$= \frac{\hat{\hat{K}}^A(t)^{\psi} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g \cdot \hat{\hat{K}}^A(t) - \hat{c}(t)}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \hat{\hat{K}}^A(t)}$$

It follows that:

$$\text{(59)} \qquad \frac{\dot{\hat{\hat{K}}}^A(t)}{\hat{\hat{K}}^A(t)} = \frac{\hat{\hat{K}}^A(t)^{\psi-1} \cdot \frac{D}{\psi} - \bar{i}(t) \cdot \frac{\psi}{1-\psi} \cdot g - \frac{\hat{c}(t)}{\hat{\hat{K}}^A(t)}}{\bar{i}(t) + \frac{\partial \bar{i}(t)}{\partial \hat{\hat{K}}^A(t)} \cdot \hat{\hat{K}}^A(t)}$$

IV.E. Transition Path

To show that $\bar{i}(t)$ increases as $\hat{\hat{K}}^A(t)$ increases on the transition path, I first derive the following relationship between the growth rates in $\bar{i}(t)$, $K^A(t)$ and g that must hold:

$$g^{\gamma}(t) = \phi(t)\left[g^{K^A}(t) + g\right] \tag{60}$$

where:

(61)

$$\phi(t) = \frac{1-\bar{i}(t)}{[\bar{i}(t)-\bar{i}(t)^2]\cdot y(t)+1} \quad \text{and} \quad y(t) = \left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} - \frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right]$$

and $\phi(t)$ has the three properties: $\phi(t) \geq 0$; $\phi(t) \leq 1$; and $\lim_{\bar{i}(t)\to 1}\phi(t) = 0$. The expression in (60) implies that the growth rate in the equilibrium cut-off, $g^{\gamma}(t)$, is proportional to the sum of the growth rate in the advanced capital stock $g^{K^A}(t)$ and the productivity of $K^A(t)$, $\frac{\psi}{1-\psi}\cdot g$ where the constant of proportionality $\phi(t)$ is time dependent with the three features set out above. I now derive this expression for $g^{\gamma}(t)$ and these three properties of $\phi(t)$.

Deriving $g^{\gamma}(t)$

To derive the expression for $g^{\gamma}(t)$, note that the equilibrium condition, $A^H(\bar{i}(t),t) = C(\bar{i}(t),t)$ can be re-arranged as:

$$a^H(\bar{i}(t),t)\cdot\bar{i}(t) = [1-\bar{i}(t)]\cdot a^A(\bar{i}(t),t)\cdot\frac{K^A(t)}{L^H} \tag{62}$$

Taking time derivatives of (62) implies a further condition that must hold in any t:

(63)

$$\dot{\bar{i}}(t)\cdot\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} + \frac{a_2^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} + \frac{\dot{\bar{i}}(t)}{\bar{i}(t)} = -\frac{\dot{\bar{i}}(t)}{[1-\bar{i}(t)]} + \dot{\bar{i}}(t)\cdot\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)} + \frac{a_2^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)} + \frac{\dot{K}^A(t)}{K^A(t)}$$

and substituting in the expression for the growth rates implies that:

$$g^{\gamma}(t)\left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}-\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right]+\frac{1}{[1-\bar{i}(t)]}\right]=g+g^{K^A}(t) \tag{64}$$

and so:

$$g^{\gamma}(t)=\phi(t)\left[g^{K^A}(t)+g\right] \tag{65}$$

where:
(66)

$$\phi(t)=\left[\bar{i}(t)\cdot\left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)}-\frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right]+\frac{1}{[1-\bar{i}(t)]}\right]^{-1}=\frac{1-\bar{i}(t)}{[\bar{i}(t)-\bar{i}(t)^2]\cdot y(t)+1}$$

Deriving the Three Properties of $\phi(t)$

First to see that $\phi(t)\geq 0$ note that the weakly positive slope of $A^H(i,t)$ implies $\forall i,t$:[25]

$$A_1^H(i,t)\geq 0 \tag{67}$$

which, given the definition of $A^H(i,t)$, can be re-written as:

$$\frac{a_1^H(i,t)\cdot a^A(i,t)-a_1^A(i,t)\cdot a^H(i,t)}{[a^A(i,t)]^2}\geq 0 \tag{68}$$

Since the denominator of the expression in (68) is always positive, this implies that the following must hold:

$$\begin{gathered} a_1^H(i,t)\cdot a^A(i,t)\geq a_1^A(i,t)\cdot a^H(i,t)\\ \frac{a_1^H(i,t)}{a^H(i,t)}\geq\frac{a_1^A(i,t)}{a^A(i,t)} \end{gathered} \tag{69}$$

Therefore so long as the $A^H(i,t)$ schedule is weakly positively sloped, $y(t)$ remains weakly positive, the denominator in the expression for $\phi(t)$ in (66)

[25]This is Assumptions 2 and 3.

remains positive, and so $\phi(t)$ also remains weakly positive. To see the second property, that $\phi(t) \leq 1$, consider the following proof by contradiction. Assume instead that $\phi(t) > 1$. The expression for $\phi(t)$ in (66) then implies:

$$\begin{aligned}
\bar{i}(t) \cdot \left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} - \frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right] + \frac{1}{[1-\bar{i}(t)]} &< 1 \\
\bar{i}(t) \cdot [1-\bar{i}(t)] \cdot y(t) &< 1 - \bar{i}(t) - 1 \\
\bar{i}(t) \cdot y(t) - \bar{i}(t)^2 \cdot y(t) &< -\bar{i}(t)
\end{aligned} \tag{70}$$

But since $\bar{i}(t) \in [0,1]$ this condition cannot hold. If $\bar{i}(t) = 0$, then the result is a contradiction since (70) requires that $0 < 0$. Similarly if $\bar{i}(t) > 0$ then (70) requires:

$$y(t) < -\frac{1}{1-\bar{i}(t)} < 0 \tag{71}$$

which is not possible since $\bar{i}(t) \in [0,1]$ and $y(t) \geq 0$. The third property, that $\lim_{\bar{i}(t)\to 1} \phi(t) = 0$, follows from the fact that:

$$\lim_{\bar{i}(t)\to 1} \left[\frac{1}{1-\bar{i}(t)}\right] = \infty \tag{72}$$

and that:

$$\lim_{\bar{i}(t)\to 1} \left[\bar{i}(t) \cdot \left[\frac{a_1^H(\bar{i}(t),t)}{a^H(\bar{i}(t),t)} - \frac{a_1^A(\bar{i}(t),t)}{a^A(\bar{i}(t),t)}\right]\right] \geq 0 \tag{73}$$

And so $\lim_{\bar{i}(t)\to 1} \phi(t) = 0$, given (66), (72) and (73).

$\bar{i}$ During Transition

The proof that $\frac{d\bar{i}(t)}{d\hat{\hat{K}}^A(t)} \geq 0$ during the conjectured transition in Figure V now follows from two relationships. The first is implied by the fact that if

$\hat{\hat{K}}^A(t)$ is increasing, (21) implies:

$$g^{K^A}(t) > g^\gamma(t) + \frac{\psi}{1-\psi} \cdot g \tag{74}$$

where $g^{K^A}(t)$ is the growth rate in the stock of advanced capital and $g^\gamma(t)$ is again the growth rate in the equilibrium cut-off $\bar{i}(t)$. The second relationship is that set out in (60). Combining these two sets of relationships implies that if $\hat{\hat{K}}^A(t)$ is increasing then:

$$\frac{1-\phi(t)}{\phi(t)} \cdot g^\gamma(t) > \frac{1}{1-\psi} \cdot g \tag{75}$$

And so:

$$g^\gamma(t) > \frac{\phi(t)}{(1-\psi)\cdot(1-\phi(t))} \cdot g \tag{76}$$

Given the properties of $\phi(t)$ derived before, and that $g > 0$, this implies that if $\hat{\hat{K}}^A(t)$ is increasing then:

$$g^\gamma(t) > 0 \tag{77}$$

i.e. $\bar{i}(t)$ is also increasing. (76) also implies $\lim_{\bar{i}(t)\to 1} g^\gamma(t) = 0$ given the properties of $\phi(t)$.

IV.F. Explicit Solution for $\bar{i}(t)$

Given (34), since $A^H(i,t) = C(i,t)$, it follows that in any t:

$$\frac{a^H(i,t)\cdot L^H}{(a^A(0,t) - b\cdot\bar{i}(t))\cdot(1-\bar{i}(t))\cdot a^A(0,t)^{\frac{\psi}{1-\psi}}} = \hat{\hat{K}}^A(t) \tag{78}$$

and so:

(79)

$$\left[b\cdot a^A(0,t)^{\frac{\psi}{1-\psi}}\right]\cdot\bar{i}(t)^2 - \left[a^A(0,t)^{\frac{1}{1-\psi}} + b\cdot a^A(0,t)^{\frac{\psi}{1-\psi}}\right]\cdot\bar{i}(t) + \left[a^A(0,t)^{\frac{1}{1-\psi}} - \frac{a^H(i,t)\cdot L^H}{\hat{\hat{K}}^A(t)}\right] = 0$$

This can be solved for $\bar{i}(t)$ in the traditional way, given its quadratic form.

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Figure I: The Relative Productivity Schedule

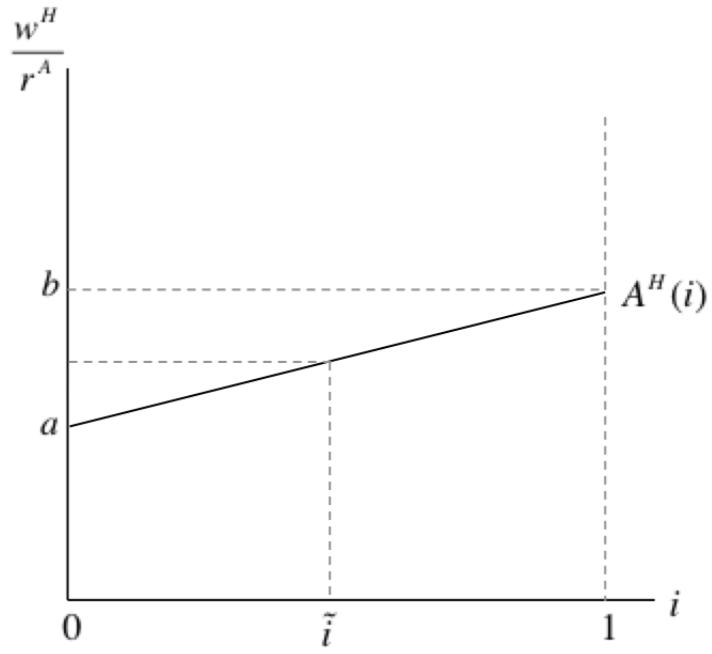

Figure II: The Zero-Profit Schedule

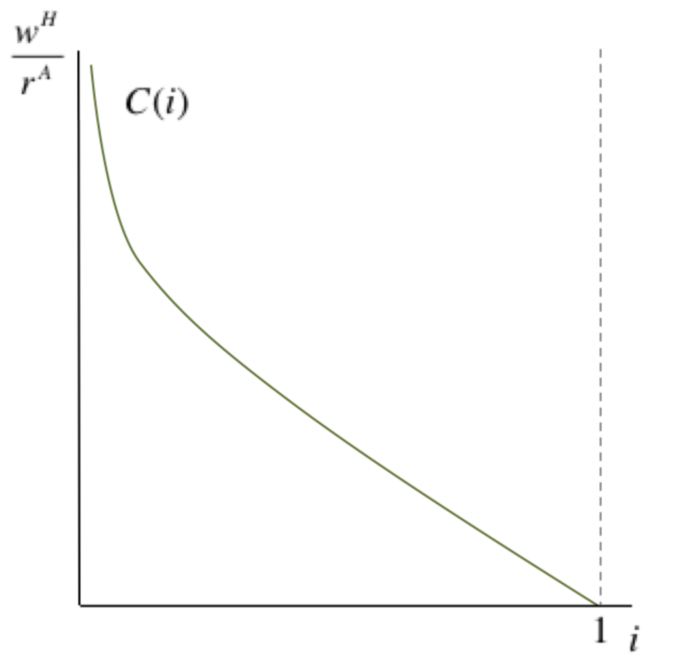

Figure III: Market Equilibrium

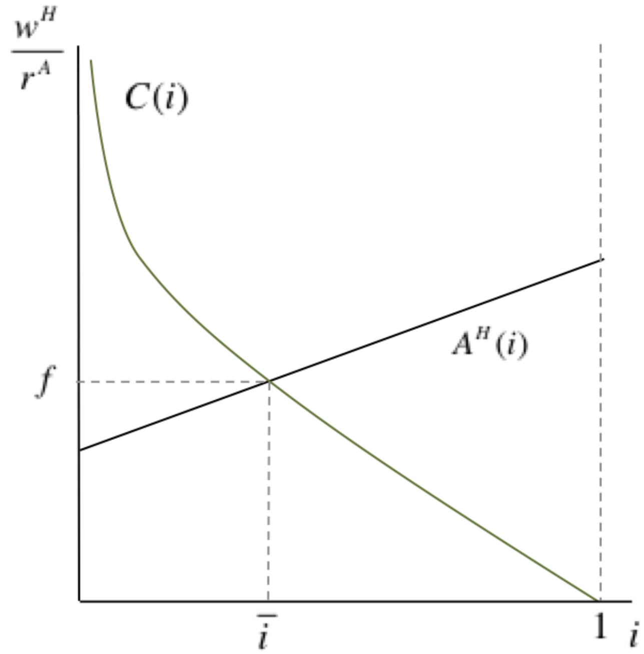

Figure IV: A New Market Equilibrium

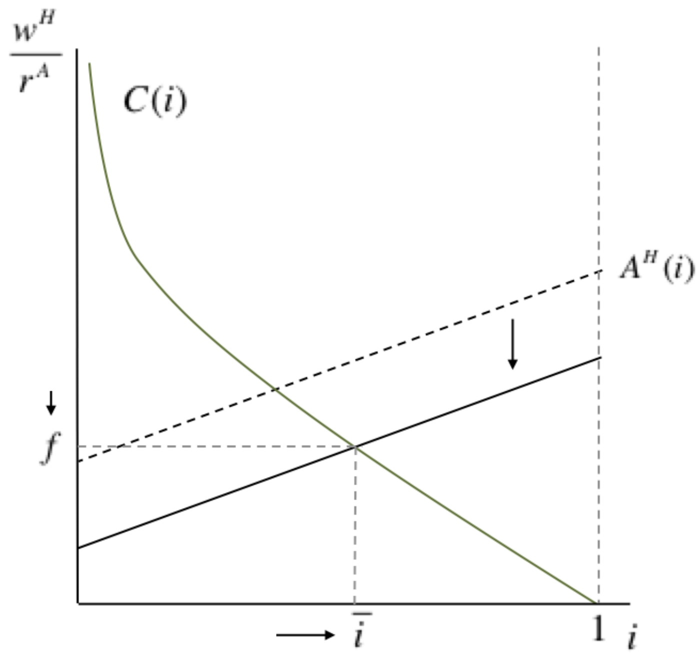

Figure V: Dynamic Equilibrium

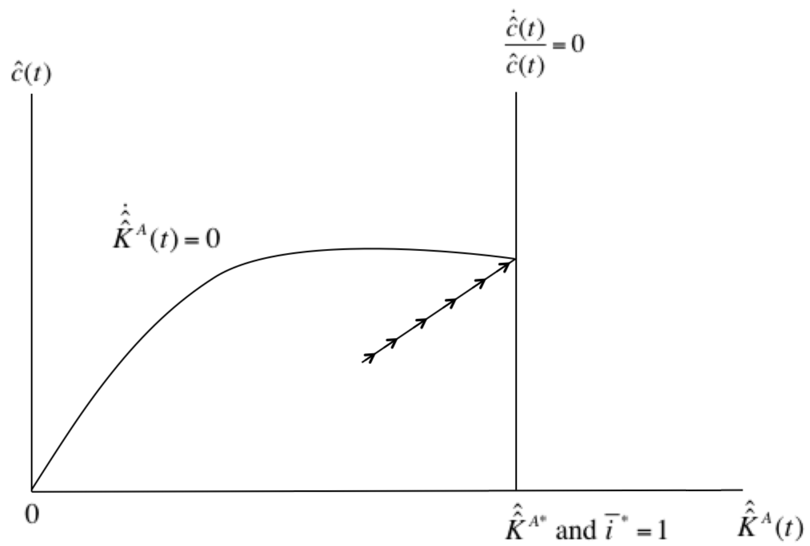

Figure VI: Simulation of a Dynamic Equilibrium