Full Implementation and Belief Restrictions*

Mariann Ollár  Antonio Penta
University of Pennsylvania, University of Wisconsin-Madison,
Econ. Dept. and Warren Center Department of Economics

October 27, 2014

Abstract

We introduce a framework to study the problem of full implementation under general restrictions on agents’ beliefs, which we call $\Delta$-Implementation. First we provide a characterization of the properties of the mechanisms that achieve $\Delta$-Implementation. These conditions relate the possibility of achieving full implementation to the strength of the strategic externalities induced by the mechanism. We then study how to achieve full implementation via the design of simple transfer schemes. In these context, the general conditions for $\Delta$-Implementation suggest a simple design principle, in which belief restrictions are used to weaken the strategic externalities induced by the baseline transfers of the belief-free benchmark. Importantly, our results require minimal information on agents’ beliefs, which regard simple moments of the distribution of types. These moment conditions arise naturally in applications and in practical problems of mechanism design.

Keywords: Full Implementation, Robust Mechanism Design, Rationalizability, Belief Restrictions, Moment Conditions, Nice Games, Uniqueness

1 Introduction

The problem of multiplicity is a key concern for the design of real-world mechanisms and institutions. Unless all the solutions of a mechanism are consistent with the outcome the designer wishes to implement, the designer may not confidently assume that the proposed mechanism will perform well. This is a well known criticism to the partial implementation approach to mechanism design, which merely requires that there exists one strategy profile consistent with the chosen solution concept that guarantees desirable outcomes. The full implementation approach (cf., Maskin, 1999) overcomes the problem of multiplicity. But in pursuit of greater generality, the existing literature has typically adopted rather complicated mechanisms.\(^1\) Thus, while it addresses an important practical concern, the full implementation literature overall has provided little insight into how real-world institutions could be designed to avoid the problem of multiplicity.

Another well-known limitation of the classical (Bayesian) approach to mechanism design, particularly important from a practical viewpoint, is the excessive reliance on common knowledge

\(^*\)We are particularly grateful to Laurent Mathevet, Stephen Morris, Ken Hendricks, Larbi Alaoui, George Mailath, Andy Postlewaite and William Sandholm for the helpful comments. We also thank seminar audiences at Stanford, NYU, UPenn, UW-Madison, Pompeu Fabra and at the SAET Conferences in Paris and Tokyo.

\(^1\)See Jackson (1992) for an influential criticism to the tail-chasing mechanisms typically used in this literature.
assumptions. This criticism, often referred to as the ‘Wilson doctrine’, has recently received considerable attention in the literature on robust implementation. It is fair to say, however, that the promise of the Wilson doctrine, “[...] to conduct useful analyses of practical problems [...]” (Wilson, 1987), is still far from being fulfilled. This is due to two main limitations: On the one hand, most of this literature has focused on environments in which the designer has no information about the agents’ beliefs. This extreme assumption represents a useful benchmark to address foundational questions, but significantly limits the relevance of the theory for practical problems of mechanism design. On the other hand, as far as full implementation is concerned (e.g. Bergemann and Morris (2009a), or Penta (2011)), the main focus thus far has been on identifying conditions under which a given mechanism achieves implementation in a robust sense, but it offers little guidance as to how to design such a mechanism, given the objectives of the designer.

In this paper we address these points pursuing a more pragmatic approach to full implementation, based on mechanisms with a clear economic interpretation and which rely on more realistic assumptions of common knowledge, intermediate between the classical and the ‘belief-free’ approaches. More specifically, we study the full implementation problem in environments with transferable utility (TU) and interdependent values, and restricting ourselves to using direct mechanisms, which only elicit players’ payoff-relevant information. Direct mechanisms and TU-environments are the main realm of the partial implementation literature. Studying full implementation in these settings is important for a number of reasons: First, it facilitates the comparison with the literature on partial implementation, by making transparent what features of an incentive compatible mechanism may or may not be problematic from the full implementation viewpoint; Second, because with these restrictions the mechanism design problem boils down to designing transfer schemes which have a clear economic interpretation and deliver insights that are easily portable to the design of real-world institutions and incentive schemes.

For the sake of illustration, consider the problem of efficient implementation with interdependent values. In environments with single-crossing preferences, the generalized VCG mechanism of Cremer and McLean (1985) guarantees partial implementation of the efficient allocation in an ex-post equilibrium, with essentially no restrictions on the strength of the preference interdependence. Hence, independent of the agents’ beliefs, truthful revelation (hence efficiency) is always achievable as part of a Bayes-Nash equilibrium (cf. Bergemann and Morris, 2005). The problem with this mechanism is that it typically admits also inefficient equilibria, which can only be ruled out if the interdependence in agents’ valuations is not too strong (cf. Bergemann and Morris, 2009). The reason is that, when preference interdependencies are strong, the VCG mechanism induces strong strategic externalities (players’ best responses are largely affected by changes in the opponents’ strategies). Strong strategic externalities in turn generate multiplicity of equilibria, hence failure of full implementation. The key idea that we pursue is to use information about agents’ beliefs to weaken the strategic externalities of the VCG mechanism, so as to achieve full implementation even with strong preference interdependence. We show that, if information about agents’ beliefs is available, strong preference interdependence need not coincide with strong strategic externalities. Clearly, in reducing the strategic externalities, we should also make sure that incentive compatibility is preserved (if not in the ex-post sense, at least for the beliefs consistent with the designer’s

---

2On the ‘belief free’ approach to mechanism design, see Bergemann and Morris (2005, 2009a, 2009b, 2011) for static mechanisms and Mueller (2012a,b) and Penta (2011) for dynamic ones. A thorough account of this literature is provided by Bergemann and Morris (2012). We discuss the related literature more extensively in Section 1.1.
information (cf. Mathevet, 2010)). There is thus a tension between the robustness of the partial implementation result (achieved by the VCG mechanism in an ex-post equilibrium), and the possibility to achieve full implementation of the efficient decision rule. Depending on the nature and amount of information about agents’ beliefs, the underlying strategic externalities may be weakened, and full implementation achieved regardless of the preference interdependence.

While efficient implementation is one of our leading examples, our analysis covers general implementation problems with interdependent values, under varying assumptions on agents’ beliefs. To this end, we adopt the solution concept of \( \Delta \)-Rationalizability (Battigalli and Siniscalchi, 2003), which generalizes rationalizability to games with incomplete information and general belief restrictions. The resulting notion of ‘\( \Delta \)-Implementation’ (Sections 2 and 3) therefore provides a unified framework to study full implementation under general belief restrictions, thereby allowing for varying degrees of robustness. Besides the obvious theoretical advantages of maintaining such a level of generality, we will argue that the flexibility of \( \Delta \)-Rationalizability is also convenient from an applied viewpoint. In Section 3.2 we provide a full characterization of the \( \Delta \)-Implementable mechanisms, based on two conditions: \( \Delta \)-Incentive compatibility and \( \Delta \)-Contractivity. In Section 6 we relate these concepts to existing notions of incentive compatibility and monotonicity, particularly to those provided by Bergemann and Morris (2009a) and Oury and Tercieux (2012) to characterize belief-free and ICR-Implementation, respectively.

The question of how to obtain \( \Delta \)-Incentive Compatible and \( \Delta \)-Contractive mechanisms through the design of simple payment schemes is considered in Section 4. Our design strategy consists of two steps. First we derive the ‘canonical transfers’, a generalization of well-known necessary conditions for ex-post incentive compatible payment schemes. Depending on the environment, and particularly on the strength of the preference interdependence, the canonical transfers may induce overly strong strategic externalities, which are problematic from the viewpoint of full implementation. The second part of our design then exploits the belief restrictions to reduce the strategic externalities, so as to induce a contractive mechanism which guarantees uniqueness. The conditions that guarantee full implementation relate the strength of the preference interdependence to the information embedded in the belief restrictions. This ‘information’, in particular, takes the form of restrictions on simple moments of the distribution of types that arise naturally in applications, both with and without a common prior.

In Section 5 we show how these general results can be applied to important special cases, such as environments that satisfy standard single-crossing conditions. First we show that, in the common prior models commonly considered in the classical and applied literature, full implementation is always possible if types are affiliated or independently distributed (Section 5.1). Our construction suggests a simple design principle: start out with the ex-post incentive compatible transfers, and

---

3 The idea of modifying ex-post incentive compatible transfers using information about beliefs is based on a clever insight of Mathevet (2010). Apart from the broad idea, however, there are important differences. In particular, Mathevet (2010) focuses on partial implementation, and modifies the baseline transfers in order to achieve supermodularity. In contrast, our design aims at inducing contractive best responses, to achieve full implementation. Furthermore, Mathevet (2010) focuses on independent private values environments, whereas we allow for interdependent values, correlation as well as non Bayesian environments. Further connections with the literature are discussed in Section 1.1.

4 For instance, as the belief restrictions are varied, \( \Delta \)-Implementation includes as special cases both Bergemann and Morris’ (2009a, 2011) belief-free implementation and Oury and Tercieux’s (2012) ICR-Implementation. The first is an important benchmark, in that it represents the most demanding notion of robustness with respect to agents’ beliefs. The second characterizes ‘continuous implementation’, an important property of local robustness for partial implementation (see also Oury (2013)).
then compensate each agent for a proper measure of the strategic externality he is subject to, given the reports. To avoid that agents misreport their type in order to inflate their compensation, each agent $i$ is also asked to pay a fee equal to the expected value of the compensation, given his type:

$$t_i(\theta) = \underbrace{t_i^{EPIC}(\theta)}_{\text{canonical transfers}} + \underbrace{CSE_i(\theta_i, \theta_{-i})}_{\text{compensation for strategic externality (depends on everybody’s report)}} - \underbrace{\mathbb{E}(CSE_i|\theta_i)}_{\text{belief-based adjustment: expected compensation (only depends on i’s report)}}.$$

The first term we add to the canonical transfers reduces the strategic externalities and ensures that the mechanism is contractive; the last term, derived from the designer’s information about agents’ beliefs, restores incentive compatibility. Full implementation follows.

Importantly, these implementation results do not rely on the full strength of the common prior assumption: common knowledge of certain summary statistics of the types’ distribution suffices. This is true more generally for our construction: our general results achieve full implementation whenever some moments of the types’ distribution are common knowledge (cf. Section 5.2). Such summary statistics can be estimated from previous data on the performance of the mechanism. As recently argued by Deb and Pai (2013), this is a desirable property for a mechanism, and it is guaranteed here thanks to the combination of the full implementation requirement and the robustness entailed by the solution concept. From a different perspective, these results indirectly shed some light on the kind of information it may be useful for the designer to disclose. We discuss this and other extensions in Section 7.

1.1 Related Literature

Our work is related to several strands of the literature in game theory and mechanism design. We briefly discuss the most closely related literature.

Solution Concept. The solution concept that we use, Δ-Rationalizability, was introduced by Battigalli (2003) and Battigalli and Siniscalchi (2003). It generalizes several versions of rationalizability for incomplete information games, including the ‘belief-free’ rationalizability studied by Bergemann and Morris (2009) and Dekel, Fudenberg and Morris’ (2007) ‘interim correlated rationalizability’ (ICR). ICR has also been studied by Oury and Tercieux (2012), Penta (2013) and Weinstein and Yildiz (2007, 2011, 2013). Battigalli et al. (2011) provide a thorough analysis of Δ-Rationalizability and its connections with belief-free, ICR and other versions of rationalizability.

Nice Games. At a more technical level, our construction exploits the notion of ‘nice mechanisms’, which extends Moulin’s (1984) idea of nice games to encompass environments with incomplete information. Our implementation results are based on general uniqueness results we establish for ‘nice games’, more extensively discussed in Ollár and Penta (2014). Nice games are convenient analytical tools, particularly if rationalizability is adopted as solution concept. For a recent application of (complete information) nice games, see Weinstein and Yildiz (2011).

Full Implementation. Within the vast literature on (full) implementation, the closest papers are Bergemann and Morris (2009a) and Oury and Tercieux (2012), which study implementation in ‘belief free’ rationalizability and ICR, respectively. Both ‘belief free’ and ICR-Implementation are special cases of Δ-Implementation. Thus, characterizations of both notions of implementation can be obtained as special cases of ours, with the proviso that Oury and Tercieux (2012) do not restrict
attention to direct mechanisms, as we do (cf. Section 6). The restriction to direct mechanisms is also shared by Bergemann and Morris (2009a), while Bergemann and Morris (2011) study belief-free implementation in general mechanisms. Within the classical literature, Jackson (1991) and Postlewaite and Schmeidler’s (1986) Bayesian Monotonicity are also connected to $\Delta$-Contractivity (cf. Section 6). From a conceptual viewpoint, our departure from that literature is inspired by Jackson’s (1992) critique of unbounded mechanisms. We push the concern for ‘relevance’ a bit further, requiring that full implementation is achieved via simple transfer schemes.

Mechanism Design in TU-Environments. TU-environments are the typical domain of the partial implementation literature. Within this area, the closest works are those that allow for interdependent values (e.g., Cremer and McLean (1985, 1988), Dasgupta and Maskin (2000), McLean and Postlewaite (2004). In recent years, a growing literature has revisited standard results, imposing extra desiderata inspired by more practical considerations. The already mentioned paper by Deb and Pai (2013) is one such example, which pursues symmetry of the mechanism. Mathevet (2010) and Mathevet and Taneva (2013) instead pursue supermodularity. In those papers, the extra desiderata are achieved by adding a belief-dependent component to some baseline payments, very much as we attain full implementation appending an extra term to the canonical transfers. One difference is that those papers maintain that types are independently distributed, whereas we allow more general correlations, as well as weaker restrictions on beliefs. At a more technical level, our design results in a contractive mechanism. Given our concern with full implementation, contractivity is a more convenient property than supermodularity. Healy and Mathevet (2013) also pursue contractivity of the mechanism, though in a complete information setting.

Robust Mechanism Design. As already mentioned, most of the literature on robust mechanism design has focused on the belief-free case. See, for instance, Bergemann and Morris (2005, 2009 and 2011) for static mechanism design, and Müller (2012a,b) and Penta (2011) for dynamic mechanism design. Kim and Penta (2012) explore partial implementation with interdependent values, maintaining some restrictions on beliefs. Lopomo, Rigotti and Shannon (2013) also explore partial implementation with belief restrictions analogous to ours, but focus on single agent problems and consider a different notion of robustness. Artemov, Kumimoto and Serrano (2013) also maintain some restrictions on beliefs, but focus on virtual implementation. Different approaches to robust mechanism design have been recently put forward by Yamashita (2013a,b), Börgers and Smith (2012,2013), Carroll (2013) and Wolitzky (2014).

Implementation and the ‘price of anarchy’. The partial implementation approach often argues that the truth-telling efficient equilibrium is plausible as it is intuitive for players to play that equilibrium instead of some other equilibria. The validity of this argument clearly depends on the particular instance. A growing literature quantifies the potential efficiency loss due to multiplicity in auction environments. Constant bounds are known for the ‘price of anarchy’ (the welfare ratio between efficient and worst Nash Equilibrium outcomes) in complete information settings. For incomplete information, Roughgarden (2012) shows that the price of anarchy can be arbitrarily

---

5 D’Aspremont, Cremer and Gerard-Varet (2005) also studied full implementation in environments with transferable utility, but they resort to unbounded mechanisms of the kind criticized above. Duggan and Roberts (2002) fully implement the efficient allocation of pollution via transfers, but under complete information and richer reports.

6 McLean and Postlewaite (2002) also explore related ideas in environments without transferable utility.

7 Early examples of this principle are the mechanisms of D’Aspremont and Gerard-Varet (1975) and of Cremer and McLean (1985), which append the baseline VCG mechanism with a belief-based component in order to achieve budget balance and surplus extraction, respectively.
large. If the design guarantees full implementation of the efficient outcome, then the price of
anarchy is minimized, equal to 1.

2 Model

Environments and Mechanisms. We consider standard environments with transferable utility. We denote by \( I = \{1, \ldots, n\} \) the set of agents, by \( X \) the set of (common) social outcomes and by \( t_i \in \mathbb{R} \) the private transfer to agent \( i \in I \). Agents’ preferences depend on the realization of the state of the world \( \theta \in \Theta = \times_{i \in I} \Theta_i \). When \( \theta \) is realized, agent \( i \) privately observes the \( i \)-th component, \( \theta_i \in \Theta_i \). We refer to \( \theta_i \in \Theta_i \) as agent \( i \)’s payoff type (or just as ‘type’), and let \( \theta_{-i} \in \Theta_{-i} = \times_{j \neq i} \Theta_j \) denote the type profile of \( i \)’s opponents. For each \( i \in I \), we let \( u_i : X \times \mathbb{R} \times \Theta \to \mathbb{R} \) denote player \( i \)’s utility function, and we assume that there exists a function \( v_i : X \times \Theta \to \mathbb{R} \) such that

\[
u_i (x, t_i, \theta) = v_i (x, \theta) + t_i \]

for every \( (x, t_i, \theta) \in X \times \mathbb{R} \times \Theta \). We refer to \( v_i (\cdot) \) as agent \( i \)’s valuation function. We maintain throughout that \((\Theta_i)_{i \in I}\) are convex and compact subsets of the real line and that \( X \) is a convex and compact subset of a Euclidean space. This model accommodates general externalities in consumption, including both pure cases of private and public goods.

The tuple \( \mathcal{E} = (I, (\Theta_i, u_i)) \) defines the ‘payoff environment’, and we assume it is common knowledge among the agents. Payoff types thus represent agents’ information about preferences. If \( v_i \) is constant in \( \theta_{-i} \) for every \( i \), then the environment is one of private values. If not, the environment has interdependent values.

A decision rule (or allocation rule) is a mapping \( d : \Theta \to X \), which assigns to each payoff state the social outcome that the designer wishes to implement. We say that an allocation rule is responsive if for any \( i \), \( \theta_i \) and \( \theta'_i \) such that \( \theta_i \neq \theta'_i \), there exists \( \theta_{-i} \in \Theta_{-i} \) such that \( d(\theta_i, \theta_{-i}) \neq d(\theta'_i, \theta_{-i}) \). We impose the following assumptions on \((\mathcal{E}, d)\):

Assumption 1 (Smoothness): (i) \( d : \Theta \to X \) is responsive and twice continuously differentiable; (ii) for every \( i \), \( v_i \) is three times continuously differentiable in all the arguments.

In general, a mechanism is a tuple \( \mathcal{M} = \langle (M_i)_{i \in I}, g \rangle \), where \( g : \times_{i \in I} M_i \to X \times \mathbb{R}^n \). For the reasons discussed in the introduction, we restrict ourselves to using direct mechanisms, in which the sets of messages are \( M_i = \Theta_i \), and the common component \( x \in X \) is chosen according to \( d \). A direct mechanism is thus uniquely determined by a transfer scheme \( (t_i)_{i \in I}, t_i : M \to \mathbb{R} \), which specifies the (possibly negative) transfer to agent \( i \), for each possible profile of reports \( m \in M \) (to distinguish the report from the state, we maintain the notation \( m \) even though \( M = \Theta \).

Any direct mechanism \( \mathcal{M} \) induces a (belief-free) game \( G^\mathcal{M} = \langle I, (\Theta_i, M_i, U_i)_{i \in I} \rangle \), where \( I \) is the set of players, \( \Theta_i \) the set of \( i \)’s payoff types, \( M_i \) is the set of \( i \)’s actions and ex-post payoff functions \( U_i : M \times \Theta \to \mathbb{R} \) are such that

\[
U_i (m; \theta) = v_i (d (m), \theta) + t_i (m).
\]

For every \( \theta_i \in \Theta_i \), \( \mu_i \in \Delta (\Theta_{-i} \times M_{-i}) \) and \( m_i \in M_i \), we let \( EU_{\theta_i}^\mu_i (m_i) \) denote player \( i \)’s
expected payoff from message \( m_i \), if \( i \)'s type is \( \theta_i \) and his conjectures are \( \mu_i \):

\[
EU_{\theta_i}^{\mu_i}(m_i) := \int_{\Theta_{-i} \times M_{-i}} U_i(m_i, m_{-i}; \theta_i, \theta_{-i}) \, d\mu_i.
\]

We also define \( BR_{\theta_i}(\mu_i) := \arg \max_{m_i \in M_i} EU_{\theta_i}^{\mu_i}(m_i) \).

**Belief Restrictions.** We model the assumptions on agents’ beliefs separately from the environment \( E \). This is because, whereas information about beliefs may be useful in designing a mechanism, agents’ beliefs are not directly relevant to the designer’s objectives, \( d \). We model ‘belief restrictions’ as sets of possible beliefs for each type of every player. Formally, \( B = ((B_{\theta_i})_{\theta_i \in \Theta_i})_{i \in I} \) where \( B_{\theta_i} \subseteq \Delta(\Theta_{-i}) \) for each \( \theta_i \in \Theta_i \) and \( i \in I \), assumed common knowledge. We maintain throughout that each \( B_{\theta_i} \) is non-empty, closed and convex. If \( B \) and \( B' \) are such that \( B_{\theta_i} \subseteq B'_{\theta_i} \) for all \( \theta_i \) and \( i \), we write \( B \subseteq B' \).

This formulation is fairly general. For instance, if \( B_{\theta_i} \) is a singleton for every \( \theta_i \) and \( i \), then the pair \((E, B)\) is a standard Bayesian environment, in which agents’ hierarchies of beliefs are uniquely pinned down by their payoff types. The further special case of a common prior model requires that \( B_{\theta_i} = \{ b_{\theta_i} \} \) are such that there exists \( p \in \Delta(\Theta) \) s.t. \( b_{\theta_i} = p(\cdot | \theta_i) \in \Delta(\Theta_{-i}) \) for each \( i \) and \( \theta_i \), whereas if \( B_{\theta_i} = B_{\theta'_i} \) for all \( i \) and all \( \theta_i, \theta'_i \in \Theta_i \) we obtain the case of independent types (cf. Example 1). At the opposite extreme, if \( B_{\theta_i} = \Delta(\Theta_{-i}) \) for every \( \theta_i \) and every \( i \), then there are no commonly known restrictions on beliefs, and the pair \((E, B)\) coincides with the belief-free environments that are common in the literature on robust mechanism design.\(^8\) Such ‘vacuous restrictions’ are thus denoted by \( B^{BF} \). Our model also accommodates settings, intermediate between the Bayesian and belief-free cases, in which some common knowledge restrictions are maintained but not to the point that belief hierarchies are uniquely determined by the payoff types. In those cases, the tuple \( B \) represents the designer’s *partial information* about agents’ beliefs. Our results apply to all of these cases.

For reasons that will be illustrated in the next example, we will distinguish between the belief restrictions in \( B \) and the beliefs with respect to which full implementation may be obtained. From this viewpoint, it is useful to think of \( B \) as the *most* that the designer is willing assume about agents’ beliefs. Clearly, if \( B \subseteq B' \), then \( B' \) entails weaker restrictions than \( B \).

### 2.1 Leading Examples: key insights and their generalizations

**Example 1 (Full Implementation in a Common Prior Model)** Consider an environment with two agents, \( i = 1, 2 \). The planner chooses some quantity \( x \in X \equiv \mathbb{R}_+ \) of a public good, with cost of production \( c(x) = \frac{1}{2}x^2 \). Players’ valuation functions are \( v_i(x, \theta) = (\theta_i + \gamma \theta_j) x \), where \( \gamma \geq 0 \) is a parameter of preference interdependence: if \( \gamma = 0 \), this is a private-value setting; if \( \gamma > 0 \), values are interdependent. Now, suppose that the planner knows that agents’ types are i.i.d. draws from a uniform distribution over \( \Theta_i \equiv [0, 1] \), denoted by \( v_{\Theta_i} \), and that this is common knowledge among the agents. This is a standard common prior environment, with independently distributed types and interdependent values. In this model, the planner’s information about agents’ beliefs is represent by belief restrictions \( B = ((B_{\theta_i})_{\theta_i \in \Theta_i})_{i \in I} \) such that \( B_{\theta_i} = \{ v_{\Theta_j} \} \) for every \( i, j \neq i \) and \( \theta_i \in \Theta_i \).

The objective of the social planner is to implement the efficient level of public good, that is \( d^* (\theta) = (1 + \gamma) (\theta_1 + \theta_2) \). It is well known that this decision rule can be partially implemented through the VCG mechanism, with transfers

\[
t^VCG_i (m) = -(1 + \gamma) (0.5m_i^2 + \gamma m_i m_j).
\] (1)

Given the VCG mechanism, for any pair \((\theta_j, m_j)\) of player \(j\)’s type and report, the ex-post best-reply function for type \(\theta_i\) of player \(i\) is

\[
BR^VCG_{\theta_i} (\theta_j, m_j) = \text{proj}_{[0,1]} (\theta_i + \gamma (\theta_j - m_j)).
\] (2)

Observe that for any \(\gamma \geq 0\) and for any realization of \(\theta\), truthful revelation \((m_i (\theta_i) = \theta_i)\) is a best response to the opponent’s truthful strategy \((m_j (\theta_j) = \theta_j)\). This is the ex-post incentive compatibility of the VCG mechanism. Partial implementation of the efficient allocation is thus guaranteed independent of agents’ beliefs. Furthermore, if \(\gamma < 1\), equation (2) is a contraction, and its iteration delivers truthful revelation as the only rationalizable strategy. In this case, the VCG mechanism also guarantees full robust implementation (Bergemann and Morris (2009a)). If \(\gamma \geq 1\), on the other hand, the VCG mechanism fails to robustly implement the efficient allocation rule. (In the general symmetric case with \(n\) agents, it can be shown that no mechanism achieves belief-free full implementation if \(\gamma \geq 1 / (n - 1)\).)

Hence, with weak interdependence in valuations, the designer need not rely on the common prior: the VCG mechanism ensures full implementation in the belief-free model \(\mathcal{B}^F \supset \mathcal{B}\). If the interdependence is strong, however, belief-free implementation is impossible, even under the \(\mathcal{B}\)-restrictions. For instance, if \(\gamma = 2\), the strategy profile \((\hat{m}_1 (\theta_1) = 1, \hat{m}_2 (\theta_2) = 0)\) is also a Bayes Nash equilibrium (BNE) of the VCG mechanism. To see this, consider the interim best reply of type \(\theta_i\), given the common prior and the opponent’s strategy \(\hat{m}_j : \Theta_j \rightarrow M_j\):

\[
BR^VCG_{\theta_i} (\hat{m}_j (\cdot)) = \text{proj}_{[0,1]} (\theta_i + \gamma [E (\theta_j | \theta_i) - E (\hat{m}_j (\theta_j) | \theta_i)]).
\]

If \(\gamma = 2\) and \(j\) always reports 0 (resp., 1), then \(i\)’s best reply is to report as high (as low) as possible. Furthermore, this equilibrium is inefficient, as it implements \(x = 3\) regardless the state.

The source of this multiplicity of equilibria, when interdependence in valuations are strong, is that the VCG mechanism determines strong strategic externalities: if \(\gamma\) is large, players’ best responses are largely affected by changes in the opponents’ strategies.

Being designed to achieve ex-post incentive compatibility, the VCG mechanism clearly ignores any information about agents’ beliefs. We propose next a different set of transfers, which do exploit some information contained in the common prior. Namely, that \(E (\theta_j | \theta_i) = \frac{1}{2} \) for all \(\theta_i\) and \(i\):

\[
t^* (m) := -(1 + \gamma) \left( \frac{1}{2} m_i^2 + \gamma m_i E (\theta_j | \theta_i) \right) = -(1 + \gamma) \left( \frac{1}{2} m_i^2 + \gamma m_i \frac{1}{2} \right).
\] (3)

These transfers induce the following the best response function:

\[
BR^*_{\theta_i} (\hat{m}_j (\cdot)) = \text{proj}_{[0,1]} \left( \theta_i + \gamma \left[ E (\theta_j | \theta_i) - \frac{1}{2} \right] \right).
\] (4)
Since, under the common prior, \( \mathbb{E}(\theta_j | \theta_i) = \frac{1}{2} \) for all \( \theta_i \), the term in square brackets cancels out for all types. Truthful revelation therefore is strictly dominant, independent of the strength of preference interdependence, \( \gamma \). In fact, this would be the case for any beliefs that satisfy the ‘moment condition’ \( \mathbb{E}(\theta_j | \theta_i) = \frac{1}{2} \) for all \( \theta_i \). Hence, full implementation is guaranteed not just for the common prior \( \mathcal{B} \), but for the weaker restrictions \( \mathcal{B}' = ((\mathcal{B}_{\theta_i})_{\theta_i \in \Theta_i})_{i \in I} \) defined as \( \mathcal{B}_{\theta_i} := \{ b_i \in \Delta(\Theta_j) : \int \theta_j \cdot db_i = \frac{1}{2} \} \). Furthermore, since truthful revelation in this mechanism is dominant, given \( \mathcal{B}' \), such restrictions need not even be common knowledge among the agents: as long as \( \mathbb{E}(\theta_j | \theta_i) = \frac{1}{2} \) for all \( \theta_i \) and \( i \), full implementation obtains independent of higher order beliefs.

The previous example is a standard Bayesian implementation problem, in which the planner’s information is represented by a common prior model, \( \mathcal{B} \). As illustrated above, however, the designer may achieve implementation without necessarily relying on the full strength of the common prior model. If \( \gamma < 1 \), the VCG mechanism ensures belief-free implementation, that is for all beliefs consistent with the vacuous restrictions \( \mathcal{B}^{\text{BF}} \supset \mathcal{B} \). If \( \gamma \geq 1 \), the transfers in (3) achieve full implementation for all beliefs consistent with the restrictions \( \mathcal{B}' \supset \mathcal{B} \). The precise definition of \( \mathcal{B}' \) clearly depends on the particular moment condition we used to design the transfers (that is, \( \mathbb{E}(\theta_j | \theta_i) = \frac{1}{2} \) for all \( \theta_i \) and \( i \)). This is only one of infinitely many conditions that are consistent with the designer’s information \( \mathcal{B} \). Had we used a different condition, say \( \mathbb{E}(G(\theta_{-i}) | \theta_i) = f(\theta_i) \) for all \( \theta_i \) and \( i \)”, full implementation may have obtained for different beliefs \( \mathcal{B}' \supset \mathcal{B} \); namely, \( \mathcal{B}' = ((\mathcal{B}_{\theta_i}')_{\theta_i \in \Theta_i})_{i \in I} \) such that \( \mathcal{B}_{\theta_i}' = \{ b_i \in \Delta(\Theta_{-i}) : \int G(\theta_{-i}) \cdot db_i = f(\theta_i) \} \).

Thus, not only the set \( \mathcal{B} \), which represents the designer’s information, need not coincide with the set of beliefs with respect to which implementation is obtained (such as \( \mathcal{B}^{\text{BF}} \) or \( \mathcal{B}' \) in the example), but the latter is itself determined by the planner’s choice of the mechanism. Both the transfers and the degree of robustness are a choice of the designer. For this reason, we distinguish between the designer’s assumptions about agents’ beliefs, represented by the belief restrictions introduced above, and the beliefs with respect to which implementation is achieved, which will be referred to as ‘\( \Delta \)-restrictions’. Our notion of ‘\( \Delta \)-implementation’ treats such \( \Delta \)-restrictions as a parameter, which can be chosen in the design of a mechanism. In Section 3 we characterize the properties of general \( \Delta \)-restrictions that ensure full implementation. We then use these general results to provide guidelines for the designer’s choice of which, of the possibly many \( \Delta \)-restrictions consistent with \( \mathcal{B} \), are useful to design transfers for full implementation. In particular, in Section 4 we develop a general design principle which consists of using properly chosen \( \Delta \)-restrictions to weaken the strategic externalities of a baseline ‘canonical’ mechanism. We show that a special kind of \( \Delta \)-restrictions (namely, moment conditions) are particularly suited to this task. In the example above, for instance, the moment condition “\( \mathbb{E}(\theta_j | \theta_i) = \frac{1}{2} \) for all \( \theta_i \) and \( i \)” enabled us to completely offset the strategic externalities of the VCG mechanism, thereby ensuring full implementation in dominant strategies. In the general case in which strategic externalities cannot be completely eliminated, our design principle pursues contractivity of the best replies, to ensure that truthful revelation is the unique rationalizable outcome (given the \( \Delta \)-restrictions). The next example illustrates the point in the context of a non-Bayesian model.

**Example 2 (Full Implementation without a Common Prior)** Consider an environment with three agents, \( i = 1, 2, 3 \), and assume that agents commonly know that types \( \theta_i \in [0, 1] \) are i.i.d. draws from some distribution \( \Phi \). The distribution itself, however, is not necessarily known by the
agents, and most importantly is unknown to the designer. This environment therefore provides an example both of non-Bayesian belief restrictions and of a situation in which the designer possibly knows less than what is commonly known by the agents.

Preferences are similar to the previous example, except that the factor of preference interdependence may be heterogeneous. That is, \( v_i(x, \theta) = (\theta_i + \gamma \theta_j + \lambda) x \) where \( \gamma, \lambda \in \mathbb{R} \) and \( x \in \mathbb{R}_+ \) denotes the quantity of public good. (We let \( j := i + 1 \text{(mod} 3) \) and \( k := i + 2 \text{(mod} 3) \)). If the cost of production is the same as in the previous example, the efficient allocation rule is given by:

\[
BR^{\text{VCG}}_{\theta_i} = \text{proj}_{[0,1]} (\theta_i + \mathbb{E} (\gamma (\theta_j - m_j) + \lambda (\theta_k - m_k) | \theta_i)).
\]

Now, suppose that \( \gamma = 4/3 \) and \( \lambda = -2/3 \). With these parameter values, any report profile is rationalizable, and full implementation fails. The following transfers instead achieve full implementation: \( t^*_i(m) = t^{VCG}_i(m) + \gamma m_j - \gamma m_k \). With these transfers, the best reply is:

\[
BR^{\text{VCG}}_{\theta_i} = \text{proj}_{[0,1]} (\theta_i + \gamma \mathbb{E} (\theta_j - \theta_k | \theta_i) + (\gamma + \lambda) \mathbb{E} (m_k - \theta_k | \theta_i))
\]

\[
= \text{proj}_{[0,1]} (\theta_i + (\gamma + \lambda) \mathbb{E} (m_k - \theta_k)).
\]

The simplification in the second line is due to the fact that, under the maintained assumptions, \( \mathbb{E} (\theta_k - \theta_j | \theta_i) = 0 \) for all \( \theta_i \). Unlike the previous example, strategic externalities are not completely eliminated in this case. However, for the values of parameters specified above, the term \( (\gamma + \lambda) = 2/3 \). Hence, the best-replies induce a contraction, which delivers truthful revelation as the only rationalizable profile. Similar to the previous example, full implementation only relies on common knowledge of the moment condition used in the design of transfers (in this case, \( \mathbb{E} (\theta_k - \theta_j | \theta_i) = 0 \) for all \( \theta_i \)). Formally, the belief restrictions in this model are such that \( B_{\theta_i} = \{ b_i \in \Delta (\Theta_{-i}) : b_i = \bigotimes_{j \neq i} \phi \text{ for some } \phi \in \Delta([0,1]) \} \), whereas the transfers \( t^* \) achieve full implementation for the weaker restrictions \( B'_{\theta_i} = \{ b_i \in \Delta (\Theta_{-i}) : \int (\theta_k - \theta_j) db_i = 0 \} \).

### 3 \( \Delta \)-Implementation

#### 3.1 Definitions

**Solution Concept.** The game \( G^M \) is a ‘belief-free’ game, in that it does not contain any information about agents’ beliefs. Restrictions on beliefs are introduced via the solution concept, \( \Delta \)-Rationalizability (Battigalli and Siniscalchi, 2003). \( \Delta \)-Rationalizability characterizes the behavioral implications of common certainty of players’ rationality and of a set of exogenous restrictions on players’ conjectures. The latter are referred to as ‘\( \Delta \)-restrictions’.

Formally, \( \Delta \)-restrictions are a collection \( \Delta = \{ (\Delta_{\theta_i})_{\theta_i \in \Theta_i} \}_{i \in I} \) such that \( \Delta_{\theta_i} \subseteq \Delta (\Theta_{-i} \times M_{-i}) \) for every \( i \) and \( \theta_i \). Unlike \( B \), which restricts agents’ beliefs about the environment, the \( \Delta \)-restrictions concern agents’ conjectures, which are also about opponents’ behavior in the mechanism. \( \Delta \)-Rationalizability consists of an iterated deletion procedure in which, for each type \( \theta_i \), a given report \( m_i \) survives the \( k \)-th round of deletion if and only if it can be justified by conjectures
in $\Delta_{\theta_i}$ that are consistent with the previous rounds of deletion:

**Definition 1 (\(\Delta\)-Rationalizability)** Fix a set of $\Delta$-restrictions. For every $i \in I$, let $R^\Delta_i = \Theta_i \times M_i$ and for each $k = 1, 2, \ldots$, let $R^\Delta_{i,k-1} = \times_{j \neq i} R^\Delta_{j,k-1}$,

$$R^\Delta_{i,k} = \{(\theta_i, m_i) : m_i \in B_{\theta_i}(\mu_i) \text{ for some } \mu_i \in \Delta_{\theta_i} \cap \Delta(R^\Delta_{-i,k-1})\},$$

and $R^\Delta_i = \bigcap_{k \geq 0} R^\Delta_{i,k-1}$.

The set of $\Delta$-rationalizable messages for type $\theta_i$ is defined as $R^\Delta_i(\theta_i) := \{m_i : (\theta_i, m_i) \in R^\Delta_i\}$.

**The $\Delta$-Restrictions.** A special case of interest is when the $\Delta$-restrictions merely capture the idea that, in the game ensuing from the mechanism, the beliefs represented in the model $B$ are common knowledge. Formally: the $\Delta$-restrictions are `equivalent to $B$', denoted by $\Delta^B = \left(\left(\Delta^{B\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i \in I}$, if for every $i \in I$ and every $\theta_i \in \Theta_i$,

$$\Delta^{B\theta_i} := \{\mu_i \in \Delta(\Theta_{-i} \times M_{-i}) : \text{marg}_{\theta_{-i}\mu_i} \in B_{\theta_i}\}, \quad (5)$$

For the reasons discussed in Section 2.1, we consider general $\Delta$-restrictions, not necessarily equivalent to the designer’s information. In particular, it will be useful to consider $\Delta$-restrictions that are `weaker than $B$': that is, $\Delta_{\theta_i} \supseteq \Delta^{B\theta_i}$ for all $i$ and $\theta_i$ (or $\Delta \supseteq \Delta^B$). Another important special case is when the $\Delta$-restrictions are vacuous, denoted by $\Delta^{BF}$ (since they are equivalent to the belief-free model). In general, we maintain the following assumption on $\Delta$-restrictions:

**Assumption 2 (‘Non Behavioral’ $\Delta$-Restrictions):** The $\Delta$-restrictions are non-behavioral: that is, there exist belief restrictions $B'$ such that $\Delta = \Delta^{B'}$.

To understand the meaning of this assumption, suppose that it is violated. Then, for some $\theta_i$, $\Delta_{\theta_i}$ restricts beliefs not only about the opponents’ types, but also about their behavior in the mechanism. If the designer has information on agents’ conjectures about others’ behavior, then he would specify $\Delta$-restrictions that violate Assumption 2. This is an interesting idea, unexplored in the implementation literature, though often implicit in more applied work.\(^9\) We point out that such behavioral restrictions can easily be accommodated into our framework, but maintain Assumption 2 throughout. Notice that, if the $\Delta$-restrictions are non behavioral, then: (i) the sets $\Delta_{\theta_i} \subseteq \Delta(\Theta_{-i} \times M_{-i})$ are closed, non-empty and convex (this follows from the definition of belief restrictions); and (ii) they are weaker than $B$ if and only if $\Delta = \Delta^{B'}$ for some $B' \supseteq B$.

**Full Implementation.** Our notion of implementation requires all the $\Delta$-Rationalizable profiles of a direct mechanism to induce outcomes consistent with the allocation rule. This notion presents several advantages. First, as the $\Delta$-restrictions are varied, $\Delta$-Rationalizability coincides with various versions of rationalizability, some of which play an important role in the literature on

\(^9\)Exogenous restrictions on behavior are common in the related literature on market design, where assuming behavioral restrictions such as linear bidding strategies is convenient when comparing performance of different mechanisms (e.g., Ausubel et al. (2013)). Other examples of implicit behavioral $\Delta$-restrictions include the introduction of ‘noise traders’ in financial market models (e.g., Kyle (1989)), the assumption that players consider the opponents’ truth-telling strategies in auctions (e.g., Deb and Pai (2013)) and that bidders do not bid above their true valuations in the empirical auctions literature (e.g., Haile and Tamer (2003)).
robustness and on implementation. Second, \(\Delta\)-Rationalizability in general is a very weak solution concept. This is important because, unlike for ‘partial’ implementation, full implementation results are stronger if obtained with respect to a weaker solution concept. Sufficient conditions for full \(\Delta\)-Implementation therefore guarantee full implementation with respect to any refinement of \(\Delta\)-Rationalizability. Finally, it can be shown that \(\Delta\)-Rationalizability characterizes the set of all Bayes-Nash equilibria (BNE) in type spaces that are consistent with the \(\Delta\)-restrictions. Full \(\Delta\)-Implementation therefore can be seen as a shortcut to analyze standard questions of Bayesian Implementation for general restrictions on beliefs (cf. Appendix A).

**Definition 2 (Full \(\Delta\)-Implementation)** Fix an allocation rule \(d\), a direct mechanism \(M = (d, t)\) and a set of \(\Delta\)-restrictions. We say that:

1. \(M\) (fully) \(\Delta\)-implements \(d\), if \(R^\Delta (\theta) \neq \emptyset\) for all \(\theta\) and \(m \in R^\Delta (\theta)\) implies \(d(m) = d(\theta)\).

2. \(M\) truthfully \(\Delta\)-implements \(d\), if \(R^\Delta (\theta_i) = \{\theta_i\}\) for all \(\theta_i\) and all \(i \in I\).

3. \(M\) implements \(d\) in (strictly) \(\Delta\)-dominant strategies, if \(R^\Delta_{i,1} (\theta_i) = \{\theta_i\}\) for all \(\theta_i\) and all \(i\).

We say that \(d\) is \(\Delta\)-Implementable (resp.: truthfully \(\Delta\)-Implementable; \(\Delta\)-DS Implementable) if there exists a direct mechanism that \(\Delta\)-implements \(d\) (resp.: truthfully \(\Delta\)-implements \(d\); implements \(d\) in \(\Delta\)-dominant strategies).

Consider point 3 first: \(\Delta\)-DS Implementation. As shown in Example 1, if truthful implementation is achieved with only one round of \(\Delta\)-rationalizability, then truthful revelation is strictly dominant for all the beliefs consistent with the \(\Delta\)-restrictions. In this case, full implementation actually obtains independent of higher order beliefs, so the \(\Delta\)-restrictions need not even be common knowledge among the agents. This concept therefore entails a very robust notion of implementation, and it is stronger than the conditions in points 1 and 2. For instance, if \(\Delta = \Delta^B\) for some Bayesian model \(B\), then \(\Delta\)-DS Implementation is equivalent to truthful revelation being strictly dominant in the interim normal form of the Bayesian game.

It is also easy to see that truthful \(\Delta\)-Implementation in general is a stronger requirement than \(\Delta\)-Implementation. The next Proposition, however, shows that the two concepts coincide under the maintained assumptions 1 and 2:

**Proposition 1** If the \(\Delta\)-restrictions are non-behavioral and \(d\) is responsive, then \(d\) is (fully) \(\Delta\)-Implementable if and only if it is Truthfully \(\Delta\)-Implementable.

**Proof.** The ‘if’ part is trivial. The ‘only if’ follows from Lemma 3 in Appendix B. 

As discussed in Section 2.1, the general notion of \(\Delta\)-Implementation is a useful theoretical tool. The next definition instead formalizes the ultimate objective of the designer, which is to achieve full implementation at least for all the beliefs in the model \(B\):

**Definition 3 (Full \(B\)-Implementation)** We say that \(d\) is \(B\)-Implementable (resp., \(B\)-DS Implementable) if it is \(\Delta\)-Implementable (\(\Delta\)-DS Implementable) for some \(\Delta \supset \Delta^B\).

\(^{10}\)In particular, if \(B\) is a Bayesian model and \(\Delta = \Delta^B\), then \(\Delta\)-rationalizability coincides with interim correlated rationalizability (ICR, Dekel, Fudenberg and Morris, 2007). ICR-Implementation has been studied by Oury and Tercieux (2012). If instead the \(\Delta\)-restrictions are vacuous (\(\Delta = \Delta^BF\)), then \(\Delta\)-rationalizability coincides with ‘belief free’ rationalizability (e.g., Bergemann and Morris, 2009). See Battigalli, Di Tillio, Grillo and Penta (2011) for a thorough analysis of the connections between \(\Delta\)-rationalizability and other solution concepts.
Hence, achieving $\Delta^B$-implementation is the minimum objective for the designer, but this definition also accounts for the possibility of achieving full implementation for a larger set of beliefs $\Delta \supseteq \Delta^B$, which would ensure a more robust result. In Example 1, for instance, depending on the parameter $\gamma$, full implementation could be obtained with respect to $\Delta^{BF}$ or $\Delta^B$, both of which are weaker than the designer’s information in that example.\footnote{The distinction between the maintained assumptions on beliefs over the environment and the beliefs with respect to which implementation is achieved is not completely new to the literature, though it typically remains implicit. For instance, within the partial implementation literature, ex-post incentive compatibility is often sought even in common prior environments. See, for instance, in Myerson (1981) and Cremer and McLean (1985,1988).}

### 3.2 Characterization of Full $\Delta$-Implementability

A close inspection of part 2 of Definition 2 should make it clear that, in order to achieve truthful $\Delta$-implementation, the truthful profile must be a mutual best response for all types, and for all conjectures allowed by the $\Delta$-restrictions. Hence, based on Proposition 1, some notion of incentive compatibility will be necessary for full $\Delta$-implementation. For any direct mechanism, and for every $i \in I$, let $C^T_i \subseteq \Delta(\Theta_{-i} \times M_{-i})$ denote the set of truthful conjectures of player $i$: that is, player $i$’s conjectures that assign probability one to his opponents reporting truthfully. Formally:

$$C^T_i = \{ \mu \in \Delta(\Theta_{-i} \times M_{-i}) : \mu(\{\theta_{-i}, m_{-i} : m_{-i} = \theta_{-i}\}) = 1 \} .$$

**Definition 4** Given $\Delta$-restrictions, a direct mechanism $M$ is $\Delta$-incentive compatible ($\Delta$-IC) if for all $\theta_i \in \Theta_i$ and for all $\mu \in \Delta_{\theta_i} \cap C^T_i$ and $\theta'_i \in \Theta_i$, $EU^i_\mu(\theta_i) \geq EU^i_{\theta'_i}(\theta'_i)$. It is strictly $\Delta$-IC if the inequality holds strictly for all $\theta'_i \neq \theta_i$.

It is easy to verify that, if $\Delta = \Delta^{BF}$, then $\Delta$-IC coincides with ex-post incentive compatibility (EPIC). If instead $\Delta = \Delta^B$, and $B$ is a standard type space, then $\Delta$-IC coincides with the standard notion of interim (or Bayesian) incentive compatibility (IIC). Clearly, the weaker the $\Delta$-restrictions, the stronger the $\Delta$-IC condition.

We will use the following notation: For any $\sigma_{-i} : \Theta_{-i} \rightarrow \Delta(M_{-i})$ and $b_i \in \Delta(\Theta_{-i})$, we let $\mu^i(b_i, \sigma_{-i}) \in \Delta(\Theta_{-i} \times M_{-i})$ denote the conjectures derived from $b_i$ and $\sigma_{-i}$. We will let $\sigma^*_i$ denote the truthful strategy ($\sigma^*_i(\theta_i) = \theta_i$ for every $\theta_i$) and write $\mu^i(b_i, \sigma^*_i)$ for $\mu^i(b_i, \sigma^*_i)$. Hence, for any $B$, $\Delta$-IC can be written as: $\forall i, \forall \theta_i, \forall b_i \in B_{\theta_i}, \forall \theta'_i : EU^i_{\theta_i}(b_i) \geq EU^i_{\theta'_i}(b_i)$. Also, let $\Sigma_i : \Theta_i \rightarrow 2^{\Theta_i} \backslash \emptyset$ denote an arbitrary non-empty valued correspondence from $i$’s types to his reports. We will write $\sigma_i \in \Sigma_i$ to signify that $\sigma_i$ is a selection from $\Sigma_i$, that is $\sigma_i(\theta_i) \in \Sigma_i(\theta_i)$ for each $\theta_i$. Similarly, we will refer to strategies as selections from $\Sigma_i$.

We provide next a general characterization of $\Delta$-Implementation, which is based on the following property of a mechanism:

**Definition 5** Let the $\Delta$-restrictions be non-behavioral, and $B' = ((B'_{\theta_i})_{i \in I})$ such that $\Delta \equiv \Delta^{B'}$. A direct mechanism is $\Delta$-contractive if for any $\Sigma \neq \sigma^*$ there exists $i \in I, \theta_i \in \Theta_i, m_i \in \Sigma_i(\theta_i)$ and $\nu_i \in \Delta(M_i)$ such that: $EU^i_{\theta_i}(b_i, \sigma_{-i})(\nu_i) > EU^i_{\theta_i}(b_i, \sigma_{-i})(m_i)$ for all $\sigma_{-i} \in \Sigma_{-i}$ and all $b_i \in B'_{\theta_i}$.

**Theorem 1** If the $\Delta$-restrictions are non-behavioral, a responsive allocation rule is (fully) $\Delta$-Implementable by a direct mechanism if and only if there exists a strictly $\Delta$-IC and $\Delta$-Contractive mechanism that truthfully $\Delta$-implements it.

**Proof.** (See Appendix B). \[\square\]
As the Δ-restrictions are varied, Δ-contractivity is related to several notions of monotonicity in the literature on implementation. In particular, in the Δ-restrictions are vacuous, then it coincides with Bergemann and Morris’ (2009a) contraction property; if Δ = ΔB and B is a Bayesian model, then Δ-contractivity is closely related to Oury and Tercieux’s (2012) ICR-monotonicity, which in turn is related to robust monotonicity (Bergemann and Morris, 2011) and to Bayesian monotonicity (Jackson (1991) and Postlewaite and Schmeidler (1986)). While conceptually important, these notions of monotonicity are not particularly suited to provide insights on the design of transfers for full implementation. We thus postpone that discussion to Section 6, and focus instead on more insightful sufficient conditions for Δ-contractivity of a Δ-IC mechanisms. These conditions will have a clear interpretation: namely, that bounding the strength of the strategic externalities is key to ensure Δ-contractivity, hence Δ-Implementation. To this end, we introduce the notion of a nice mechanism. Nice mechanisms extend the idea of ‘nice games’ to the incomplete information games induced by a direct mechanism. Besides allowing intuitive and easy-to-check conditions for Δ-contractivity, nice mechanisms are particularly useful to instruct the design of contractive mechanisms through transfers, which we turn to in Section 4.

3.3 Conditions for Δ-Contractivity in ‘Nice’ Mechanisms

Consider a belief-free game with incomplete information, \( G = (I, (\Theta_i, M_i, U_i))_{i \in I} \), where I is the set of players, \( \Theta_i \) is the set of i’s payoff types, \( M_i \subseteq \mathbb{R} \) is the set of i’s actions and \( U_i : M \times \Theta \to \mathbb{R} \) is the payoff function of player i. We say that a game is ‘smooth’ if payoff functions are twice continuously differentiable. We let \( D_{jk}U_i(m, \theta) \) denote the second order partial derivative of i’s payoff with respect to strategies of players j and k:

\[
D_{jk}U_i(m, \theta) := \frac{\partial^2 U_i(m, \theta)}{\partial m_j \partial m_k}.
\]

**Definition 6** Fix a set of Δ-restrictions \( \Delta = (\Delta_{\theta_i})_{\theta_i \in \Theta_i, i \in I} \). Game \( G = (I, (\Theta_i, M_i, U_i))_{i \in I} \) is ‘nice’ with respect to Δ (or Δ-nice) if it is smooth and for every \( i \in I, \theta_i \in \Theta_i \) and \( \mu \in \Delta_{\theta_i}, \) the expected payoff function \( EU^\mu_{\theta_i} : M_i \to \mathbb{R} \) is strictly concave. A mechanism \( M \) is Δ-nice if \( G^M \) is Δ-nice. For convenience, we will use the term ‘nice’ instead of ‘ΔBF-nice’.

The next proposition provides sufficient conditions for a Δ-IC nice mechanism to be Δ-contractive (hence, by Theorem 1, to guarantee truthful Δ-Implementation.)

**Theorem 2** Let \( M \) be a Δ-nice and Δ-IC direct mechanism. Then: \( M \) is Δ-contractive if one of the following holds:

1. **(Δ-Self Determination)** for each agent \( i, \) for all \( \theta_i, \) for all \( \mu \in \Delta_{\theta_i}, \) and for all \( m_i, m_i' \in M_i \)

\[
\left| \int_{\Theta_{-i} \times M_{-i}} D_{ii}U_i(m_i', m_{-i}, \theta_i, \theta_{-i}) \, d\mu \right| > \int_{\Theta_{-i} \times M_{-i}} \sum_{j \neq i} |D_{ji}U_i(m_i, m_{-i}, \theta_i, \theta_{-i})| \, d\mu. \tag{6}
\]

---

12 The idea of ‘nice’ games was introduced by Moulin (1984) for games with complete information. For a recent application, see Weinstein and Yıldız (2011). A general analysis of nice games with incomplete information and Δ-restrictions is provided in Ollár and Penta (2014).
2. **(Ex-Post Self-Determination)** for each agent $i$, for all $\theta \in \Theta$, for all $m \in M$, and for all for all $m'_i \in M_i$,

$$|D_{m_i} U_i (m'_i, m_\cdot, \theta)| > \sum_{j \neq i} |D_{m_j} U_i (m, \theta)|.$$  \hfill (7) 

**Proof.** (See Appendix B). \hfill ■

To understand the meaning of inequalities (6) and (7), consider the first-order condition of the optimization problem of type $\theta_i$, given conjectures $m \in M_i$. Because of the strict concavity assumption implicit in the definition of nice mechanism, this condition is both necessary and sufficient for $m'_i \in \text{int}(\Theta_i)$ to be a best response to the conjectures $\mu \in \Delta_{\Theta_i}$. The second derivative $D_{m_j} U_i (m_i, m_\cdot, \theta_i, \theta_\cdot)$ therefore captures how the report of player $j$ affects the best response of player $i$, hence (for $j \neq i$) $j$’s ‘strategic externalities’ on $i$. Both conditions (6) and (7) require the ‘own effect’ (the LHS of the inequalities) to be stronger than the opponents’ effects, considered jointly (the RHS of the inequalities). These conditions therefore capture the idea that strategic externalities should not be too large, extending the main insight underlying the analogous condition in Moulin (1984).

The difference between the two conditions is that the first requires the Self-determination property to hold for all conjectures that are consistent with the $\Delta$-restrictions, while the second is an ex-post requirement. Clearly, the two conditions coincide if $\Delta = \Delta^{BF}$. In general, however, condition (7) is stronger than (6), hence the theorem could be stated in terms of the latter alone. We present (7) nonetheless because it is often easier to check in applications.

4 Designing Contractive Mechanisms through Transfers

In accordance with the literature on implementation, Theorem 1 provides a full characterization of the general properties of the mechanisms that ensure full implementation, but it is not very helpful for understanding how the transfers should be designed to ensure $\Delta$-contractivity. Theorem 2, on the other hand, suggests that nice mechanisms can be used to guarantee $\Delta$-contractivity, provided that the strategic externalities are adequately bounded. In the following we exploit this insight to explicitly construct transfers that achieve full implementation. Assumptions 1 and 2 will be maintained throughout this Section.

In Section 4.1 we consider belief-free implementation, that is, when $\Delta = \Delta^{BF}$. This is the most demanding notion $\Delta$-Implementation, and in many situations it would not be possible. When possible, however, it is convenient to adopt a mechanism that achieves it, because it entails full robustness of the result. We introduce the *canonical transfers*, and show that they characterize the mechanisms that achieve belief-free implementation. Hence, if the canonical transfers induce overly strong strategic externalities, belief-free implementation is impossible. Full implementation may still be possible if information about beliefs is used. In Section 4.2 we introduce a natural class of $\Delta$-restrictions (the ‘moment conditions’), which are particularly suited to design transfers.
for full implementation. The transfers for full implementation are obtained adding a belief-based term to the canonical transfers, to reduce the underlying strategic externalities. The resulting mechanism is ‘nice’, and full implementation follows from Theorem 2. (Applications are discussed in Section 5.)

4.1 Canonical Transfers and Belief Free Implementation

In this Section we consider the most demanding notion of robustness, that is when the Δ-restrictions are vacuous. Since Δ-IC coincides with ex-post incentive compatibility, Theorem 1 implies that strict ex-post incentive compatibility (EPIC) is necessary for belief-free truthful implementation (see also Bergemann and Morris, 2009). We thus focus on the question of how to design transfers that fully implement \( d \), if possible. Consider the following transfers: for each \( i \in I \) and \( m \in \Theta \), let

\[ t_i^* (m) = -v_i (d(m), m) + \int_{\Theta}^{m_i} \frac{\partial v_i (d(s_i, m_{-i}), \theta_i, m_{-i})}{\partial \theta_i} \bigg|_{\theta_i = s_i} ds_i. \tag{8} \]

We will refer to \( t^* = (t_i^* (\cdot))_{i \in I} \) as the ‘canonical transfers’, and to the direct mechanism \( \mathcal{M} = (d, t^*) \) as the ‘canonical mechanism’. In the canonical mechanism, agents pay their valuation as entailed by the reports profile (treated as truthful) minus the ‘total own preference effect’. This way, agents’ payments fully coincide with the ‘total allocation effect’ of their report, given the opponents’.

Canonical transfers generalize several mechanisms: if \( d \) is the efficient allocation rule, then \( t^* \) coincides with the VCG transfers; in auction environments, it specializes to the incentive compatible auction of Myerson (1981) for private values and Li (2013) and Roughgarden and Talgam-Cohen (2013) for interdependent values. Proposition 2 below shows that the canonical transfers characterize the direct mechanisms that achieve belief-free full implementation. The proof of this result is based on the following Lemma, which generalizes analogous results for the above mentioned special cases. We report it here because it has intrinsic interest from the viewpoint of partial implementation (proofs are in Appendix C):

Lemma 1 Suppose that \( \mathcal{M} = (d, t) \) is EPIC and \( t \) is differentiable. Then, for every \( i \) and for every \( m \), there exists a function \( \tau_i : \Theta_{-i} \rightarrow \mathbb{R} \) such that \( t_i^* (m) = t_i^* (m) + \tau_i (m_{-i}) \).

Proposition 2 Allocation rule \( d \) is belief-free implementable by a differentiable direct mechanism if and only if the canonical mechanism is belief-free truthfully implementable.

In many environments of economic interest (e.g., environments with ‘single crossing’ preferences, as in Section 5) the canonical mechanism induces a nice game. Hence, combining our results,
it follows that if in such environments ex-post incentive compatibility is possible, full implementation can only fail due to the canonical mechanism inducing overly strong strategic externalities.

We provide next a measure of such strategic externalities. For any \( i \in I \), let \( V_i : \Theta \times \Theta \to \mathbb{R} \) be such that for any \((m, \theta) \in \Theta \times \Theta\):

\[
V_i(m, \theta) := \left( \frac{\partial v_i(d(m), \theta)}{\partial d} - \frac{\partial v_i(d(m), m)}{\partial d} \right) \frac{\partial d(m)}{\partial \theta_i}.
\]

For every \( i \in I \), define the ‘contractivity gap’ as:

\[
CG_i := \max_{\theta, m} \left( \sum_{j \neq i} \left| \frac{\partial V_i(m, \theta)}{\partial m_j} \right| - \left| \frac{\partial V_i(m, \theta)}{\partial m_i} \right| \right).
\]

**Corollary 1** Suppose that the canonical mechanism is nice. Then: If the allocation rule is EPIC but not belief-free fully implementable, then \( CG_i > 0 \) for some \( i \).

To understand this result, notice that \( V_i(m, \theta) \) is nothing but the derivative of the ex-post payoff function of the canonical mechanism with respect to \( i \)'s type, evaluated at state \( \theta \), when the reported profile is \( m \). The ‘contractivity gap’ therefore measures the maximal difference between the opponents’ ability to jointly affect this derivative and player \( i \)'s own effect, evaluated across all possible combinations of states and reports. Hence, \( CG_i < 0 \) means that \( i \)'s own effect on the first-order condition of the canonical mechanism always dominates the combined strategic externalities at all states and reports. The result then follows from Theorem 2.

### 4.2 Full Implementation via Moment Conditions

In environments in which ex-post incentive compatibility is possible and the canonical mechanism is nice, such as environments with single-crossing preferences (cf. Section 5), failure to achieve belief-free implementation is due to the existence of positive contractivity gaps. In these cases, information about beliefs may be useful to restore the contraction property for full implementation. In general, however, ex-post incentive compatibility may also be problematic.\(^{15}\) In that case, information about beliefs should be used to ensure both contractiveness and incentive compatibility.

Our analysis exploits an important kind of information about beliefs, represented by ‘moment conditions’. Moment conditions arise naturally in many problems of mechanism design, and are particularly useful for the design of transfers for full implementaiton.

#### 4.2.1 Information about Beliefs: Moment Conditions

**Definition 7** A \( B \)-consistent moment condition is defined by a collection of functions \( \rho = (G_i, f_i)_{i \in I} \) s.t. for every \( i \in I \), \( G_i : \Theta_{-i} \to \mathbb{R} \) and \( f_i : \Theta_i \to \mathbb{R} \) are twice continuously differentiable and for every \( i, \theta_i \) and \( b_i \in B_{\theta_i} \):

\[
\int_{\Theta_{-i}} G_i(\theta_{-i}) \cdot db_i = f_i(\theta_i).
\]

We let \( \varphi(B) \) denote the set of \( B \)-consistent moment conditions.

\(^{15}\) That the very notion of EPIC is too restrictive is a well-known criticism to the belief-free approach. See, for instance, Jehiel et al. (2010).
In words, a $\mathcal{B}$-consistent moment condition represents the idea that, given $\mathcal{B}$, it is common belief that agent $i$’s conditional expectation (conditional on $\theta_i$) of some moment $G_i(\theta_{-i})$ of the opponents’ types is described by some function $f_i$ of $i$’s type. Moment conditions are natural objects in a variety of settings. The following examples illustrate how moment conditions are implicit in standard models.

**Example 3 (Fundamental Value Models)** Consider a model in which types can be decomposed into a fundamental component and a noise component, i.e. $\theta_i = \theta_0 + \varepsilon_i$ where $\theta_0$ is drawn from a normal distribution and $\varepsilon_i$’s are i.i.d. across agents and independent of $\theta_0$ (cf. Rostek and Weretka (2012)). This model entails many moment conditions. For example, irrespective of further details about the involved distributions, it is common knowledge in this model that $\mathbb{E}(\theta_i - \theta_k|\theta_i) = 0$ for any $\theta_i$ and $l, k \neq i$. This is represented by setting $G_i(\theta_{-i}) = \theta_k - \theta_l$ for some $l, k \neq i$ and $f_i(\theta_i) = 0$ for any $\theta_i$. Examples for such information models include financial models with intrinsic values (e.g., Grossman and Stiglitz (1980) and Hellwig (1980)) and common value auctions.

**Example 4 (Unobserved Heterogeneity)** Suppose that types $\theta_i$ are i.i.d. draws from a distribution $F_\eta$, where $\eta$ is a parameter drawn from some distribution $H$. The realization of $\eta$ is observed by the agents but not by the designer (e.g., Aradillas-Lopez et al., 2013). Then, the moment condition with $G_i(\theta_{-i}) = \theta_k - \theta_l$ and $f_i(\theta_i) = 0$ holds in this environment, regardless of the specification of $F$ and $H$.

**Example 5 (Spatial Values)** Consider an environment with two distinct groups of agents (e.g., by geographic location, technology, etc.). Types for the groups are drawn from distribution $F$, with mean $\mathbb{E}_F$. Agents are independently assigned to group $1$ with probability $p$. Agents inherit the type of the group. An agent’s group and type are his private information (cf. Ausubel and Baranov (2010)). This information structure admits the moment equation $\mathbb{E}(\theta_j|\theta_i) = p(i)\theta_i + (1-p(i))\mathbb{E}_F(v_j)$, where $p(i) = p$ if $i$ belongs to group $1$, and $(1-p)$ otherwise. The moment condition thus obtains setting $G_i(\theta_{-i}) = \theta_j$ for some $j \neq i$ and $f_i(\theta_i) = p(i)\theta_i + (1-p(i))\mathbb{E}_F$.

Moment conditions arise even more naturally in many real-world settings. In these settings, knowledge of the expected value of some conditional moments of the distribution is a much more basic and realistic kind of information than the one typically assumed by the standard approach, which requires common knowledge of the prior distribution. Consider the following examples:

**Example 6 (Uncorrelated types without a prior)** Suppose that the designer has data showing no significant correlations across agents. His information, however, does not include the entire distribution of players’ types, but only some moments of that distribution. In this case, the designer’s information $\mathcal{B}$ itself consists of ‘moment conditions’. For example, if types are uncorrelated, for each $i$ and $j$, and for every $\theta_i$, we have that $\mathbb{E}(\theta_j|\theta_i) = y_j$ for some $y_j \in \mathbb{R}$. In this case, $G_{-i}(\theta_{-i}) = \theta_j$ and $f_i(\theta_i) = y_j$.

**Example 7 (Estimation-based Conditions)** Consider a situation in which past data facilitate conditional predictions of agents’ types in the form of linear regressions. Linear regressions are nothing but moment conditions, with $G_{-i}(\theta_{-i}) = \theta_j$ for $j \neq i$ and $f_i(\theta_i) = a_i\theta_i + b_i$. Alternatively, past data may only report aggregate statistics of the distributions, so that only conditional expectations of the average of types may be allowed. In this case, a moment condition is obtained letting $G_i(\theta_{-i}) = \frac{1}{n-1}\sum_{j \neq i} \theta_j$, and so on.
In fact, econometric methods typically provide a description of the environment in terms of conditional moments of the distributions, rather than a single ‘common prior’. In these cases, the very belief-restrictions $\mathcal{B}$ can be specified as the set of all beliefs consistent with such moment conditions, taken as a primitive. Examples 6 and 7 are instances of this kind of situations.

Observe that, in general, any belief restriction entails common knowledge of some moment conditions (that is, $\varrho (\mathcal{B}) \neq \emptyset$ for any $\mathcal{B}$). At a minimum, condition (10) is trivially satisfied for any constant functions $G_i (\cdot) = f_i (\cdot) = \gamma$. If $\mathcal{B}$ is vacuous (i.e., in a belief-free environment) only such trivial moment restrictions are commonly known. In general, the stronger the belief-restrictions (i.e., the smaller the sets $\mathcal{B}$), the richer the set of moment conditions:

**Remark 1** The $\varrho (\cdot)$ correspondence is decreasing: i.e., $\varrho (\mathcal{B}') \subseteq \varrho (\mathcal{B})$ if $\mathcal{B} \subseteq \mathcal{B'}$.

In particular, common prior models are ‘maximal’ in the set of moment conditions they entail: if $\mathcal{B}$ is a common prior model, any collection of functions $G_i : \Theta_i \rightarrow \mathbb{R}$, satisfies $(G_i, f_i)_{i \in I} \in \varrho (\mathcal{B})$ for $f_i^G (\theta_i) := \mathbb{E} (G_i (\theta_i) | \theta_i)$.

For any $\rho = (G_i, f_i)_{i \in I} \in \varrho (\mathcal{B})$, we define the $\Delta$-restrictions derived from $\rho$ as $\Delta^\rho := \left( \{ \Delta^\rho_{\theta_i} \}_{i \in I} \right)$ such that for every $i$ and $\theta_i$,

$$\Delta^\rho_{\theta_i} = \{ \mu \in \Delta (\Theta_i \times M_i) : E^\mu (G_i (\theta_i)) = f_i (\theta_i) \}.$$  

It is easy to verify that $\Delta^\rho$ satisfies Assumption 2 on the $\Delta$-restrictions, hence $\Delta^\rho_i \cap C_i^T \neq \emptyset$ for all $\theta_i$ and $i$. Also, by construction, $\mathcal{B} \subseteq \Delta^\rho$ if $\rho \in \varrho (\mathcal{B})$.

### 4.2.2 Designing Transfers for Uniqueness

In the following we will design transfers and provide conditions for full $\Delta^\rho$-Implementation. Since $\Delta^\mathcal{B} \subseteq \Delta^\rho$ whenever $\rho \in \varrho (\mathcal{B})$, it follows that achieving $\Delta^\rho$-Implementation for some $\rho \in \varrho (\mathcal{B})$ also ensures $\mathcal{B}$-Implementation. Given that $\mathcal{B}$-Implementation is the ultimate goal, and that $\varrho (\mathcal{B})$ in general contains several moment conditions, the key question is to understand which $\rho \in \varrho (\mathcal{B})$ is convenient to choose in the design of the mechanism. Our conditions are formulated precisely to inform this choice. (The proof is in Appendix C.)

**Theorem 3** Allocation rule $d : \Theta \rightarrow X$ is (Fully) $\mathcal{B}$-Implementable by a direct mechanism if there exists a $\mathcal{B}$-consistent moment condition $\rho = (G_i, f_i)_{i \in I} \in \varrho (\mathcal{B})$ that satisfies the following conditions. For all $i$, for all $\theta_i \in \Theta_i$, for all $m_i, m_i' \in M_i$ and for all $\mu \in \Delta^\rho_{\theta_i}$:

1. $\int_{\Theta_i \times M_i} \left| \frac{\partial V_i (m_i', m_i - \theta_i, \theta_i)}{\partial m_i} - f_i' (m_i') \right| d\mu > \sum_{j \neq i} \int_{\Theta_i \times M_i} \left| \frac{\partial V_i (m_i, m_i - \theta_i, \theta_i)}{\partial m_j} + \frac{\partial G_i (m_i)}{\partial m_j} \right| d\mu$.

2. $\int_{\Theta_i \times M_i} \frac{\partial V_i (m_i, m_i - \theta_i, \theta_i)}{\partial m_i} d\mu < f_i' (m_i)$.

Moreover, for $\rho = (G_i, f_i)_{i \in I} \in \varrho (\mathcal{B})$ that satisfies the two conditions, the following transfers guarantee Full $\Delta^\rho$-Implementation:

$$t^\rho_i (m) = t^*_i (m) + G_i (m_i - m_i') + \int_{m_i'}^{m_i} f_i (s) \, ds.$$  

(11)
We also provide a stronger, ‘ex-post’ version of these conditions, which is often easier to check in applications:

**Remark 2** The conditions of Theorem 3 are satisfied if for all $i$, for all $\theta \in \Theta$, for all $m_{i-} \in M$ and for all $m_i, m_i' \in M_i$:

1. \[ \left| \frac{\partial V_i(m_i', m_{i-}, \theta)}{\partial m_i} - f_i'(m_i) \right| > \sum_{j \neq i} \left| \frac{\partial V_i(m_i, m_{i-}, \theta)}{\partial m_j} + \frac{\partial G_i(m_{i-})}{\partial m_j} \right|, \]

2. \[ \frac{\partial V_i(m, \theta)}{\partial m_i} < f_i'(m_i) \]

Theorem 3 states two properties of moment conditions that are useful to guarantee full implementation. To understand what these are, let us consider the stronger versions stated in Remark 2. First, notice that if the contractivity gap (9) is negative for all $i$, then Condition 1 is satisfied by any trivial moment condition, in which $f_i$ and $G_i$ are constant functions. Since such trivial moment conditions are consistent with any belief restrictions, full implementation is guaranteed by the canonical mechanism in the belief-free sense. Now, suppose that the contractivity gap is positive for some agent. Condition 1 clarifies which properties of beliefs can be used to create a wedge between the preference interdependence and the strategic externalities: a moment condition in which the derivative of $f_i$ has the opposite sign of $\partial V_i / \partial m_i$ can be used to increase the ‘own effect’; the ‘external effects’ instead can be weakened by moment functions $G_i$ with derivatives that contrast the externality in the canonical mechanism. Condition 2 instead requires that the ‘own effect’ in the canonical mechanism is bounded above by the derivative of the $f_i$ function.

To gain further insights on how these conditions contribute to the full implementation result, it is useful to consider the transfers that guarantee full implementation (eq. 11). With these transfers, the first order derivative of $\theta_i$‘s expected payoff, given $\mu \in \Delta (\Theta_{-i} \times M_{-i})$, is:

\[ \frac{\partial \text{EU}_\theta^\mu (m_i)}{\partial m_i} = \int_{\Theta_{-i} \times M_{-i}} \left( \frac{\partial v_i (d(m), \theta)}{\partial d} - \frac{\partial v_i (d(m), m)}{\partial d} \right) \frac{\partial d(m)}{\partial m_i} + G_i (m_{-i}) - f_i (m_i) \ d\mu. \]

This shows that for any truthtelling conjecture $\mu \in \Delta_{\theta_i} \cap C^T_i$, the report $m_i = \theta_i$ is an extremal point. This does not necessarily result in $\Delta^\theta$-IC, as that depends on the second order conditions as well. Condition 2 in Theorem 3, however, guarantees that the ensuing mechanism is nice, which ensures that the second order conditions are satisfied. This mechanism therefore is $\Delta^\theta$-IC (hence $B$-IC). Full implementation follows from the fact that Condition 1 in Theorem 3 also guarantees the Self-Determination condition of Theorem 2.

Theorem 3 is constructive in the sense that it pins down a precise design principle: the designer shall start out with the canonical transfers, and then add a new term which is based on suitable moment-conditions. ‘Suitable’ here means that the term added to the canonical transfers ought to guarantee niceness of the mechanism and reduce the strategic externalities. We illustrate the versatility of this general principle in the next section.

## 5 Applications: Single-Crossing Environments

In this Section we illustrate how the general results of Theorem 3 can be applied to special cases of economic interest, under different assumptions on agents’ beliefs. An important class of envi-
environments are those that satisfy the following single-crossing condition (SCC.1): for every $i \in I$, and $(x, \theta)$, $\partial^2 v_i (x, \theta) / \partial x \partial \theta_i > 0$.

The next lemma generalizes standard results on ex-post (partial) implementation:

**Lemma 2** In environments that satisfy (SCC), the canonical mechanism is EPIC if and only if the allocation rule is strictly increasing: $\partial d (\theta) / \partial \theta_i > 0$ for every $\theta$ and every $i$.

**Proof.** (See Appendix D)

Joint with Corollary 1, Lemma 2 implies that in SCC-environments with increasing allocation rules, failure to achieve belief-free full implementation is possible only if the canonical mechanism is not nice or due to positive contractivity gaps. In those cases, information about beliefs may be used to achieve full implementation. These environments therefore present an interesting trade-off between the robustness of the partial implementation result, obtained by the canonical mechanism in a belief-free sense, and the possibility of achieving full implementation: the mechanism that achieves the latter will necessarily exploit information about beliefs, and therefore reduce the robustness of the partial implementation result.

To simplify the analysis, we consider quadratic SCC-environments:

**Definition 8** An ‘SCC-environment’ satisfies the following conditions: (SCC.0) $0 < \partial d (\theta) / \partial \theta_i < \infty$ for every $\theta$ and every $i$ and (SCC.1) $0 < \partial^2 v_i (x, \theta) / \partial x \partial \theta_i < \infty$ for all $i, x$ and $\theta$. An environment is ‘quadratic’ if (Q.1) for all $i, j, k \in I$ and all $(x, \theta) \in X \times \Theta$, $\partial^3 v_i (x, \theta) / \partial x \partial \theta_j \partial \theta_k = 0$ and (Q.2) $\partial^2 d (\theta) / \partial \theta_i \partial \theta_j = 0$ for all $i, j \in I$ and $\theta \in \Theta$.

Under these assumptions, for any $i, j \in I$ and $\theta, m \in \Theta$:

$$\frac{\partial V_i (m, \theta)}{\partial m_j} = - \left( \frac{\partial^2 v_i}{\partial x \partial \theta_i} (d (m), m) \right) \frac{\partial d (m)}{\partial \theta_j}.$$ 

Furthermore, for $j = i$, we have $\partial V_i (m, \theta) / \partial m_j < 1$, hence the canonical mechanism is nice.

Conditions (SCC.0) and (SCC.1) are standard assumptions for general SCC-environments (cf. Lemma 2). Conditions (Q.1) and (Q.2) are satisfied, for instance, by environments in which agents’ valuations are quadratic and the allocation rule maximizes a linear functional of agents’ valuations (the efficient rule would be a special case). While rather special in the context of our paper, these assumptions are extremely common in the applied literature on unit demand auctions, divisible good auctions, finance, etc. Conditions (Q.1) and (Q.2), however, are not essential to our analysis, and can be relaxed (see Section 5.3).

### 5.1 Common Prior Models

#### 5.1.1 Independent Types

In an independent common prior model, for any $G_i : \Theta \rightarrow \mathbb{R}$, the condition $\mathbb{E} (G_i (\theta_{-i}) | \theta_i) = f_i (\theta_i)$ holds true with $f_i : \Theta_i \rightarrow \mathbb{R}$ s.t. $f_i' = 0$ (just by the definition of independence). Hence, since $G_i$ can be chosen freely in common prior models, independence leaves us huge leeway in manipulating the external effects on the RHS of Condition 1 of Theorem 3, without affecting the LHS. The ex-post condition of Remark 2 therefore can be rewritten as:

$$\frac{\partial V_i (m, \theta)}{\partial m_j} = - \left( \frac{\partial^2 v_i}{\partial x \partial \theta_i} (d (m), m) \right) \frac{\partial d (m)}{\partial \theta_j}. $$
\[
\min_{m_i \in M_i} \left| \left( \frac{\partial^2 v_i}{\partial x \partial \theta_j} (d(m), m) \right) \right| > \sum_{j \neq i} \left| \left( -\frac{\partial^2 v_i}{\partial x \partial \theta_j} (d(m), m) \right) + \frac{\partial G_i(m_{-i})}{\partial m_j} \frac{\partial d(m)}{\partial \theta_i} \right| \tag{12}
\]

Condition (Q.1) in Definition 8 guarantees that for any \( i \) and \( j \in I \), there exists \( K_{ij} \in \mathbb{R} \) such that \( -\frac{\partial^2 v_i}{\partial x \partial \theta_j} (d(m), m) = K_{ij} \) for all \( m \). Inequality (12) therefore simplifies to:

\[
|K_{ii}| > \sum_{j \neq i} \left| K_{ij} + \frac{\partial G_i(m_{-i})}{\partial m_j} \frac{\partial d(m)}{\partial \theta_i} \right|. \tag{13}
\]

Notice that in this case we can choose the moment condition to completely neutralize the strategic externalities, setting the RHS of (13) equal to zero. That is, if \( \tilde{G}_i \) is chosen so that

\[
\frac{\partial \tilde{G}_i}{\partial m_j} (m_{-i}) = -K_{ij} \frac{\partial d}{\partial \theta_i} (m) \text{ for all } m \text{ and } j \neq i, \tag{14}
\]

or

\[
\tilde{G}_i (m_{-i}) = -\sum_{j \neq i} K_{ij} \frac{\partial d}{\partial \theta_i} (m) \cdot m_j. \tag{15}
\]

(Equations (14) and (15) are well-defined because (Q.2) guarantees that \( \frac{\partial d}{\partial \theta_i} (m_i, m_{-i}) \) is constant in \( m_i \). Hence, the following Proposition holds:

**Proposition 3** Full implementation is always possible in quadratic SCC-environments with independent common prior. In particular, let \( B \) be an independent common prior model. For any \( i \in I \), let \( \tilde{G}_i : \Theta_{-i} \rightarrow \mathbb{R} \) be defined as in (15) and let \( \tilde{f}_i (\theta_i) := \mathbb{E} (\tilde{G}_i (\theta_{-i}) | \theta_i) \). Then the transfers

\[
t_i (m) = t_i^C (m) + \tilde{G}_i (m_{-i}) m_i - \int_{m_i}^{m_i} \tilde{f}_i (s_i) ds_i \tag{16}
\]

ensure \( \Delta^\rho \)-DS Implementation. Since \( \rho = (\tilde{f}_i, \tilde{G}_i)_{i \in I} \in \wp (B) \), full \( B \)-Implementation follows.

To understand the logic of the mechanism, first notice that the function \( \tilde{G}_i (m_{-i}) \) constructed above is nothing but a measure of the strategic externality that other players impose on \( i \). The transfers in (16) therefore are such that, starting from the canonical mechanism, player \( i \) is compensated for the total strategic externality he is subject to. The last term in (16) is nothing but the expected value of such compensation, when \( i \) reports \( m_i \). This term is added to prevent the agent from misreporting his type, in order to inflate the implied compensation for the strategic externality. Hence, the first term eliminates the strategic externalities, and second ensures incentive compatibility. Full Implementation follows.

---

\(^{16}\)Substituting for the constant \( K_{ij} = -\frac{\partial^2 v_i}{\partial x \partial \theta_j} (d(m), m) \) in (15), we obtain:

\[
\tilde{G}_i (m_{-i}) = \sum_{j \neq i} \left( \frac{\partial^2 v_i}{\partial d \partial \theta_j} (d(m), m) \cdot m_j \right) \cdot \frac{\partial d}{\partial \theta_i} (m).
\]

The term in parenthesis represents the effect of \( j \)'s report on \( i \)'s marginal utility for the public good, and is multiplied by the impact of \( i \)'s report on the allocation. Overall, this is the total strategic externality that player \( i \) is subject to, for each increment of his own report.
5.1.2 Affiliated Types

Suppose that $i = j$ for all $i$ and $j$, and that types are affiliated (Milgrom and Weber, 1982). Under the maintained assumptions for quadratic SCC-environments, and if valuations are supermodular (that is, for all $i,j$ and $x$ and $\theta$, $0 < \partial^2 v_i(x,\theta)/\partial x \partial \theta_i < \infty$ (SCC.2)), the moment functions $\bar{G}_i : \Theta_{-i} \to \mathbb{R}$ defined in (15) are such that $\partial (\bar{G}_i(m))/\partial m_j > 0$ for all $m$ and $j \neq i$. If types are affiliated, Theorem 5 in Milgrom and Weber (1982) implies that $E(\bar{G}_i(\theta_{-i}) | \theta_i)$ is an increasing function of $\theta_i$. Hence, letting $\bar{f}_i(\cdot) \equiv E(\bar{G}_i(\theta_{-i}) | \cdot)$, the moment condition $\rho = (\bar{G}_i, \bar{f}_i) \in \varrho(\mathcal{B})$ satisfies $\bar{f}_i > 0$ for all $i$. By construction, $\bar{G}_i$ is such that the RHS of Condition 1 in Theorem 3 is equal to zero. Since $\bar{f}_i > 0$, SCC implies that the LHS of the same condition is (strictly) positive. The condition for full $\Delta^\rho$-Implementation therefore is satisfied, which implies the following:

**Proposition 4** Full implementation is always possible in quadratic SCC-environments with supermodular valuations and affiliated types. In particular, for any $i \in I$, let $\bar{G}_i : \Theta_{-i} \to \mathbb{R}$ be defined as in (15) and $\bar{f}_i : \Theta_{-i} \to \mathbb{R}$ be s.t. $\bar{f}_i(\theta_i) \equiv E(\bar{G}_i(\theta_{-i}) | \theta_i)$ for each $\theta_i$. Then the transfers in (11) with $\rho = (\bar{f}_i, \bar{G}_i)_{i \in I}$ ensure $\Delta^\rho$-DS Implementation. Since $\bar{f}_i > 0$, SCC implies that the LHS of the same condition is (strictly) positive. The condition for full $\Delta^\rho$-Implementation therefore is satisfied, which implies the following:

5.1.3 Equivalence of EPIC and DS-Implementation

The construction above can also be used to derive an interesting equivalence between ex-post and (interim) dominant strategy incentive compatibility (iDSIC):

**Proposition 5** Under assumptions Q.1-2 and SCC.2, in a common prior environment with independently distributed or affiliated types, an allocation function is EPIC-Implementable if and only if it is interim DS-implementable.

**Proof.** (See Appendix D.)

The proof of this result is very simple. First, we show that an allocation rule is iDSIC only if it is increasing. The ‘only if’ part of the proposition then follows immediately from Lemma 2. The ‘if’ direction follows from the discussion above: if the allocation rule is EPIC, Lemma 2 implies that it is increasing, hence condition SCC.0 is satisfied. Propositions 3 and 4 in turn imply that the allocation rule is iDSIC.

This result is somewhat related to an important result by Manelli and Vincent (2010, MV), recently generalized by Gershkov et al. (2013). MV’s results show that, in Bayesian environments with private values, for any (interim) Incentive Compatible mechanism there is an ‘equivalent’ mechanism that is Dominant Strategy Incentive Compatible. Given the restriction to private values, one way of interpreting this result is as an equivalence between ‘partial’ and ‘full’ implementation in direct mechanisms. From this viewpoint, Proposition 5 can be seen as a generalization of that insight to Bayesian environments with interdependent values.\(^{17}\) We should point out, however, that MV’s notion of equivalence is different from ours. In particular, MV define two mechanisms as ‘equivalent’ if they deliver the same interim expected utilities for all agents and the same ex-ante expected social surplus. Here instead we maintain the traditional notion of equivalence, which requires that the mechanisms induce the same ex-post allocation. (As shown by Gershkov et al.

\(^{17}\)We are grateful to Stephen Morris for this insight.
(2013), equivalence results à la MV do not extend beyond environments with linear utilities and independent types.)

5.2 Moments Conditions without a Prior

In real-world problems of mechanism design, the designer’s information typically does not take the form of a common prior distribution on agents’ types. For instance, when the designer’s information is based on econometric estimates, the belief restrictions $B$ are naturally represented directly in terms of a set of moment conditions (cf. Section 4.2). In this section we show how Theorem 3 can be used in these non-Bayesian settings as well.

For instance, suppose that the only information available to the designer concerns the conditional averages $\mathbb{E} (\theta_j | \theta_i)$. For each $i, j$, let $\varphi_{ij} : \mathcal{L}_i \to \Theta_j$ be such that, for each $\theta_i \in \Theta_i$, $\varphi_{ij} (\theta_i) := \mathbb{E} (\theta_j | \theta_i)$. For simplicity, assume that these functions $\varphi_{ij}$ are differentiable. Then, the designer’s information is represented by belief restrictions $B_{\text{ave}} = (B_{\text{ave}}^i)_{\theta_i \in \Theta_i} \subseteq B^i = (B^i_{\beta})_{\beta \in \Delta (\Theta_i)}$ such that $B_{\text{ave}}^i = \{ \beta \in \Delta (\Theta_i) : \mathbb{E}_\beta (\theta_j) = \varphi_{ij} (\theta_i) \}$, for each $i \in I$ and $\theta_i \in \Theta_i$.

For any affine transformation $\tilde{\gamma}_i : \Theta_{-i} \to \mathbb{R}$ of $i$’s opponents’ types, let $\tilde{\varphi}_i : \Theta_i \to \mathbb{R}$ be such that $\tilde{\varphi}_i (\theta_i) := \tilde{\gamma}_i ((\varphi_{ij} (\theta_i))_{j \in (1 \setminus i)})$. Because of the linearity of the $\mathbb{E} (\cdot)$ operator, it is easy to see that the collection $\{(\tilde{\gamma}_i, \tilde{\varphi}_i)_{i \in I}\}$ defines a $B_{\text{ave}}$-consistent moment condition. Next notice that, in quadratic SCC environments, the function $G_i : \Theta_{-i} \to \mathbb{R}$ (eq. 15) is indeed linear, and increasing if (SCC.2) holds. Hence, letting $\tilde{G}_i \equiv \tilde{\gamma}_i$ and $\tilde{f}_i \equiv \tilde{\varphi}_i$ we have that $\rho = (\tilde{G}_i, \tilde{f}_i)_{i \in I} \in \varrho (B_{\text{ave}})$, and $\tilde{f}_i$ is non-decreasing if so are the functions $\varphi_{ij}$. Then, the next result follows from Theorem 3 for the same reasons as Proposition 4:

**Proposition 6** Let $B_{\text{ave}}$ be such that, for each $i, j$, the functions $\varphi_{ij}$ are non-decreasing. Then, in a quadratic SCC environment with supernormal valuations (SCC.2), the mechanism defined in (11) with $\rho = (\tilde{G}_i, \tilde{f}_i)_{i \in I}$ and $\tilde{G}_i$ defined as in (15), ensures $\Delta^0$-DS Implementation. Since $\rho = (\tilde{G}_i, \tilde{f}_i)_{i \in I} \in \varrho (B_{\text{ave}})$, $B_{\text{ave}}$-DS Implementation follows.

5.3 Extensions

The logic of Propositions 3, 4 and 5 extends well beyond the cases of common prior models with independent or affiliated types. To see this, notice that for $\tilde{G}_i : \Theta_{-i} \to \mathbb{R}$ defined in eq. (15), the maintained assumptions for quadratic SCC-environments guarantee that the RHS of Condition 1 in Theorem 3 is equal to zero. Affiliation or independence further guarantee that the conditional moment $\mathbb{E} (\tilde{G}_i (\theta_{-i}) | \theta_i)$ is (weakly) increasing in $\theta_i$, hence the moment condition $\rho = (\tilde{f}_i, \tilde{G}_i)_{i \in I}$ can be used with no risk of upsetting the LHS of that condition. This argument, however, remains valid whenever $\mathbb{E} (\tilde{G}_i (\theta_{-i}) | m_i) < \frac{\partial \mathbb{E} (m, \theta)}{\partial m_i}$ for all $m$, which ensures that both conditions for contractions are still satisfied by $\rho = (\tilde{f}_i, \tilde{G}_i)_{i \in I}$. In Proposition 6, the assumption that functions $\varphi_{ij}$ are non-decreasing plays the same role as the assumptions of independence and affiliation in the common prior models, and can be weakened similarly.

Conditions (Q.1) and (Q.2) may also be weakened in Propositions 3, 4 and 6. In the argument above, we used these conditions to ensure that $\partial V_i / \partial m_j < 0$ and that $\tilde{G}_i$ could be designed to completely neutralize the strategic externalities of the canonical mechanism. Clearly, $\partial V_i / \partial m_j < 0$ can be guaranteed by weaker conditions. If (Q.2) is violated, however, then we may not be able to choose $\tilde{G}_i$ to completely offset the strategic externalities: if $\frac{\partial V_i'}{\partial \theta_i} (m)$ varies with $m_i$, the same $\tilde{G}_i (\cdot)$
cannot set the RHS identically equal to zero. But if $|\partial^2 d/\partial \theta_i \partial \theta_j|$ and the variations of the valuations’ concavity are small relative to $|\partial V_i/\partial m_i|$, then $\tilde{G}_i$ can be chosen so that the RHS is bounded above by $|\partial V_i/\partial m_i|$, and the argument goes through essentially unchanged. The only difference is that Full $\Delta^R$-Implementation would not be obtained with one round of $\Delta$-rationalizability only. That is, we would have Full $\Delta$-Implementation, but not $\Delta$-DS Implementation.

6 $\Delta$-Contractivity and Monotonicity

To compare $\Delta$-contractivity to the related notions introduced in the implementation literature, it is useful to reformulate some of our concepts in the context of that literature, which typically considers general (non quasilinear) environments. In such frameworks, the space of outcomes is given by an abstract set $A$. agents’ preferences are defined as $u_i : A \times \Theta \rightarrow \mathbb{R}$, and social choice functions are $f : \Theta \rightarrow A$. A mechanism is a tuple $((M_i)_{i \in T}, g)$ s.t. $g : M \rightarrow A$, and it is direct if $g = f$ and $M_i = \Theta_i$ for all $i$. To see the connection with our framework, let $A = X \times \mathbb{R}^n$. Given $d : \Theta \rightarrow X$, any direct mechanism $M = (d, t)$ in our setting induces a function $f^M : \Theta \rightarrow A$ in the general setting: $f^M$ is such that $f^M(\theta) = (d(\theta), t(\theta))$ for each $\theta \in \Theta$. That $M$ truthfully $\Delta$-implements $d$ in our model therefore is the same as saying that $f^M$ is truthfully $\Delta$-Implemented by a direct mechanism in the general setting. For the sake of the argument, we will slightly abuse notation as follows: for any $\alpha \in \Delta(A)$, we write $u_i(\alpha, \theta)$ instead of $\int u_i(\alpha, \theta) \, d\alpha(\alpha)$ and for any $f : \Theta \rightarrow A$ and $\nu_i \in \Delta(\Theta_i)$ we write $f(\nu_i, \theta_{\neg i})$ for the measure over $A$ induced by $\nu_i$, given $\theta_{\neg i}$.

Most of the literature on full implementation does not impose the restriction that it should be achieved through direct mechanism. An important exception is Bergemann and Morris (2009a), who consider rationalizable implementation through direct mechanisms in belief-free environments, under some restrictions on agents’ preferences $u_i : A \times \Theta \rightarrow \mathbb{R}$ (in particular, preferences are required to admit monotone aggregators and satisfy a strict single crossing property). As already mentioned, belief-free environments in our model correspond to the special case in which $\Delta = \Delta^{BF}$. It is easy to show that, if preferences satisfy the restrictions in Bergemann and Morris’ (2009a), $\Delta^{BF}$-contractivity is equivalent to Bergemann and Morris’ (2009) contraction property.

Another special case of interest is when $\Delta = \Delta^{B}$ and $B$ is a Bayesian model (i.e., $B_{\theta_i} = \{b_{\theta_i}\}$ for every $\theta_i$). As shown by Battigalli et al. (2011), in that case $\Delta$-Rationalizability coincides with interim correlated rationalizability (ICR, Dekel et al. (2007)). Full $\Delta$-Implementation therefore coincides with ICR-implementation, studied by Oury and Tercieux (2012). Oury and Tercieux (2012) show that a condition known as ‘ICR-monotonicity’ is a necessary and (with minimal extra assumptions) sufficient for ICR-Implementation. ICR-monotonicity is a Bayesian version of a notion introduced by Bergemann and Morris (2011) in the ex-post environment, and strengthens Postlewaite and Schmeidler (1986) and Jackson’s (1991) Bayesian monotonicity to account for the weaker solution concept. For the sake of the argument, we recall both Jackson’s and Oury and Tercieux’s notions of monotonicity.

**Definition 9** A deception is a collection of mappings $\Sigma = (\Sigma_i)_{i \in T}$, where each $\Sigma_i : \Theta_i \rightarrow 2^{\Theta_i}$. A deception $\Sigma$ is acceptable for $f : \Theta \rightarrow A$ if, for all $\theta \in \Theta$, and for all $\theta' \in \Sigma(\theta)$, $f(\theta) = f(\theta')$. A deception $\Sigma$ is unacceptable if it is not acceptable.
Definition 10 (Bayesian Monotonicity) The SCF $f : \Theta \to A$ is Bayesian monotonic if, for every unacceptable single-valued deception $\sigma : \Theta \to \Theta$, there exist $i, \theta_i \in \Theta_i$ and $y : \Theta \to A$ such that

$$\int u_i ((y \circ \sigma)(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \, d\theta_i(\theta_{-i}) > \int u_i ((f \circ \sigma)(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \, d\theta_i(\theta_{-i})$$

while

$$\int u_i (y(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \, d\theta_i(\theta_{-i}) \leq \int u_i (f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \, d\theta_i(\theta_{-i}).$$

Similar to Maskin’s monotonicity, Bayesian monotonicity ensures elimination of undesirable equilibria. For instance, suppose that some unacceptable deception $\beta$ is played, so that $f \circ \beta \neq f$. Under Bayesian monotonicity, there would be at least one player who, given $\beta$, has the incentives to signal the deception and induce an outcome according to function $y$ rather than $f$. The second condition ensures that the same player would not have an incentive to induce $y$ if the opponents were reporting truthfully.

Definition 11 (ICR-Monotonicity) The SCF $f : \Theta \to A$ is ICR-monotonic if, for every unacceptable deception $\Sigma$, there exist $i$, $\theta_i \in \Theta_i$ and $\theta'_i \in \Sigma_i(\theta_i)$ such that, for every $\mu_i \in \Delta(\Theta_{-i} \times \Theta_{-i})$ that satisfies the properties: (i) $(\theta_{-i}, \theta'_{-i}) \in \text{supp}(\mu_i) \Rightarrow \theta'_{-i} \in \Sigma_{-i}(\theta_{-i})$; (ii) $\text{mary}_{\theta_{-i}} \mu_i = b_\theta$; there exists $y^* : \Theta_{-i} \to \Delta(A)$ such that

$$\int u_i (y_i^*(\theta'_{-i}), \theta_i, \theta_{-i}) \, d\mu_i > \int u_i (f(\theta'_i, \theta_{-i}), \theta_i, \theta_{-i}) \, d\mu_i \quad (17)$$

while for all $\theta_i \in \Theta_i$,

$$\int u_i (y_i^*(\theta'_{-i}), \theta_i, \theta_{-i}) \, d\theta_i(\theta_{-i}) \leq \int u_i (f(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) \, d\theta_i(\theta_{-i}). \quad (18)$$

ICR-monotonicity extends the idea of Bayesian monotonicity, except that rather than giving players the incentives to signal a deviation from truthful reporting, some player has the incentive to signal sets of non-truthful reports whenever it is the case (17), and only if it is the case (18). The existence of $y^* : \Theta_{-i} \to \Delta(A)$ in the definition ensures that at least one type of some player would find it profitable to signal the deception and choose according to $y^*$ rather than $f$.

In the definitions above, functions $y^*$ and $y$ need not have any relation with the SCF $f$. The reason is that both notions of implementation do not impose any restriction on the feasible mechanisms. Using richer message spaces, and letting the outcome function change from $f$ to the designated $y^*$, then (under typically weak extra assumptions) such monotonicity conditions are also sufficient for the implementation results. If the restriction to direct mechanism is imposed, however, the ‘reward function’ $y^*$ should itself be achievable as part of the direct mechanism, through some player’s unilateral deviation from the candidate deception:

---

In Oury and Tercieux (2012), the reward $y^*$ is required to induce a degenerate lottery over $A$ (that is, $y^* : \Theta_{-i} \to A$). We allow for the possibility of non-degenerate lotteries, as done by Bergemann and Morris (2011). Clearly, this change does not affect the necessity result, as the version with non-degenerate lotteries is weaker. Degenerate lotteries in general make sufficiency arguments easier. Given the restriction to direct mechanisms, however, the weaker version with non-degenerate lotteries is also sufficient in our setting.
Definition 12 Function \( y^* : \Theta \rightarrow \Delta (A) \) is a ‘directly feasible’ reward for type \( \theta_i \) if there exists \( \nu_i \in \Delta (\Theta_i) : f (\nu_i, \theta_i) = y^* (\theta_i) \) for all \( \theta_i \in \text{supp} (b_{\theta_i}) \). Let \( Y^*(\theta_i) \subseteq [\Delta (A)]^{|\Theta|-1} \) denote the set of type \( \theta_i \)’s directly feasible rewards.

This notion provides precisely the missing link between ICR-Monotonicity and \( \Delta \)-Contractivity:

**Definition 13 (direct ICR-Monotonicity)** The SCF \( f : \Theta \rightarrow A \) is direct ICR-monotonic if, for every unacceptable deception \( \Sigma \), there exist \( i, \theta_i \in \Theta_i \) and \( \theta'_i \in \Sigma_i (\theta_i) \) such that, for every \( \mu_i \in \Delta (\Theta_i) \times \Theta_i \) that satisfies the properties: (i) \( (\theta_i, \theta'_i) \in \text{supp} (\mu_i) \Rightarrow \theta'_i \in \Sigma_i (\theta_i) \) and (ii) \( \text{marg}_{\Theta_i \times \mu_i} = b_{\theta_i} \); there exists \( y^* \in Y^*(\theta_i) \) that satisfies the inequalities in (17) and (18).

**Proposition 7** Let \( d : \Theta \rightarrow X \) be responsive, \( M = (d, t) \) and \( B \) be such that satisfies \( |B_{\theta_i}| = 1 \) for every \( \theta_i \) and \( i \). Then the following are equivalent:

1. \( M \) is \( \Delta^B \)-IC and \( \Delta^B \)-contractive

2. \( f^M \) satisfies direct ICR-Monotonicity.

**Proof.** (See Appendix D.3).

7 Conclusion

Mechanism design is concerned with identifying conditions under which there exist institutions (or mechanisms) which guarantee socially desirable outcomes through the decentralized interaction of rational agents. The relevance of the theory therefore depends on the nature of the solution concepts adopted to model such interactions, and on the properties of the mechanisms that are used to achieve the results.

In this paper we developed an approach to full implementation based on the solution concept of \( \Delta \)-Rationalizability (Battigalli and Siniscalchi, 2003), which allows us to study implementation under general restrictions on agents’ beliefs. Our approach subsumes as special cases the important notions of belief-free and ICR-implementation (cf. Bergemann and Morris (2009a) and Oury and Tercioux (2012), respectively) as well as accommodate more realistic assumptions on players’ beliefs, intermediate between the ‘belief-free’ and the classical Bayesian benchmarks. We also showed that this approach is convenient to achieve full implementation through mechanisms that have a clear economic interpretation and are as simple as those developed by the partial implementation literature. This is an important innovation on the literature on full implementation, which also has the advantage of providing a bridge between two important branches of the literature which have typically proceeded in parallel.

While largely inspired by the literature on belief-free mechanism design, we departed from it in important ways. In particular, by treating belief restrictions as parameters (an important conceptual innovation of our approach) we showed that often minimal information on agents’ beliefs may suffice to overcome the difficulties of the belief-free approach. This information takes the form of restrictions on simple moments of the distribution of types. ‘Moment conditions’ arise naturally in many economic contexts, and are extremely useful for the design of transfers for full implementation. At the same time, our results show that the methodology developed by the belief-free literature can be extended to address more applied problems of mechanism design, overcoming important limitations of the traditional approach to full implementation.
Another important difference with respect to the belief-free literature is that the main focus thus far has been on identifying conditions under which a given mechanism achieves implementation in a robust sense (e.g., Bergemann and Morris (2009a) or Penta (2011)), offering very little guidance on the design of transfers to fully implement a given objective of the designer. In contrast, we provided an explicit construction of transfers which suggests a general design principle: First, consider the ‘canonical transfers’, which generalize well known necessary conditions for ex-post implementation. Then, modify these transfers adding a belief-based component designed in order to weaken the strategic externalities which may otherwise impair the full implementation result. The resulting mechanism is contractive, and induces truthful revelation as the only rationalizable outcome.

The basic design strategy of modifying baseline transfers adding a belief-dependent component is also shared by recent work by Mathevet (2010) and Mathevet and Taneva (2013). The difference is that here we pursue contractivity of the best reply, rather than supermodularity of the mechanism. From a technical viewpoint, our implementation results are based on the notion of a nice mechanism, which extends the Moulin’s (1981) notion of nice games to incomplete information environments. Our uniqueness results for nice games with incomplete information (further developed in Ollár and Penta (2014)) are useful game theoretic concepts that extend beyond mechanism design. (For instance, they may be used to extend the analysis of Weinstein and Yildiz (2011) beyond the complete information case). Besides ensuring uniqueness, the contractivity of the best replies which we pursued has other important properties, such as (1) small sensitivity to small perturbations of the moment conditions, (2) small sensitivity to lower levels of common belief in rationality, (3) small sensitivity to small misspecifications of the domain. All these properties point to a broader concept of robustness outlined in the Wilson doctrine. The first point, in particular, is important because it ensures that small mispecifications of the moment conditions result in an outcome that is proportionately close to the desired one. It is important to note that ‘closeness’ here is in terms of the natural allocation space, as opposed to the probabilistic notion of the virtual implementation literature. Though beyond the scope of this paper, this point also suggests that the fundamental logic of our construction can also be extended to moment conditions with inequalities. Further developing these results is part of ongoing research.

The weakness of the solution concept and the generality of our framework, both in terms of preferences and belief restrictions, ensure that our results accommodate several important special cases. For instance, for the case in which the belief restrictions are derived from a standard type space, our notion of implementation coincides with ICR-Implementation. Since most of the classical literature on implementation focuses on Bayesian environments, this is an important special case of our analysis. Clearly, since ICR is a superset of Bayes Nash equilibrium (BNE), our sufficient conditions also guarantee full implementation in BNE (e.g., Jackson (1991)), as well as an explicit design of the transfer scheme that achieves that. Also, Oury and Tercieux (2012) have shown an important connection between partial and full implementation. Namely, say that a social choice function is ‘continuously partially implemented’ at a given type in the universal type space, if it is incentive compatible for all types in its neighborhood. Building on important work by Weinstein and Yildiz (2007), Oury and Tercieux (2012) show that this is possible if and only if the SCF is fully implemented in ICR. Hence, our sufficient conditions for \(\Delta\)-Implementation are also sufficient for ‘continuous (partial) implementation’.
In Section 5 we discussed some implications of our general results for important special cases, such as environments with single-crossing preferences, with and without common priors. In common prior environments, we provided sufficient conditions for full implementation with independent and correlated types, as well as an equivalence of partially and fully implementable allocation rules that is somewhat reminiscent of the important equivalence result of Manelli and Vincent (2010). In environments with quadratic preferences, our construction indeed ensures that strategic externalities are completely eliminated, thereby achieving dominant strategy implementation. When this is the case, our results have implications for another important topic of recent research: max-min implementation (cf. Wolitzky (2014), or Carroll (2013) for a related setting). Wolitzky (2014), in particular, characterizes max-min implementability in environments with belief-restrictions. Max-min implementability is a natural consequence of our full implementation if the latter is achieved in dominant strategies: since $\Delta$-DS Implementation implies that truth-telling is a best reply to any beliefs consistent with the belief restrictions, it is also a best reply to the ‘worst case’ beliefs, as required by max-min implementation. Thus, indirectly, our results show that moment conditions can be useful for max-min implementation as well.

Finally, we note that our construction sheds some light on at least two novel important research questions. In particular, Deb and Pai (2013) recently argued that an important topic for future research is the design of mechanisms that only use properties of the distribution which can be estimated from previous performances of the mechanism. The moment conditions we use in our design can be thought of precisely as summary statistics of the designer’s data. Our mechanisms therefore satisfy the property advocated by Deb and Pai, and the notions of $\Delta$-Incentive Compatibility and $\Delta$-Implementation provide the analytical framework to address that problem, from the viewpoint of partial and full implementation, respectively. The question raised by Deb and Pai is related to a second one, that is understanding what kind of information on a mechanism’s past performance may be useful for the designer to disclose. This question is extremely relevant for practical mechanism design. For instance, the recent development of online trading platforms has provided the designers of those platforms with a huge quantity of data on the distribution of users’ preferences, strategies, etc. Some of this information is clearly used by the platforms to better target the advertising campaigns, but very little is understood on the potential to use such information to shape the very strategic interaction in such mechanisms. If, as in these examples, the designer of the mechanism has information on the distributions that may not be commonly known by the agents, he or she may decide to publicly disclose part of that information if this could improve the outcome of the mechanism. The conditions we provide to achieve contractivity of the mechanism point at the properties of the moment conditions that can be conveniently made public: once common knowledge of particular moment conditions is established, transfers may be suitably designed to guarantee full implementation.
Appendix

A  Equivalent Approaches to Full $\Delta$-Implementation

We define the set of type spaces consistent with $\mathcal{B}$ as follows:

**Definition 14** A ($\Theta$-based) type space is a tuple $T = (T_i, \hat{\theta}_i, \tau_i)_{i \in I}$ such that $T_i$ is a compact set of player $i$'s types; $\hat{\theta}_i : T_i \rightarrow \Theta_i$ is a measurable function assigning to each type a payoff type, and $\tau_i : T_i \rightarrow \Delta(T_{-i})$ is a belief function. A type space is consistent with belief restrictions $\mathcal{B}$ (or, $\mathcal{B}$-consistent) if the belief functions $(\tau_i)_{i \in I}$ satisfy the following: for every $i \in N$ and for every $t_i \in T_i$, there exists $\beta \in \mathcal{B}_{\hat{\theta}(t_i)}$ such that for any measurable $E \subseteq \Theta_{-i}$,

$$
\tau_i(t_i) \left[ \{ t_{-i} \in T_{-i} : \hat{\theta}_{-i}(t_{-i}) \in E \} \right] = \beta(E).
$$

Equation (19) is a consistency condition, which requires that type $t_i$'s beliefs about the opponents' types are consistent with the belief restrictions $\mathcal{B}$. It can be shown that any hierarchy of beliefs consistent with the $\mathcal{B}$ restrictions can find an implicit representation as a type in an $\mathcal{B}$-consistent type space.

**Proposition 8** Fix an environment $\mathcal{E}$, belief restrictions $\mathcal{B}$ and a direct mechanism $\mathcal{M}$. Let the $\Delta = \Delta^\mathcal{B}$. For every $\theta_i$, $R^\Delta_i(\theta_i)$ characterizes the set of messages that payoff type $\theta_i$ would play in some BNE for some $\mathcal{B}$-consistent type space.

(This Proposition generalizes an analogous result by Battigalli and Siniscalchi's (2003), who proved it for games with finite actions. The argument, however, is essentially the same.)

B  Omitted Proofs From Sections 2 and 3.2

B.1  Proof of Theorem 1

The proposition follows immediately from the following lemmata.

**Lemma 3** Let the $\Delta$-restrictions be non-behavioral. Then: If $d$ responsive and $\Delta$-implementable by a direct mechanism, then it is truthfully $\Delta$-implementable by a strictly $\Delta$-IC mechanism.

**Proof.** Fix the mechanism $\mathcal{M} = (d, t)$, and suppose that it $\Delta$-Implements $d$, and let $R^\Delta$ denote the set of $\Delta$-rationalizable reports in mechanism $\mathcal{M}$. Then: (a) for all $i$ and $\theta_i$, $R^\Delta_i(\theta_i) \neq \emptyset$, and (b) for all $\theta, \theta' \in \Theta$, $\theta' \in R^\Delta(d) \implies d(\theta) = d(\theta')$.

**Step 1:** We show that for any $\theta_i \neq \theta'_i$, $R^\Delta_i(\theta_i) \cap R^\Delta_i(\theta'_i) = \emptyset$. Suppose not, and let $m_i \in R^\Delta_i(\theta_i) \cap R^\Delta_i(\theta'_i)$. By definition of implementation, for any $\sigma_{-i} \in R^\Delta_{-i}$, we must have that $d(m_i, \sigma_{-i}(\theta_i)) = d(\theta_i, \theta_{-i})$ and $d(m_i, \sigma_{-i}(\theta'_i)) = d(\theta'_i, \theta_{-i})$ for any $\theta_{-i}$. But $d(\theta_i, \theta_{-i}) = d(\theta'_i, \theta_{-i})$ for all $\theta_{-i}$ contradicts the fact that $d$ is responsive.

**Step 2:** We show that for any $i$, for any $\theta_i \in \Theta_i$: $|R^\Delta_i(\theta_i)| \leq 1$. Suppose not, then there exist $i \in I$ and $\theta'_i \neq \theta_i \in \Theta_i$ such that $\theta_i, \theta'_i \in R^\Delta_i(\theta'_i)$ and $\theta_i \neq \theta'_i$. For each $i \in I$, let $Y_i := \bigcup_{\theta_i \in \Theta_i} R^\Delta_i(\theta_i)$. Then, by Step 1, for every $i \in I$ there exists an onto function $f_i : Y_i \rightarrow \Theta_i$ such that: (1) $\theta_i \in R^\Delta_i(f_i(\theta_i))$ for any $\theta_i \in Y_i$; (2) for any $\theta \in Y$, $d(\theta) = d(f(\theta))$. Under the absurd
hypothesis, \( \theta_i, \theta'_i \in Y_i \) are such that \( \theta_i \neq \theta'_i \) and \( f_i(\theta_i) = f_i(\theta'_i) \). But then \( d(f_i(\theta_i), f_{-i}(\theta_{-i})) = d(f_i(\theta'_i), f_{-i}(\theta_{-i})) \) for any \( \theta_{-i} \in Y_{-i} \). Using (2), it follows that \( d(\theta_i, \theta_{-i}) = d(\theta'_i, \theta_{-i}) \) for all \( \theta_{-i} \in Y_{-i} \). This would contradict Responsiveness unless there exists \( \theta''_i \in \Theta_{-i} \setminus Y_{-i} : d(\theta_i, \theta''_i) \neq d(\theta'_i, \theta''_i) \). But because \( f_{-i} \) is onto, there exists \( \theta''_{-i} \in Y_{-i} : f_{-i}(\theta''_{-i}) = \theta''_i \), but if \( f_i(\theta_i) = f_i(\theta'_i) \), then \( d(f_i(\theta_i), f_{-i}(\theta''_{-i})) = d(f_i(\theta'_i), f_{-i}(\theta''_{-i})) \), which by (2) implies that \( d(\theta_i, \theta''_{-i}) = d(\theta'_i, \theta''_{-i}) \), a contradiction.

**Step 3:** We show that there exists a mechanism that truthfully \( \Delta \)-implements \( d \). From step 2, for every \( i \in I \) there exists a one-to-one function \( i_i : \Theta_i \to \Theta_i \) such that \( R^{\Delta}_i(\theta_i) = \{i_i(\theta_i)\} \) and \( d(\theta) = d(i(\theta)) \) for each \( \theta \in \Theta_i \), where \( i(\theta) \equiv (i_i(\theta_i))_{i \in I} \). We let \( \iota_i(\Theta_i) = \bigcup_{\theta_i \in \Theta_i} i_i(\theta_i) \). Let \( \tilde{R}^{\Delta} \) denote the set of rationalizable strategies in the mechanism \( \hat{M} \) that is identical to \( M \) except that each \( i \)'s action space is set equal to \( i_i(\Theta_i) \). Clearly, \( \emptyset \neq \tilde{R}^{\Delta}_i \subseteq R^{\Delta}_i \) hence \( \tilde{R}^{\Delta}_i(\theta_i) = \{i_i(\theta_i)\} \) for every \( i \) and \( \theta_i \). Now consider the mechanism \( \tilde{M} = (d, t') \) s.t. \( t' = t \circ \iota \), and let \( \hat{\tilde{R}}^{\Delta} \) denote the \( \Delta \)-rationalizable strategies in \( \hat{\tilde{M}} \). Clearly, this mechanism is strictly \( \Delta \)-IC, hence \( \theta_i \in \tilde{R}^{\Delta}_i(\theta_i) \) for each \( i \) and \( \theta_i \). We will show that, in fact, for any \( k = 0, 1, \ldots, \iota \) \( \hat{\tilde{R}}^{\Delta,k}_i(\theta_i) \subseteq \tilde{R}^{\Delta,k}_i(\theta_i) \), that is \( \theta'_i \in \tilde{R}^{\Delta,k}_i(\theta_i) \) implies \( i_i(\theta'_i) \in \hat{\tilde{R}}^{\Delta,k}_i(\theta_i) \). Once this is proven, it follows that \( \tilde{R}^{\Delta}_i(\theta_i) = \{\theta_i\} \), for if there exists \( \theta'_i \neq \theta_i \) s.t. \( \theta'_i \in \tilde{R}^{\Delta}_i(\theta_i) \), then \( i_i(\theta'_i) \in \hat{\tilde{R}}^{\Delta}_i(\theta_i) \) and \( i_i(\theta'_i) \neq i_i(\theta_i) \), contradicting \( \tilde{R}^{\Delta}_i(\theta_i) = \{i_i(\theta_i)\} \). The proof is by induction. For \( k = 0 \) the condition \( \iota \) \( \hat{\tilde{R}}^{\Delta,k}_i(\theta_i) \subseteq \tilde{R}^{\Delta,k}_i(\theta_i) \) is trivially satisfied. For the inductive step, suppose that the statement is true up to \( k = 1 \). We show that \( i_i \left( \tilde{R}^{\Delta,k}_i(\theta_i) \right) \subseteq \hat{\tilde{R}}^{\Delta,k}_i(\theta_i) \). Let \( \theta'_i \in \tilde{R}^{\Delta,k}_i(\theta_i) \), then there exists \( \mu \in \Delta_{\theta_i} \cap \tilde{R}^{\Delta,k-1}_i \) :

\[
\theta'_i \in \arg \max_{\theta'_i} \int \left[ u_i \left( d(\theta''_{-i}, \theta'_i), \theta_{-i}, \theta_i \right) + t'_i \left( \theta''_{-i}, \theta'_i \right) \right] d\mu(\theta''_{-i}, \theta_{-i})
\]

\[
= \arg \max_{\theta'_i} \int \left[ u_i \left( i(\theta''_{-i}, \theta'_i), \theta_{-i}, \theta_i \right) + \iota_i \left( i(\theta''_{-i}, \theta'_i) \right) \right] d\mu(\theta''_{-i}, \theta_{-i})
\]

Now, let \( \mu \in \Delta(\Theta_{-i} \times \Theta_{-i}) \) s.t. \( \mu(i(\theta), \theta) = \tilde{\mu}(\theta) \). Under the inductive assumption, and if \( \Delta_{\theta_i} \) entails no behavioral restrictions, \( \mu \in \Delta_{\theta_i} \cap \tilde{R}^{\Delta,k-1}_i \). We want to show that

\[
\iota_i \left( \theta'_i \right) \in \arg \max_{\iota_i \left( \theta'_i \right) \in \iota_i \left( \Theta_i \right)} \int \left[ u_i \left( d(\theta''_{-i}, \theta'_i), \theta_{-i}, \theta_i \right) + t_i \left( \theta''_{-i}, \iota_i \left( \theta'_i \right) \right) \right] d\mu(\theta''_{-i}, \theta_{-i})
\]

Suppose not. Then \( \exists \theta''_i \in \iota_i \left( \Theta_i \right) : \)

\[
\int \left[ u_i \left( d(\theta''_{-i}, \theta''_i), \theta_{-i}, \theta_i \right) + t_i \left( \theta''_{-i}, \theta''_i \right) \right] d\mu(\theta''_{-i}, \theta_{-i}) > \int \left[ u_i \left( d(\iota_i(\theta''_i), \theta_{-i}, \theta_i) + t_i \left( \iota_i(\theta''_i) \right) \right) \right] d\mu(\theta''_{-i}, \theta_{-i})
\]

But, letting \( \theta''''_i = \iota_i^{-1}(\theta''_i) \), we can write the two sides of this inequality as follows:

**LHS:** \( \int \left[ u_i \left( d(\theta''_{-i}, \theta''_i), \theta_{-i}, \theta_i \right) + t_i \left( \theta''_{-i}, \theta''_i \right) \right] d\mu(\theta''_{-i}, \theta_{-i}) \)

\( = \int \left[ u_i \left( \iota_i(\theta''_{-i}), \theta_{-i}, \theta_i \right) + t_i \left( \iota_i(\theta''_{-i}) \right) \right] d\tilde{\mu}(\theta''_{-i}, \theta_{-i}) \)

\( = \int \left[ u_i \left( \iota_i(\theta''_{-i}), \theta_{-i}, \theta_i \right) + t'_i \left( \theta''_{-i}, \theta''_i \right) \right] d\tilde{\mu}(\theta''_{-i}, \theta_{-i}) \)

31
RHS: \[
\int \left[ u_i \left( d \left( \theta_{-i}' \mid \theta_i \right), \theta_{-i} \mid \theta_i \right) + t_i \left( \theta_{-i}' \mid \theta_i \right) \right] d\mu \left( \theta_{-i}' \mid \theta_i \right)
\]
\[
= \int \left[ u_i \left( d \left( \theta_{-i}' \mid \theta_i \right), \theta_{-i} \mid \theta_i \right) + t_i \left( \theta_{-i}' \mid \theta_i \right) \right] d\mu \left( \theta_{-i}' \mid \theta_i \right)
\]
\[
= \int \left[ u_i \left( d \left( \theta_{-i}' \mid \theta_i \right), \theta_{-i} \right) + t_i \left( \theta_{-i}' \mid \theta_i \right) \right] d\mu \left( \theta_{-i}' \mid \theta_i \right)
\]

Hence,
\[
\int \left[ u_i \left( d \left( \theta_{-i}' \mid \theta_i \right), \theta_{-i} \right) + t_i \left( \theta_{-i}' \mid \theta_i \right) \right] d\mu \left( \theta_{-i}' \mid \theta_i \right)
\]
\[
> \int \left[ u_i \left( d \left( \theta_{-i}' \mid \theta_i \right), \theta_{-i} \right) + t_i \left( \theta_{-i}' \mid \theta_i \right) \right] d\mu \left( \theta_{-i}' \mid \theta_i \right)
\]

which contradicts 20. Thus, \( \hat{R}^\Delta_i (\theta_i) = \{ \theta_i \} \) for each \( \theta_i \), which is truthful Full \( \Delta \)-Implementation. That \( \hat{M} \) is strictly \( \Delta \)-IC follows trivially. \( \blacksquare \)

**Lemma 4** Under Assumption 2, if \( d \) is responsive and truthfully \( \Delta \)-implementable, then \( \Delta \)-contraction property holds.

**Proof.** Truthful implementation implies that for any \( \theta_i' \in \Theta_i \), \( m_i \neq \theta_i' \), and \( b_i \in \partial \theta_i' \).

\[
EU^\mu_{\theta_i} (b_i) \left( \theta_i' \right) > EU^\mu_{b_i} (m_i) \quad (21)
\]

Fix \( \Sigma \neq \sigma^* \), and let \( k \) be the largest \( k \) such that for every \( i \), \( \theta_i' \in \Sigma_i \left( \theta_i \right) \), \( R^\Delta_i (\theta_i') = \{ \theta_i' \} \subseteq R^\Delta_i (\theta_i) \). Such \( k \) exists, because \( R^\Delta_i (\theta_i) = M_i \) (by definition) and \( R^\Delta_i (\theta_i) = \{ \theta_i \} \) (by truthful implementation) for all \( \theta_i \in \Theta_i \). Hence, \( \exists i, \theta_i, \theta_i' \in \Sigma_i \left( \theta_i \right) \):

\[
\theta_i' \in R^\Delta_i (\theta_i) \quad \text{and} \quad \theta_i' \notin R^\Delta_i (\theta_i),
\]

but this means that for all \( b_i \in \partial \theta_i \) and for all \( \sigma_{-i} \in R^\Delta_{-i} \), there exists \( m_i : EU^\mu_{\theta_i} (b_i, \sigma_{-i}) (m_i) > EU^\mu_{\theta_i} (b_i, \sigma_{-i}) (\theta_i') \). In other words: define the operator \( L \) s.t. \( \mu \mapsto L \left( \mu \right) \) where \( L \left( \mu \right) : M_i \rightarrow \mathbb{R} \) is such that \( L \left( \mu \right) (m_i) = EU^\mu_{\theta_i} (\theta_i') - EU^\mu_{b_i} (m_i) \); then there exists no \( \mu \in \Delta_{\theta_i} \) such that \( L \left( \mu \right) (m_i) \geq 0 \) for all \( m_i \in M_i \). Notice that \( L \) is a linear operator from \( \Delta_{\theta_i} \rightarrow \mathbb{R}^{M_i} \). Since \( \Delta_{\theta_i} \) is closed and convex, the image \( L \left( \Delta_{\theta_i} \right) := \bigcup_{\mu \in \Delta_{\theta_i}} L \left( \mu \right) \) is also closed and convex. Moreover, it is disjoint from the positive orthant of \( \mathbb{R}^{M_i} \). Hence there is a nonzero functional \( \phi \) separating these two convex sets, such that

\[
\phi \left( L \left( \mu \right) \right) < 0 \quad \text{for all} \quad \mu \in \Delta_{\theta_i} \quad \text{and}
\]
\[
\phi \left( T \right) > 0 \quad \text{for all} \quad T \in \mathbb{R}^{M_i}_{++}.
\]

By these two properties, the normalization of \( \phi \) gives a nonnegative probability measure \( \nu_i \in \Delta (M_i) \) such that \( L \left( \mu \right) < 0 \) for all \( \mu \in \Delta_{\theta_i} \). That is, \( EU^\mu_{\theta_i} (b_i, \sigma_{-i}) (\nu_i) > EU^\mu_{b_i} (b_i, \sigma_{-i}) (\theta_i') \) for all \( b_i \in B_i \) and for all \( \sigma_{-i} \in R^\Delta_{-i} \). This claim remains true also for \( \sigma_{-i} \in R^\Delta_{-i} \), that is

\[
EU^\mu_{\theta_i} (b_i) (\nu_i) > EU^\mu_{\theta_i} (b_i) (\theta_i') \quad \text{for all} \quad b_i \in \partial \theta_i.
\]

Furthermore, by definition of \( k \), we have that \( \sigma_{-i} \left( \theta_{-i}' \right) \in R^\Delta_{-i} (\theta_{-i}) \) for any \( \sigma_{-i} \in \Sigma_{-i} \) and for
any \( \theta_i' \). Thus, we have established that:

\[
\forall \sigma \neq \sigma^*, \exists i, \theta_i, \theta_i' \in \Sigma_i(\theta_i) : \\
EU_{\theta_i}^{\mu_i^*(b_i, \sigma_{-i})}(\nu_i) > EU_{\theta_i'}^{\mu_i^*(b_i, \sigma_{-i})}(\nu_i') \quad \text{for all } \sigma_{-i} \in \Sigma_{-i} \text{ and } b_i \in \vartheta_i.
\] (22)

which is precisely the contraction property.

**Lemma 5** If \( d \) is strictly \( \Delta \)-IC and satisfies \( \Delta \)-contractivity, then it is truthfully \( \Delta \)-Implementable.

**Proof.** Clearly, strict \( \Delta \)-IC implies that \( \sigma^* \in R^\Delta \). We next to show that in fact \( R^\Delta = \{ \sigma^* \} \).

Suppose not, then \( \Delta \)-contractivity implies that there exists \( i, \theta_i, \theta_i' \in R^\Delta(\theta_i) \) and \( \nu_i \in \Delta(\Theta_i) \) such that \( EU_{\theta_i}^{\mu_i^*(b_i, \sigma_{-i})}(\nu_i) > EU_{\theta_i'}^{\mu_i^*(b_i, \sigma_{-i})}(\nu_i') \) for all \( b_i \in \vartheta_i, \sigma_{-i} \in R^\Delta_{-i} \) and \( b_i' \in \vartheta_i' \). But this implies that \( \theta_i' \) is dominated by \( v_i \) for all beliefs concentrated on \( \theta_i R^\Delta_{-i} \), hence \( \theta_i' \notin R^\Delta(\theta_i) \), a contradiction. ■

**B.2 Proof of Theorem 2**

Let us assume that the longest distance between the fixed strategy and another rationalizable one, \( l := \max_{i, \theta_i} \left\{ \max_{m_i \in R^\Delta_i(\theta_i)} |m_i - \theta_i| \right\} \neq 0 \). We prove the result by contradiction.

Let \( i, \theta_i^* \) and \( m_i^* \in R^\Delta_i(\theta_i^*) \) be s.t. \( |m_i^* - \theta_i^*| = l \). Since \( m_i^* \in R^\Delta(\theta_i^*) \), \( \exists \mu \in \Delta(R^\Delta_{-i}) : m_i^* = \arg\max_{m_i} EU_{\theta_i}^{\mu_i^*}(m_i) \). By \( \Delta \)-IC we also know that \( \theta_i \in R^\Delta_i(\theta_i) \) for all \( \theta_i \) and \( i \), hence \( C^T_i \subseteq R^\Delta_{-i} \). Let \( \mu^* \in C^T_i : \text{marg}_{\theta_i \mu}^T = \text{marg}_{\theta_i \mu} \). By the assumed \( \Delta \)-niceness of the mechanism, best responses are unique and described as the minimizer of the absolute value of the derivative of the expected utility function. We examine the difference in the first derivative of the expected utility that justifies \( m_i^* \) and the first order condition that justifies \( \theta_i^* \):

\[
\frac{\partial EU_{\theta_i}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = m_i^*} - \frac{\partial EU_{\theta_i'}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = \theta_i^*} = \int_{M_{-i} \times \theta_{-i}} \frac{\partial U_i(m_{-i}, m_i, \theta_i^*, \theta_{-i})}{\partial m_i} \bigg|_{m_i = m_i^*} \, d\mu
\]

\[
- \int_{M_{-i} \times \theta_{-i}} \frac{\partial U_i(m_{-i}, m_i, \theta_i^*, \theta_{-i})}{\partial m_i} \bigg|_{m_i = \theta_i^*} \, d\mu^* \quad \text{(24)}
\]

We consider two cases:

**Case 1:** \( m_i^* > \theta_i^* \). Since \( EU_{\theta_i}^{\mu_i^*}(m_i) \) is strictly concave and maximized at \( m_i^* \), whereas \( EU_{\theta_i'}^{\mu_i^*}(m_i) \) is strictly concave and maximized at \( \theta_i^* \), if \( m_i^* > \theta_i^* \) it follows that \( \frac{\partial EU_{\theta_i}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = m_i^*} \geq 0 \)

and \( \frac{\partial EU_{\theta_i'}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = \theta_i^*} > 0 \), whereas: \( \frac{\partial EU_{\theta_i}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = \theta_i^*} \leq 0 \) and \( \frac{\partial EU_{\theta_i'}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = m_i^*} \). Hence:

\[
\frac{\partial EU_{\theta_i}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = m_i^*} - \frac{\partial EU_{\theta_i'}^{\mu_i^*}(m_i)}{\partial m_i} \bigg|_{m_i = \theta_i^*} \geq 0.
\]
Letting \( b_{\theta_i} (\theta_{-i}, m_{-i}, m_i) = \left( \frac{\partial u_i(m_{-i}, s_i, \theta_{-i})}{\partial m_i} \bigg|_{s_i=m_i} \right) \), this can be rewritten as:

\[
\int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, m_i^*) \, d\mu - \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, \theta_i^*) \, d\mu^* \geq 0
\]

Next, we separate the own effect from external effects by adding and subtracting \( \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, \theta_i^*) \, d\mu \). Rearranging terms, we obtain

\[
B_i := \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, \theta_i^*) \, d\mu - \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, \theta_i^*) \, d\mu^* \\
\geq \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, \theta_i^*) \, d\mu - \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, m_i^*) \, d\mu =: A_i
\]

By the mean value theorem, there exists \( m_i' \in [\theta_i^*, m_i^*] \) such that

\[
A_i = \left( \int_{M_{-i} \times \Theta_{-i}} \frac{\partial b_{\theta_i} (\theta_{-i}, m_{-i}, m_i)}{\partial m_i} \bigg|_{m_i=m_i'} \, d\mu \right) \cdot (\theta_i^* - m_i^*)
\]

Since \( (\theta_i^* - m_i^*) < 0 \) and the strict concavity of the expected payoffs, both terms are negative, thus \( A \) can be written as

\[
A_i = \left| \int_{M_{-i} \times \Theta_{-i}} D_{i} U_i (m_{-i}, m_i', \theta_i^*, \theta_{-i}) \cdot d\mu \right| \cdot l
\]

Since \( \text{marg}_{\theta_{-i}} \mu^* = \text{marg}_{\theta_{-i}} \mu \) and \( \mu^* \in C_i^T \), the term \( B \) can be written as:

\[
B_i = \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, m_{-i}, \theta_i^*) \, d\mu - \int_{M_{-i} \times \Theta_{-i}} b_{\theta_i} (\theta_{-i}, \theta_{-i}, \theta_i^*) \, d\mu
\]

which by a mean-value Cauchy-Schwarz inequality, is bounded by

\[
B_i \leq \left( \int_{M_{-i} \times \Theta_{-i}} \sum_{j \neq i} \frac{\partial b_{\theta_i} (\theta_{-i}, m_{-i}, m_i)}{\partial m_j} \bigg|_{m_i=\theta_i^*} \cdot |\theta_j - m_j| \, d\mu \right) \cdot l
\]

Since \( A_i \leq B_i \), we have that

\[
\left| \int_{M_{-i} \times \Theta_{-i}} D_i U_i (m_{-i}, m_i', \theta_i^*, \theta_{-i}) \cdot d\mu \right| \leq \left| \int_{M_{-i} \times \Theta_{-i}} \sum_{j \neq i} D_{ij} U_i (m_{-i}, m_i', \theta_i^*, \theta_{-i}) \cdot d\mu \right|
\]

which contradicts the \( \Delta \)-Self Determination condition for \( i \).

**Case 2:** If \( m_i^* < \theta_i^* \), proceed similarly until contradicting the Self Determination condition.
C Proofs from Section 4

Lemma 1: Suppose that $M = (d, t)$ is EPIC and $t$ is differentiable. Then, for every $i$ and for every $m$, there exists a function $\tau_i : \Theta_{-i} \rightarrow \mathbb{R}$ such that $t_i (m) = t_i^* (m) + \tau_i (m_{-i})$.

Proof: A necessary condition for truthful revelation to be a best response to the opponent truthful revelation at every state (that is, EPIC) is that the following first-order condition is satisfied for every $i$ and every $\theta$:

$$\frac{\partial v_i (d (\theta), \theta)}{\partial x} \frac{\partial d (\theta)}{\partial \theta_i} + \frac{\partial t_i^* (\theta)}{\partial \theta_i} = 0$$

hence

$$\frac{\partial t_i^* (\theta)}{\partial \theta_i} = - \frac{\partial v_i (d (\theta), \theta)}{\partial x} \frac{\partial d (\theta)}{\partial \theta_i}.$$

Integrating over $m_i$, it follows that, for any $\theta = (\theta_i, \theta_{-i})$

$$t_i^* (\theta_i, \theta_{-i}) = - \int_{0}^{\theta_i} \frac{\partial v_i (d (s, \theta_{-i}), s, \theta_{-i})}{\partial x} \frac{\partial d (s, \theta_{-i})}{\partial \theta_i} ds + K$$

(25)

Now, for every $i$, define the function $\varpi_i : \Theta \rightarrow \mathbb{R}$ s.t. $\forall (\theta_i, \theta_{-i}) \in \Theta_i \times \Theta_{-i}, \varpi_i (\theta_i, \theta_{-i}) = v_i (d (\theta_i, \theta_{-i}), \theta_i, \theta_{-i})$, and notice that

$$\frac{\partial \varpi_i (\theta_i, \theta_{-i})}{\partial \theta_i} = \frac{\partial v_i (d (\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial x} \frac{\partial d (\theta_i, \theta_{-i})}{\partial \theta_i} + \frac{\partial v_i (d (\theta_i, \theta_{-i}), \theta_i, \theta_{-i})}{\partial \theta_i},$$

hence (25) can be rewritten as

$$t_i^* (\theta_i, \theta_{-i}) = - \int_{0}^{\theta_i} \frac{\partial \varpi_i (s, \theta_{-i})}{\partial \theta_i} ds + \int_{0}^{\theta_i} \frac{\partial v_i (d (s, \theta_{-i}), s, \theta_{-i})}{\partial \theta_i} ds + K$$

(26)

$$= - v_i (d (\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) + \int_{0}^{\theta_i} \frac{\partial v_i (d (s, \theta_{-i}), s, \theta_{-i})}{\partial \theta_i} ds + K + v_i (d (0, \theta_{-i}), 0, \theta_{-i}).$$

(27)

The result follows letting $\tau_i (\theta_{-i}) = K + v_i (d (0, \theta_{-i}), 0, \theta_{-i})$ for every $\theta_{-i}$.

Proposition 2: Allocation rule $d$ is belief-free implementable by a differentiable direct mechanism if and only if the canonical mechanism is belief-free truthfully implementable.

Proof: The ‘only if’ part is trivial. For the ‘only if’, suppose that $d$ is truthfully belief-free implemented by $M = (d, t)$. Results in Bergemann and Morris (2009) imply that $M$ is EPIC, hence by Lemma 1 transfers $t$ can be written as $t_i (m) = t_i^* (m) + \tau_i (m_{-i})$ for some $\tau_i : \Theta_{-i} \rightarrow \mathbb{R}$. It follows that the ex-post best-responses generated by $M$ and by the canonical mechanism are identical, but this implies that also the sets of (belief-free) Rationalizable strategies are identical for the two mechanisms. Hence, if $M$ truthfully implements $d$, so does the canonical mechanism.

C.1 Proof of Theorem 3

Consider the mechanism with transfers as in eq. (11). Observe that Condition 2 in the Theorem guarantees niceness of the mechanism. By strict concavity, truthelling is best response to any allowed conjecture concentrated on the truthelling profile, thus the mechanism is $\Delta$-IC. Condition 1 in the Theorem implies the $\Delta$-Self Determination Condition of Theorem 2. The result thus
follows from Theorem 2.

D Proofs from Sections 5 and 6

D.1 Proof of Lemma 2

Lemma 2: If the environment satisfies the single-crossing condition. Then: (1) The canonical mechanism is EPIC if and only if the allocation rule is strictly increasing: \( \frac{\partial d(\theta)}{\partial \theta_i} > 0 \) for every \( \theta \) and every \( i \).

Proof: To prove (1), notice that truthful revelation satisfies the (necessary) first-order conditions in the canonical mechanism, in that \( V_i(\theta, \theta) = 0 \) for all \( \theta \in \Theta \). Taking the second order derivative of the ex-post payoff function, and simplifying, we obtain:

\[
\frac{\partial^2 U^*_i}{\partial^2 m_i} (d(\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) = - \frac{\partial^2 v_i(d(\theta), \theta)}{\partial x \partial \theta_i} \frac{\partial d(\theta)}{\partial \theta_i}.
\]

Since the SCC implies that \( \frac{\partial^2 v_i(d(\theta)), \theta)}{\partial x \partial \theta_i} > 0 \), truthful revelation is uniquely optimal only if \( \frac{\partial d(\theta)}{\partial \theta_i} > 0 \).

D.2 Proof of Proposition 5

Proposition 5: Under assumptions Q.1-2 and SCC.2, in a common prior environment with independently distributed or affiliated types, an allocation function is EPIC-Implementable if and only if it is iDSIC-implementable.

Proof: As explained in the text, the proof of the result follows from Lemma 2 and Propositions 3 and 4, provided that we prove the following Lemma.

Lemma 6 Under assumptions Q.1-2 and SCC.2, if \( d \) is iDSIC, then it is strictly increasing.

Proof. \( \forall \theta_i \in \Theta_i, \forall m \in \Theta, \) define

\[
U_i(m, \theta_i) := \int_{\theta_{-i}} v_i(d(m), \theta_i, \theta_{-i}) \cdot B_{\theta_i}(\theta_{-i})
\]

A necessary condition for truthful revelation to be an (interim) best response independent of the opponents’ strategies is: \( \forall \theta_i \in \Theta_i, \forall \theta_{-i} \in \Theta_{-i}, \)

\[
t^*_i(\theta_i, \theta_{-i}) = - \int_0^{d_i} \frac{\partial^2 U_i(s_i, \theta_{-i}, \theta_i)}{\partial m_i} ds + K.
\]

Substituting these transfers and taking the first order conditions of the \( \theta_i \)'s optimization problem in the resulting mechanism, it is easy to see that truthful revelation satisfies the (necessary) first-order conditions. Under the maintained assumptions Q.1 and Q.2, the second order derivative of the interim payoff, for each \( \theta_i \in \Theta_i \) and \( m_{-i} \in \Theta_{-i} \), simplifies to:

\[
(S.O.C.) \int_{\theta_{-i}} \frac{\partial^2 U^*_i}{\partial^2 m_i} (d(m_i, m_{-i}), \theta_i, \theta_{-i}) \cdot B_{\theta_i}(\theta_{-i}) = - \frac{\partial d(m)}{\partial \theta_i} \cdot \int_{\theta_{-i}} \frac{\partial^2 v_i(d(m), \theta_i, \theta_{-i})}{\partial x \partial \theta_i} B_{\theta_i}(\theta_{-i}).
\]

Since SCC.2 implies that \( \int_{\theta_{-i}} \frac{\partial^2 v_i(d(m), \theta)}{\partial x \partial \theta_i} B_{\theta_i}(\theta_{-i}) > 0 \), truthful revelation is uniquely optimal only if \( \frac{\partial d(m)}{\partial \theta_i} > 0 \).
D.3 Proof of Proposition 7

Lemma 7 If \( \Sigma \) and \( \Sigma' \) are acceptable for \( f \), then \( \Sigma^* := \Sigma \cup \Sigma' \) defined as \( \Sigma^*(\theta) = \Sigma(\theta) \cup \Sigma'(\theta) \) for each \( \theta \) is also acceptable for \( f \).

**Proof.** This is trivial. Let \( \theta, \theta' : \theta' \in \Sigma^*(\theta) \). By definition of \( \Sigma^* \), it must be \( \theta' \in \Sigma(\theta) \) or \( \theta' \in \Sigma'(\theta) \). One way or the other, if \( \Sigma \) and \( \Sigma' \) are both acceptable, \( f(\theta') = f(\theta) \), hence \( \Sigma^* \) is acceptable.

Lemma 8 If \( f \) is responsive, any deception \( \Sigma \neq \sigma^* \) is unacceptable for \( f^\mathcal{M} \).

**Proof.** The proof is based on three steps. The first two are essentially the same as Steps 1 and 2 in Lemma 3.

**Step 1:** For any \( \theta_i \neq \theta'_i \), \( \Sigma_i(\theta_i) \cap \Sigma_i(\theta'_i) = \emptyset \). Suppose not, and let \( \theta^*_i \in \Sigma_i(\theta_i) \cap \Sigma_i(\theta'_i) \). Then, for any \( \theta_{-i} \) and \( \theta'_{-i} \in \Sigma_{-i}(\theta_{-i}) \), \( f(\theta^*_i, \theta_{-i}) = f(\theta_i, \theta_{-i}) \) and \( f(\theta^*_i, \theta'_{-i}) = f(\theta'_i, \theta'_{-i}) \). But \( f(\theta_i, \theta_{-i}) = f(\theta'_i, \theta_{-i}) \) for all \( \theta_{-i} \) contradicts responsiveness.

**Step 2:** For any \( i \), for any \( \theta_i \in \Theta_i : |\Sigma_i(\theta_i)| = 1 \). Suppose not, then there exist \( i \in I \) and \( \theta^*_i, \theta_i, \theta'_i \in \Theta_i \) such that \( \theta_i, \theta'_i \in \Sigma_i(\theta^*_i) \) and \( \theta_i \neq \theta'_i \). For each \( i \in I \), let \( Y_i := \bigcup_{\theta_i \in \Theta_i} \Sigma_i(\theta_i) \). Then, by Step 1, for every \( i \in I \) there exists an onto function \( \gamma_i : Y_i \rightarrow \Theta_i \) such that: (1) \( \theta_i \in \Sigma_i(\gamma_i(\theta_i)) \) for any \( \theta_i \in Y_i \); (2) for any \( \theta \in Y_i \), \( f(\theta) = f(\gamma_i(\theta)) \). Under the absurd hypothesis, \( \theta_i, \theta'_i \in Y_i \) are such that \( \theta_i \neq \theta'_i \) and \( \gamma_i(\theta_i) = \gamma_i(\theta'_i) \). But then \( f(\gamma_i(\theta_i), \gamma_{-i}(\theta_{-i})) = f(\gamma_i(\theta'_i), \gamma_{-i}(\theta'_{-i})) \) for any \( \theta_{-i} \in Y_{-i} \). Using (2), it follows that \( f(\theta_i, \theta_{-i}) = f(\theta'_i, \theta_{-i}) \) for all \( \theta_{-i} \in Y_{-i} \). This would contradict Responsiveness unless there exists \( \theta''_{-i} \in \Theta_{-i} \cup \Sigma_{-i} : f(\theta_i, \theta''_{-i}) \neq f(\theta'_i, \theta''_{-i}) \).

**Step 3:** Suppose that \( \Sigma \) is acceptable. Since \( \sigma^* \) is trivially acceptable, Lemma 7 implies that \( \Sigma' = \Sigma \cup \{\sigma^*\} \) is also acceptable. But if \( \Sigma \neq \sigma^* \), \( \Sigma' \) is such that \( \exists i, \theta_i : |\Sigma_i'(\theta_i)| > 1 \), contradicting Step 2.

**Proposition 7:** Let \( d : \Theta \rightarrow X \) be responsive, \( \mathcal{M} = (d, t) \) and the \( \Delta \)-restrictions be derived from \( \mathcal{B} \) that satisfies \( |B_0| = 1 \) for every \( \theta_i \) and \( i \). Then the following are equivalent:

1. \( \mathcal{M} \) is \( \Delta \)-IC and \( \Delta \)-contractive
2. \( f^\mathcal{M} \) satisfies direct ICR-Monotonicity.

**Proof:** First of all, notice that if \( d \) is responsive, so is \( f^\mathcal{M} \), hence any deception \( \Sigma \neq \sigma^* \) is unacceptable for \( f^\mathcal{M} \) (Lemma 8).

(1) \( \Rightarrow \) (2) is trivial: if \( \mathcal{M} \) is \( \Delta \)-contractive, then for any \( \Sigma \neq \sigma^* \) there exists \( i, \theta_i, \theta'_i \in \Sigma_i(\theta_i) \) and \( \nu^*_i \in \Delta(\Theta_i) \) such that: \( EU^\mathcal{M}_\theta(b_{\sigma_i}, \sigma_{-i})(\nu_i) > EU^\mathcal{M}_\theta(b_{\sigma_i}, \sigma_{-i})(\theta'_i) \) for all \( \sigma_{-i} \in \Sigma_{-i} \), but this is just inequality (17) for \( y^* \in Y^*(\theta_i) \) obtained as \( f(\nu^*_i, \theta_{-i}) = y^*(\theta_{-i}) \). Inequality (18) follows trivially from \( \Delta \)-IC.

(2) \( \Rightarrow \) (1) direct ICR-Monotonicity implies that there exists \( y^* \in Y^*(\theta_i) \) that satisfies (17), but this implies that \( \exists \nu^*_i \in \Delta(\Theta_i) \) such that: \( EU^\mathcal{M}_\theta(b_{\sigma_i}, \sigma_{-i})(\nu_i) > EU^\mathcal{M}_\theta(b_{\sigma_i}, \sigma_{-i})(\theta'_i) \) for all \( \sigma_{-i} \in \Sigma_{-i} \), which is \( \Delta \)-contractivity. Furthermore, for any \( i, \theta_i, \theta'_i \neq \theta_i \), consider the deception

\[
\Sigma_j(\theta_j) = \begin{cases} 
\{\theta_i, \theta'_i\} & \text{if } (j, \theta_j) = (i, \theta_i) \\
\theta_j & \text{otherwise}
\end{cases}
\]
This deception is unacceptable, hence by direct ICR-monotonicity, there exists $y^* \in Y^* (\theta_i)$ such that for any $\theta_{-i}$,

$$\int u_i (y_i^* (\theta_{-i}), \theta_i, \theta_{-i}) d\theta_i > \int u_i (f (\theta'_i, \theta_{-i}), \theta_i, \theta_{-i}) d\theta_i,$$

and

$$\int u_i (f (\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) d\theta_i \geq \int u_i (y_i^* (\theta_{-i}), \theta_i, \theta_{-i}) d\theta_i, \tag{28}$$

$$\int u_i (f (\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) d\theta_i \geq \int u_i (y_i^* (\theta_{-i}), \theta_i, \theta_{-i}) d\theta_i, \tag{29}$$

Together, this implies:

$$\int u_i (f (\theta_i, \theta_{-i}), \theta_i, \theta_{-i}) d\theta_i > \int u_i (f (\theta'_i, \theta_{-i}), \theta_i, \theta_{-i}) d\theta_i,$$

Since this holds for any $i, \theta_i, \theta'_i$, $\Delta$-IC follows. 

**REFERENCES**


