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STABLE PARTITIONS FOR GAMES WITH NON-TRANSFERABLE UTILITIES AND EXTERNALITIES

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Stable partitions for games with non-transferable utilities and externalities*

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Abstract We provide a model of coalitional bargaining with claims in order to solve games with non-transferable utilities and externalities. We show that, for each such game, payoff configurations exist which will not be renegotiated. In the ordinal game derived from these payoff configurations, we can find a partition in which no group of players has an incentive to jointly change their coalitions. For games without externalities this partition and the corresponding payoffs constitute a strong Nash equilibrium in a strategic form game with complete information. We use our model to provide a common framework for a variety of solutions for cooperative games, bargaining problems, and bankruptcy problems.

Keywords: Games with non-transferable utilities in partition function form, Bargaining with claims, Ordinal games, Core stable partitions, Non-cooperative coalition formation

JEL Classification: C71, C78, G34

1 Introduction

The most general class of cooperative games are those with non-transferable utility and externalities. Especially in the context of industrial organizations, they are relevant: not only do they allow the consideration of synergies (and, hence, non-linear payoff functions), they also enable us to take the indirect influence of a cooperation between companies on an outside party's payoffs via market mechanisms into account. However, the theory of these games is rather meager and even generalizations of well known solutions for games with only one of these features are far from trivial. The main scope of this paper is to propose a completely new approach for general games. The solutions a motivated in three different ways: first, they provide stable partitions in an ordinal coalition formation game; second, they are the result of bargaining and are robust with respect to renegotiation; third, for games without externalities, they deliver a strong equilibrium in a non-cooperative coalition formation game. Before we

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discuss the theoretical model in detail, we illustrate the shortcomings of previous models with an example that attracted a lot of attention in 2014.

1.1 The Alstom case

Alstom is a French corporation specializing in power generation and rail infrastructure. On April 24 the news portal Bloomberg reported that the American corporation General Electrics planned to acquire Alstom and that negotiations were already taking place.[1] Although this report was neither confirmed nor denied by any of the parties, three days later the German group Siemens released a statement to the press that "a letter has been submitted to the Board of Alstom to signal [Siemens'] willingness to discuss future strategic opportunities".[2] On April 29 Siemens announced their intention to make an offer for Alstom provided that access to Alstom's data room is granted for four weeks.[3] Alstom announced on May 6 that a binding offer had been made by General Electric,[4] and a joint offer of Siemens and the Japanese corporation Mitsubishi was made on June 16.[5] General Electric revised its offer on June 19;[6] Siemens and Mitsubishi revised theirs on June 20.[7] On June 26 Alstom's Board of Directors recommended the acceptance of the offer of General Electrics.[8]

At first sight, this three player game appears similar to the classical glove game where one player has a left glove (here Alstom) and two players have a right glove each (here General Electric and the consortium of Siemens and Mitsubishi). Profit can be made only from selling a pair of gloves. But classical solutions such as the Shapley value (Shapley, 1953) or the core (Gillies, 1959; Aumann, 1967) do not make a good prediction here: According to the Shapley value, the division of the surplus in a coalition of Alstom and General Electric should not be affected by the presence of Siemens and Mitsubishi. According to the core, on the other hand, the mere existence of Siemens in the game (Siemens was a player at least from April 29 on) should prohibit any agreement between Alstom and General Electric in which Alstom would not get at least

[1] Kirchfeld, Campbell, and McCracken, *General Electric Said in Talks to Buy Frances Alstom*, April 24, 2014, Bloomberg

[2] Press Release AXX201404.31, *Siemens signals Alstom willingness to discuss*, April 27, 2014, Siemens

[3] Press Release AXX201404.33, *Siemens will make an offer to Alstom*, April 29, 2014, Siemens

[4] Press Release, *Alstom is considering the proposed acquisition of its Energy activities by GE and the creation of a strong standalone market leader in the rail industry*, May 6, 2014, Alstom

[5] Joint Press Release AXX201406.46, *Mitsubishi Heavy Industries and Siemens provide a compelling proposal for Alstom*, June 16, 2014, Mitsubishi and Siemens; see also: Ad-hoc Announcement according to 15 WpHG (Securities Trading Act) *Siemens provides a proposal for Alstom together with Mitsubishi Heavy Industries*, June 16, 2014

[6] Press Release, *GE Announces Energy and Transport Alliance with Alstom*, June 19, 2014, General Electric

[7] Joint Press Release AXX201406.50, *Mitsubishi Heavy Industries and Siemens specify proposal to Alstom*, June 20, 2014, Mitsubishi and Siemens

[8] Press Release, *Alstom Board of Directors recommends General Electrics offer*, June 26, 2014

the same utility as from the best possible deal with Siemens and Mitsubishi. In fact, the presence of Siemens affected the eventual agreement between Alstom and General Electric, but the initial offer of General Electric did not account for a potential cooperation between Alstom, Siemens, and Mitsubishi and was revised only after a bid has actually been made by Siemens and Mitsubishi. It therefore seems that negotiations within a coalition are influenced by some *proposed* payoff allocations in other coalitions rather than by purely *potential* surpluses in other coalitions. This will be the starting point of our analysis.

1.2 Bargaining with claims

In a simple bargaining problem (Nash, 1950), players bargain how to allocate surpluses they can jointly achieve. In particular, the grand coalition is the only coalition that can achieve any surplus and each player has the option to stay alone. The payoff she receives when staying alone is her disagreement point. A game assigns to each coalition S a set of payoff allocations that are feasible for S, i.e. that could be achieved by the members of S provided they cooperate. Such a game can be considered as a vector of simple bargaining problems, one for each (embedded) coalition, if we define her disagreement point in each of these bargaining problems to be the payoff she would receive if she stayed alone. It therefore seems appropriate to use bargaining solutions in order to solve games – an approach that has been proposed for instance by Kalai (1977). In contrast to simple bargaining problems, in general games players have several coalitions they might join, and thus, do not only have ultimate disagreement points but also certain claims, which they consider justified, but which do not need to be satisfied. These bargaining problems with claims have been introduced by Chun and Thomson (1992). In our model claims are derived from a given payoff configuration: Players observe all their options, they derive claims for each coalition, and payoffs are renegotiated. After this negotiation players observe all their options again, derive (maybe new) claims, and renegotiate again. This is basically what happened during the bidding war between General Electric and Siemens/Mitsubishi. One result will be that we can find a payoff configuration which is robust with respect to such renegotiation. Such payoff configurations are called *consistent*.

We will impose only very weak assumptions on how players derive claims and how a bargaining rule should work in order to prove some existence results. Several specific claim functions and bargaining solutions will be proposed and discussed, and properties of the corresponding payoff configurations will be derived.

1.3 Ordinal coalition formation

Classical cooperative game theory focuses on the question: how should the surplus of the grand coalition be allocated to players? In particular, the formation of smaller coalitions is not an option. Nonetheless, the definition of the most prominent solution concepts, the core, heavily relies on the idea that subcoalitions can form and leave. For any given payoff vector x a coalition S is a deviation if there is a payoff distribution y which is feasible for S and allocates a higher payoff to each member than the original one, i.e. $y_i > x_i$ for all $i \in S$. The core consists of those payoff vectors that are feasible for the grand coalition and for which no deviation exist. Unfortunately, the core of games (even with transferable utilities and without externalities) may be empty. The assumption that the grand coalition forms has been dropped with the introduction of the coalition structure core (Aumann and Drèze, 1974; Greenberg, 1994); but non-emptiness cannot be guaranteed here either.

The crucial point in the definition of a deviation S is that there is *any* payoff distribution y which is feasible for S and allocates a higher payoff to each member. Both models remain silent about whether such a payoff allocation y is actually achieved. Recall the previous example: It turned out that a potential collaboration between Alstom, Siemens, and Mitsubishi had no effect on the offer of General Electric. Only after an offer had been made (i.e. a payoff allocation had been fixed), General Electric revised their offer.

Shenoy (1979) offered a solution to this problem: Instead of defining a payoff distribution only for the grand coalition (or a specified partition of the player set), the author applied a solution (for instance the Shapley value, Shapley, 1953) to the subgame in each coalition. For each coalition S there was now only one possible way to distribute payoffs, say y, and S was a deviation only if *these particular* payoffs exceeded the original ones for each member of S. One important question which has not been answered yet is: Can we find for each game a payoff configuration (that is a vector of payoffs for each player in each partition) such that a stable partition does exist? We will show that we can find such a payoff configuration which in addition is efficient, individually rational and anonymous, that is which satisfies those three axioms that are the basis for most characterizations of solutions of cooperative games.

1.4 Non-cooperative coalition formation

In recent years the interest in models of non-cooperative coalition formation has increased (see for instance Ray, 2007). A prominent article in this area is Chatterjee et al. (1993), where the main idea is that a player proposes both a coalition and a potential distribution of payoffs in this coalition. Members of the proposed coalition either accept and leave the game with their payoff, or reject, in which case the next player makes a new proposal. This model is heavily inspired by Rubinstein (1982), but in the presence of many players and without restrictions on the coalition structure, these games may become very complicated.

A different approach of Hart and Kurz (1983) divided the coalition formation process into two parts: First, payoffs for each player in each partition are specified. Then, each player chooses a coalition. If a coalition is chosen by all its members, it forms, otherwise players stay alone. In a strong equilibrium no group of players has a reason to change their strategies. One of the results of

Hart and Kurz (1983, 1984) was that, in general, such a strong equilibrium does not exist. We will show that for each game (without externalities) there is a payoff configuration such that this non-cooperative game has a strong equilibrium.

1.5 Further related literature

A very early solution which is defined for arbitrary partitions rather than for the grand coalition is the bargaining set of Davis and Maschler (1963) (see also Peleg, 1963, for a generalization to games with non-transferable utility). In their model, players may have objections against each other, and payoff vectors in the bargaining set are balanced in some sense. Behind our model is a similar idea, namely that the payoffs players receive and the claims they make are consistent, i.e. robust with respect to renegotiation.

Kalai and Smorodinsky (1975) proposed a bargaining solution according to which players' payoffs are proportional to the maximal payoff they could receive as long as their opponents' payoffs do not fall below their disagreement points. This highest possible payoff plays the role of what we will define more generally as a *claim* in this paper. Kalai (1977) proposed a similar proportional rule, but here the proportions do not depend on the game, but are universal constants. Kalai and Samet (1985) considered the egalitarian solution and used it to define a generalization of the Shapley value for games with non-transferable utility. We will show how our model can be refined in order to obtain this particular solution.

Claims have appeared mainly in the context of bankruptcy games such as in Aumann and Maschler (1985) or Curiel et al. (1987). A combination of bargaining problems and claims was first proposed by Chun and Thomson (1992). All these models consider the grand coalition as the only coalition in which gains or losses can be realized. Moreover, claims are independently specified and, in contrast to our model, do not emerge from the game. Related to this work is the article of Moulin (2000) which investigates rationing rules. Although only demands are explicitly defined in this model, the assumption of non-negative payoffs replaces the idea of ultimate disagreement points in Chun and Thomson (1992). Models in the same spirit have also been proposed by Hougaard et al. (2012, 2013). The crucial point of these and of our model is that players have both claims, which can be considered as expectations, and disagreement points, which are minimal payoffs they require in order to agree to join a coalition.

Based on the work of Shenoy (1979) articles of Dimitrov and Haake (2008b) and Karos (2014) derived conditions under which stable partitions in the resulting abstract games do exist. However, these articles concentrated only on the class of voting games and did not give conditions that would be both sufficient and necessary.

Ordinal coalition formation games as we introduce them here are very related to the hedonic coalition formation games of Drèze and Greenberg (1980). Unlike the games there, we consider games with externalities: that is, players do not only have preferences over coalitions but over partitions. Not much is known about the existence of core stable partitions in hedonic games (some conditions can be found for instance in Banerjee et al., 2001; Bogomolnaia and Jackson, 2002; Iehlé, 2007), but we show that for all ordinal games in our model a core stable partition must exist.

1.6 Structure of the paper

In Section 2 we introduce our model of coalitional bargaining with claims. We prove the existence of consistent payoff configurations for each game under general conditions. In Section 3 we introduce ordinal coalition formation games. We show how an ordinal game can be derived from a payoff configuration and analyze ordinal games derived from consistent payoff configurations. In particular, we give sufficient conditions for a payoff configuration to induce an ordinal game with a stable partition. In Section 4 we refine our results from Section 2 for the proportional rule. We will show that for each game there is a payoff configuration which is consistent with the proportional bargaining solution under a reasonable claim form, and that for each such payoff configuration the derived ordinal game possesses a core stable partition.

Sections 5 to 7 provide some further applications of the model and may be read independently of each other. In Section 5 we show how the Shapley value, the bargaining set, and the core are related to our model of coalitional bargaining. In Section 6 we derive a class of single-valued solutions which are based on opportunity costs and marginal contributions of players in different coalitions. In Section 7 we show that each game without externalities can be solved non-cooperatively by means of a strong equilibrium.

Section 8 concludes the paper and provides some possible avenues for further research.

2 Coalitional bargaining with claims

In this section we introduce our model of coalitional bargaining in general games. We start with basic definitions in Subsection 2.1 and introduce bargaining problems with claims in Subsection 2.2. Subsection 2.3 defines claim forms and some basic properties they may satisfy. In Subsection 2.4 we introduce our main results, namely the existence of payoff configurations that will not be renegotiated. We close with Subsection 2.5, where we derive further properties of the set of consistent payoff configurations.

2.1 Preliminaries and notation

Throughout the paper N shall be a finite set of players. A *coalition* is a nonempty subset $S \subseteq N$ and the set $\mathcal{P} = \mathcal{P}(N)$ is the collection of all coalitions. For $i \in N$ we denote by $\mathcal{P}_i$ the collection of all coalitions containing i. A *partition* is a collection $\sigma = \left\{S^1, \ldots, S^m\right\}$ of coalitions such that $S^k \cap S^l = \emptyset$ for all $k \neq l$ and $\bigcup_{k=1}^m S^k = N$. The set Σ shall be the collection of all partitions of

N, and for a coalition S we denote the set of all partitions containing S by Σ_S. For $i \in N$ and a partition σ we denote by $\sigma(i)$ the (unique) coalition $S \in \sigma$ with $i \in S$. An *embedded coalition* is a pair $(S, \sigma) \in \mathcal{P} \times \Sigma$ such that $S \in \sigma$. The collection of all embedded coalitions is denoted by $\mathcal{E}$ and the collection of all embedded coalitions (S, σ) with $i \in S$ by $\mathcal{E}_i$.

A *game* is a map V which assigns to each embedded coalition (S, σ) a nonempty, closed, convex subset $V(S, \sigma) \subseteq \mathbb{R}^S$ such that

1. $V(S, \sigma)$ is comprehensive: that is, for $x \in V(S, \sigma)$ and $y \leq x$ we have $y \in V(S.\sigma)$,[9]

2. the set $\{x \in V(S, \sigma) : x \geq x^*\}$ is bounded for each $x^* \in V(S, \sigma)$.

A game can hence have transferable or non-transferable utility, and be in characteristic function or partition function form (Thrall and Lucas, 1963). If $x \in V(S, \sigma)$ we say that x is *feasible* for (S, σ). We call a game *non-leveled* if for all (S, σ) and all $x \in \partial V(S, \sigma)$ we have $y > x$ only if $y \notin V(S, \sigma)$.

Player i's *disagreement point in* V is $d_i^V = \min_{\tau \in \Sigma_{\{i\}}} \max(V(\{i\}, \tau))$. This is the worst-case-payoff player i can achieve if she stays alone, independent of how the remaining players form their coalitions. Throughout this paper we assume that $d_S^V \in V(S, \sigma)$ for all games V and all embedded coalitions (S, σ).[10] This standard condition allows us to avoid unnecessary technicalities in what follows; it will become clear later that it is no loss of generality, see Remark 2.6.

A vector $x \in \mathbb{R}^{N \times \Sigma}$ is called a *payoff configuration*. Hence, a payoff configuration specifies in each partition for each player a payoff. For a payoff configuration x, we call $d_i(x) = \min_{\tau \in \Sigma_{\{i\}}} x_{i,\tau}$ player i's *disagreement point in* x. This is the minimal payoff player i can achieve if she stays alone, independent of how the remaining players form their coalitions. We say that x is *individually rational* if $x_{i,\sigma} \geq d_i(x)$ for all $\sigma \in \Sigma$.

For a game V let $\Delta(V) \subseteq \mathbb{R}^{N \times \Sigma}$ be the set of all payoff configurations x with $x_{S,\sigma} \in V(S, \sigma)$ for all embedded coalitions (S, σ) and $x_{i,\sigma} = \max(V(\{i\}, \sigma))$ for all $\sigma \in \Sigma_{\{i\}}$. Hence, for any embedded coalition (S, σ) we require $x_{S,\sigma}$ to be feasible for (S, σ), and if S contains only one player i, we require that $x_{i,\sigma}$ is the maximal feasible payoff. This makes sense as there is no good reason to allocate less to i than she could achieve on her own. Note that this implies $d_i(x) = d_i^V$ for all $x \in \Delta(V)$.

We say that x is *efficient in* V if $y > x$ implies $y \notin \Delta(V)$ for all $y \in \mathbb{R}^{N \times \Sigma}$. Hence, if V is non-leveled, $x \in \Delta(V)$ is efficient if and only if $x_{S,\sigma} \in \partial V(S, \sigma)$ for each embedded coalition (S, σ). We say that x is *individually rational in* V if $x_{i,\sigma} \geq d_i^V$ for all $i \in N$ and all $\sigma \in \Sigma$. In particular, $x \in \Delta(V)$ is individual rational if and only if it is individually rational in V. We denote the subset of all efficient (resp. individually rational) payoff configurations in V by $\Delta_{\text{eff}}(V)$ (resp. $\Delta_{\text{ir}}(V)$).

[9]For $x, y \in \mathbb{R}^S$ we write $x \geq y$ if $x_i \geq y_i$ for all $i \in S$, we write $x > y$ if $x \geq y$ and $x \neq y$, and we write $x \gg y$ if $x_i > y_i$ for all $i \in S$.

[10]For a vector $d = (d_i)_{i \in N} \in \mathbb{R}^N$ we write d_S for $(d_i)_{i \in S}$.

Let $\rho : N \to N$ be a permutation. For a partition $\sigma = \{S^1, \ldots, S^k\}$ we define $\rho(\sigma) = \{\rho(S^1), \ldots, \rho(S^k)\}$. For a game V we define ρV by $(\rho V)(S, \sigma) = V(\rho(S), \rho(\sigma))$ for each embedded coalition (S, σ). We say that $x \in \Delta(V)$ is *anonymous* if $x_{\rho(i),\rho(\sigma)} = x_{i,\sigma}$ for each permutation with $\rho V = V$. We denote the set of all anonymous payoff configurations in V by $\Delta_{\text{an}}(V)$.

2.2 Bargaining problems with claims

Following Chun and Thomson (1992) we define a *bargaining problem (with claims)* as a quadruple (S, X, d, c) of a coalition S with $|S| \geq 2$, a closed, convex, and comprehensive subset $X \subseteq \mathbb{R}^S$ such that $\{x \in X;\ x \geq x^*\}$ is bounded for each $x^* \in X$, a *disagreement point* $d \in X$, and a *claim point* $c \geq d$.[11] A *bargaining solution* is a map F which maps each bargaining problem (S, X, d, c) to a vector $x \in X$. The first three of the following four properties are standard and do not need much discussion.

Individual Rationality. A bargaining solution F is *individually rational* if $F(S, X, d, c) \geq d$ for each bargaining problem (S, X, d, c).

Efficiency. A bargaining solution F is *efficient* if $y > F(S, X, d, c)$ implies $y \notin X$ for all bargaining problems (S, X, d, c).

Anonymity. A bargaining solution F is *anonymous*

$$F_i(S, X, d, c) = F_{\rho(i)}\left(\rho(S), X', d_{\rho(S)}, c_{\rho(S)}\right)$$

for all bargaining problems (S, X, d, c) and all permutations ρ where $x \in X'$ if and only if $x_{\rho^{-1}(S)} \in X$.

Continuity. Let (S, X, d, c) be a bargaining problem. A bargaining solution F is *continuous in c* if for all sequences c^n with $c^n \geq d$ and $\lim_{n\to\infty} c^n = c$ we have $F(S, X, d, c^n) \to F(S, X, d, c)$. F is *continuous* if it is continuous in c for all bargaining problems (S, X, d, c).

Continuity of F ensures that the bargaining solution does not jump after a small change in c. We do not consider continuity of F in any other variable than c throughout this paper; therefore, as this cannot lead to confusion, by continuity we always mean continuity in c.

2.3 Claim forms

Let V be a game and $x \in \Delta(V)$ be a payoff configuration. Suppose that x builds the starting point for any negotiations and is known to all players – maybe from experience or because it is the result of previous negotiations, such as in the Alstom case. Then players can and will derive claims from x: If player i knows that she could receive a high payoff in a coalition T, she might use this potential payoff as a bargaining chip when negotiating payoffs in another coalition S. This

[11]Note that Chun and Thomson (1992) require $c \notin X$. We do not impose this condition.

motivates for the following definitions. Let V be a game. A *claim form* maps each game V to a function $C^V : \Delta(V) \to \mathbb{R}^{N \times \Sigma}$ with $C^V_{i,\sigma}(x) \geq d_i(x)$ for all $i \in N$ and all $\sigma \in \Sigma$. We call C^V a *claim function.* Hence, a claim form specifies in each partition a claim for each player that depends both on the game V and a previous payoff configuration $x \in \Delta(V)$. Recall that we have $d(x) = d^V$ for such x, hence, the condition $C^V_{i,\sigma}(x) \geq d_i(x)$ simply states that a player claims at least her guaranteed payoff in the game V.

We say that a claim form is *continuous* if C^V is continuous for all games V. For payoff configurations and bargaining solutions we introduced anonymity as a property which ensured that payoffs do not depend on a players' names. We do the same for claim functions.

Anonymity. A claim form C is *anonymous* if

$$C^{\rho V}_{\rho(i),\rho(\sigma)}\left(\left(x_{\rho(i),\rho(\sigma)}\right)_{i\in N,\sigma\in\Sigma}\right) = C^V_{i,\sigma}(x) \tag{1}$$

for each permutation ρ, each game V, and each $x \in \Delta(V)$.

We can interpret anonymity of a claim form as the condition that player i would derive the same claims as player j does, if i was in j's shoes. Although this seems like a strong condition on first sight, Equation (1) not only ensures that i and j change their roles with respect to the payoff configuration x, but also with respect to the game V. This means, that i would make the same claims as j if she were to face x from j's perspective and had j's utility function. Thus, anonymity is a very natural property of a claim function.

Given a payoff configuration x, a player's claim in an embedded coalition should not be entirely arbitrary. The following two related properties of claim functions give different bounds for claims. They will build one pillar of our analysis in the remainder of this section.

Weak reasonableness. A claim function C^V is *weakly reasonable in x* if

$$C^V_{i,\sigma}(x) \leq \max_{S\in\mathcal{P}_i} \left\{y_i : \, y \in V(S,\sigma), y_j \geq d^V_j \text{ for all } j \in S\right\} \tag{2}$$

for all games V, all $i \in N$, and all $\sigma \in \Sigma$. A claim function C^V is *weakly reasonable* if C^V is weakly reasonable for all $x \in \Delta(V)$. A claim form C is *weakly reasonable* if C^V is weakly reasonable for each game V.

For the second property we need the following definition. Let x be a payoff configuration, let $\sigma \in \Sigma$, and let $i \in N$. We say that $T \in \mathcal{P}_i$ is an *outside option* of i in σ if for all $j \in T \setminus \{i\}$ we have $x_{j,\tau} \geq x_{j,\sigma}$ for all $\tau \in \Sigma_T$ and $x_{j,\tau} > x_{j,\sigma}$ for some $\tau \in \Sigma_T$. That means an outside option is a coalition in which all players except i can, according to x, only gain compared to their payoffs in σ. (In particular, staying alone is an outside option.) Hence, it is rather easy for i to convince the other members of T (if any) to collaborate. We say that an outside option T of i is *positive* if $x_{i,\tau} > x_{i,\sigma}$ for all $\tau \in \Sigma_T$.

Strong Reasonableness A claim function C^V is *strongly reasonable in* x or just *reasonable* if

$$C^V_{i,\sigma}(x) \leq \max_{\tau\in\Sigma} x_{i,\tau} \text{ and} \tag{3a}$$

$$C^V_{i,\sigma}(x) > x_{i,\sigma} \text{ if and only if } i \text{ has a positive outside option in } \sigma \tag{3b}$$

for all games V, all $i \in N$, and all $\sigma \in \Sigma$. Strong reasonableness of a claim function C^V or a claim form C are defined as before.

Claim functions specify how players derive their claims in each coalition in each partition depending on all payoffs of all players in all partitions. Inequality (2) guarantees that a player claims no more than she would receive in any other coalition where payoffs are shared individually rationally. Weak reasonableness is a very general condition and related to the concepts of the core and the bargaining set (see Section 5 for details). A player i can claim a payoff even though there is no partition in which this payoff would actually be allocated to her. If C is strongly reasonable this is not possible anymore. Condition (3a) guarantees that a player claims no more than she would receive in any other partition according to x. This condition reflects our observation in the Alstom case: Alstom had a reasonable claim in the negotiations with General Electric only after Siemens and Mitsubishi made a proposal. Accordingly, this proposal caused a revised offer from General Electric. Condition (3b) states that a player can claim strictly more than she already gets only if she actually has a higher outside option according to the distribution in x. Applied to the Alstom case this condition ensures that an offer of Siemens and Mitsubishi affects Alstom's position when bargaining with with General Electric if and only if it is better than the one of General Electric.

Both axioms should be considered as minimal requirements on a claim function depending on the context. Weak reasonableness seems appropriate if players can easily make proposals how to share a surplus. This might be the case in very simple structures, where developing and negotiating proposals can be done at (almost) no costs. However, if negotiations and potential contracts are very complicated, a claim can be justified by an existing proposal of a third party rather than by the mere possibility of an unspecified collaboration with such a party.

Remark 2.1. In a positive outside option every player is at least as well off as in the partition they deviated from and for each player there is a possibility that she is even strictly better off. Hence, our definition of a positive outside option reflects a rather pessimistic view of players on the outcome of deviating. We are well aware of criticism of this point of view (for instance in Ray and Vohra, 1997). Although many results (not all) can be derived for different notions of outside options, we use this approach as it reflects the idea that the payoff player i receives in a positive outside option T of σ might serve as a bargaining chip when negotiating payoffs in $\sigma(i)$. But this bargaining chip is useful only in so far as it cannot be reduced by some *bloodthirsty behavior* (Ray and Vohra, 1997) of players in $N \setminus T$.

Note that, in general, strong reasonableness does not imply weak reasonableness (see for instance part 3 in Example 2.2). However, if C^V is strongly reasonable on $\Delta_{\text{ir}}(V)$ then C^V is also weakly reasonable on $\Delta_{\text{ir}}(V)$. As these are the payoff configurations we are dealing with most of the time, we stick to the names *weakly* and *strongly* reasonable.

Example 2.2.

1. The function C with $C_{i,\sigma}(x) = d_i(x)$ represents a claim form if we define C^V as the restriction $C_{|\Delta(V)}$ for all games V. In particular, C is continuous, anonymous, and weakly reasonable but not strongly reasonable.

2. Consider the claim form C which is defined by

$$C^V_{i,\sigma}(x) = \max_{T \in \mathcal{P}_i} \min_{\tau \in \Sigma_T} \max_{y \in V(T,\tau)} \{y_i : \, y_j \geq x_{j,\sigma} \text{ for all } j \in T \setminus \{i\}\}. \quad (4)$$

   This claim function is related to the idea of *deviations* in the definition of the core: In any partition σ player i maximizes the worst-case-payoff she can obtain in any coalition T provided that each member of T receives at least her payoff in σ. We will investigate the relation of this claim function to the core and the bargaining set in Section 5. Note that C is continuous and anonymous but neither weakly nor strongly reasonable: If x is not individual rational, Inequality (2) can easily be violated; and $C^V_{i,\sigma}(x) > x_{i,\sigma}$ is possible without i having an outside option.

3. For a payoff configuration x, a partition $\sigma \in \Sigma$, and a player $i \in N$ let

$$\mathcal{T}^*_{i,\sigma}(x) = \left\{ T \in \mathcal{P}_i : \, \min_{\tau \in \Sigma_T} x_{j,\tau} \geq x_{j,\sigma}, \max_{\tau \in \Sigma_T} x_{j,\tau} > x_{j,\sigma} \text{ for all } j \in T \setminus \{i\} \right\}$$

   be the set of *outside options* of i in σ. The function

$$C_{i,\sigma}(x) = \max_{T \in \mathcal{T}^*_{i,\sigma}(x)} \min_{\tau \in \Sigma_T} x_{i,\tau} \quad (5)$$

   is an anonymous and strongly reasonable claim form, but neither weakly reasonable (x might not be individually rational) nor continuous. We will further explore the latter observation in Example 2.8.

2.4 Consistent payoff configurations

Let C be a claim form, let V be a game, and let $x \in \Delta(V)$. Then $d(x) = d^V$ and, by definition, $C^V_{S,\sigma}(x) \geq d_S(x) = d^V_S$ for all embedded coalitions (S, σ). Hence, $\left(S, V(S,\sigma), d^V_S, C^V_{S,\sigma}(x)\right)$ is a bargaining problem for each embedded coalition (S, σ) with $|S| \geq 2$. This means that, given a claim form C, any game V can be interpreted as a collection of bargaining problems with claims. Other authors (see for instance Harsanyi, 1963; Aumann, 1967; Kalai, 1977) have used a similar interpretation of a game as a vector of bargaining problems without claims.

As claim forms can be used in order to renegotiate payoffs, we are interested in payoff configurations which are invariant with respect to such renegotiations. This idea is captured by the following definition.

Definition 2.3. Let V be a game, let C be a claim form, and let F be a bargaining solution. A payoff configuration $x \in \Delta(V)$ is *consistent (with F under C)* if

$$x_{i,\sigma} = F_i\left(\sigma(i), V(\sigma(i), \sigma), d^V_{\sigma(i)}, C^V_{\sigma(i),\sigma}(x)\right) \tag{6}$$

for all $i \in N$ and all partitions $\sigma \in \Sigma$ with $|\sigma(i)| \geq 2$. We denote by $\mathcal{K}^C_F(V)$ the collection of all payoff configurations that are consistent with F under C.

Note that for $x \in \mathcal{K}^C_F(V)$ the payoffs $x_{i,\sigma}$ are already uniquely determined for partitions with $|\sigma(i)| = 1$ as $x \in \Delta(V)$. Applying the bargaining rule F to the claims derived from x according to C will result in x again; and hence, players would not renegotiate consistent payoff configurations. We will use this subsection to derive some basic existence results. In Section 4 we will pay closer attention to a specific bargaining rule, namely the proportional bargaining solution.

It is time to show that consistent payoff configurations exist for all games V as long as bargaining solutions and claim form satisfy some mild conditions.

Proposition 2.4. *Let V be a game, let C be a continuous claim form, and let F be a continuous bargaining solution. If F is, additionally, individually rational then $\mathcal{K}^C_F(V)$ is a nonempty, closed subset of $\Delta_{\text{ir}}(V)$.*

Proof. Let $\hat{F}^V : \Delta(V) \to \Delta(V)$ be defined as

$$\hat{F}^V_{i,\sigma}(x) = \begin{cases} F_i\left(\sigma(i), V(\sigma(i)\sigma), d^V_{\sigma(i)}, C^V_{\sigma(i)}(x)\right), & \text{if } |\sigma(i)| \geq 2, \\ x_{i,\sigma}, & \text{otherwise.} \end{cases}$$

A payoff configuration $x \in \Delta_{\text{ir}}(V)$ is consistent with F if and only if $\hat{F}^V(x) = x$. It is clear that $\hat{F}^V$ is continuous since F and C are continuous. As $\hat{F}^V$ is a map from $\Delta_{\text{ir}}(V)$ into itself and as $\Delta_{\text{ir}}(V)$ is compact and convex, $\hat{F}^V$ must have a fixed point in $\Delta_{\text{ir}}(V)$ by Brouwer's fixed point theorem, that is $\mathcal{K}^C_F(V) \neq \emptyset$. Let x^n be a sequence of consistent payoff configurations with $x = \lim_{n\to\infty} x^n$. Then $\hat{F}^V(x) = \hat{F}^V(\lim_{n\to\infty} x^n) = \lim_{n\to\infty} \hat{F}^V(x^n) = \lim_{n\to\infty} x^n = x$ by continuity of $\hat{F}^V$. Hence, $\mathcal{K}^C_F(V)$ is closed. □

An example of a continuous and individually rational bargaining solutions is the *constrained egalitarian solution* (Aumann and Maschler, 1985; Curiel et al., 1987): $\tilde{E}_i(S, X, d, c) = \max(c_i - \lambda, d_i)$, where $\lambda \in \mathbb{R}$ is uniquely determined by the efficiency of $\tilde{E}$. Note that, unlike in the original definition, λ might take negative values in our model.

In Chun and Thomson (1992) claim points always lie outside X, the same is true for bankruptcy problems. This is not the case in our model. It might, therefore, happen that a bargaining solution assigns to one player more and to another player less than they claim. The next definition addresses this observation.

Fairness. A bargaining solution F is called *fair* if for all bargaining problems (S, X, d, c) there is $i \in S$ with $F_i(S, X, d, c) \geq c_i$ if and only if $F_j(S, X, d, c) \geq c_j$ for all $j \in S$.

Fairness guarantees that nobody's claim can be satisfied while someone else's is not. The *egalitarian bargaining solution*, defined as

$$E(S, X, d, c) = c + \mathbb{1}_S \cdot \max\{t;\ c + t\mathbb{1}_S \in X\},$$

is a fair and continuous bargaining solution. This solution is the natural extension of the classical egalitarian solution for cases in which the reference point (usually the disagreement point, in our cases c) lies outside X. A drawback of this solution is that it does not satisfy individual rationality, so that Proposition 2.4 does not apply.

Proposition 2.5. *Let V be a game, let C be a continuous and weakly or strongly reasonable claim form, and let F be a continuous bargaining solution. If F is, additionally, fair and efficient then $\mathcal{K}_F^C(V)$ is a nonempty, closed subset of $\Delta_{\text{eff}}(V)$.*

Proof. We prove the proposition for a weakly reasonable C, and sketch how it is done for strongly reasonable C afterwards. Let $\hat{F}^V$ be defined as in the proof of Theorem 2.4 and define

$$\begin{aligned} x_+^* &= \max_{i \in N} \max_{(S,\sigma) \in \mathcal{E}_i} \left\{x_i :\ x \in \partial V(S, \sigma) \text{ and } x_j \geq d_j^V \text{ for all } j \in S \setminus \{i\}\right\}, \\ x_-^* &= \min_{i \in N} \min \left\{ d_i(x), \min_{(S,\sigma) \in \mathcal{E}_i, |S| \geq 2} \min_{c \in \prod_{j \in S}\left[d_j(x), x_+^*\right]} \left\{F_i\left(S, V(S), d_S^V, c\right)\right\} \right\}, \end{aligned}$$

Hence, x_+^* is the highest payoff any player could receive in any partition, provided that payoffs are distributed individually rationally. The value x_-^* is the minimal payoff, any player can receive in any partition, provided that payoffs are distributed according to F and the claims of all players j lie between $d_j(x)$ and x_+^*, what is given for any weakly reasonable claim form. Let $Q \subseteq \Delta(V)$ be the set of payoff configurations with $x_-^* \leq x_{i,\sigma} \leq x_+^*$ for all partitions σ and all $i \in N$. Obviously, Q is compact and convex. We show that $\hat{F}^V(Q) \subseteq Q$. To do this let $x \in Q$ and let $(S, \sigma) \in \mathcal{E}$, with $|S| \geq 2$, arbitrary but fixed. By fairness we have to consider only the following two cases.

1. Suppose $\hat{F}_{i,\sigma}^V(x) \geq C_{i,\sigma}^V(x)$ for all $i \in S$. Clearly, $\hat{F}_{i,\sigma}^V \geq C_{i,\sigma}^V(x) \geq d_i^V \geq x_-^*$. This also implies $\hat{F}_{i,\sigma}^V \leq x_+^*$, as $\hat{F}_{j,\sigma}^V \geq d_j^V$ for all $j \in S \setminus \{i\}$.

2. Suppose now $\hat{F}_{i,\sigma}^V(x) < C_{i,\sigma}^V(x)$ for all $i \in S$. Then we have $\hat{F}_{i,\sigma}^V(x) < C_{i,\sigma}^V(x) \leq x_+^*$, as C is weakly reasonable. Further, by construction, $\hat{F}_{i,(S,\sigma)}^V(x) \geq x_-^*$.

As (S,σ) was arbitrary, we conclude that $\hat{F}^V(x) \in Q$, and the non-emptyness of $\mathcal{K}_C^F(V)$ is proved by applying Brouwer's fixed point theorem. The same arguments as in the proof of Proposition 2.4 show that $\mathcal{K}_C^F(V)$ must be closed.

If C is strongly reasonable, replace x_+^* by $x_+' = \max_{\tau\in\Sigma} x_{i,\tau}$ and adjust x_-^* and Q accordingly. The rest of the proof is identical. □

Remark 2.6. Suppose that V is a game such that $d_S^V \notin V(S,\sigma)$ for some embedded coalitions (S,σ). Then it is clear that partition σ will not form under any circumstances as coalition S would break apart. Hence, define $\Sigma^* = \Sigma^*(V)$ to be the collection of all partitions σ with $d_S^V \in V(S,\sigma)$ for all $S \in \sigma$, and define $\mathcal{E}^*$ to be the set of all embedded coalitions (S,σ) with $\sigma \in \Sigma^*$. If we now require Equation (6) to hold only for all $i \in N$ and all $\sigma \in \Sigma^*$ with $|\sigma(i)| \geq 2$, we see that the results of this paper remain valid. Hence, the condition $d_S^V \in V(S,\sigma)$ is no loss of generality.

In the next example we present a claim form which is both reasonable and continuous.

Example 2.7. Define

$$\mathcal{T}_{i,\sigma}(x) = \left\{ T \in \mathcal{P}_i : \min_{j\in T} \min_{\tau\in\Sigma_T} x_{j,\tau} \geq x_{j,\sigma} \right\}$$

for all $i \in N$, $\sigma \in \Sigma$, and $x \in \mathbb{R}^{N\times\Pi}$, and let

$$\tilde{\alpha}_{i,\sigma}(T,x) = \begin{cases} \min_{j\in T} \max_{\tau\in\Sigma_T} x_{j,\tau} - x_{j,\sigma}, & \text{if } T \in \mathcal{T}_{i,\sigma}(x), \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

be defined for all payoff configurations x and all $T \in \mathcal{P}_i$. Clearly, $\tilde{\alpha}_{i,\sigma}(T,.)$ is continuous for all x, and so is its normalized version

$$\alpha_{i,\sigma}(T,x) = \frac{\tilde{\alpha}_{i,\sigma}(T,x)}{1 + \sum_{T'\in\mathcal{P}_i} \tilde{\alpha}_{i,\sigma}(T,x)}.$$

Moreover, $\alpha_{i,\sigma}(T,x) > 0$ if and only if T is a positive outside option of i in σ. Hence, the claim form

$$C_{i,\sigma}(x) = \left(1 - \sum_{T\in\mathcal{P}_i} \alpha_{i,\sigma}(T,x)\right) x_{i,\sigma} + \sum_{T\in\mathcal{P}_i} \alpha_{i,\sigma}(T,x) \min_{\tau\in\Sigma_T} x_{i,\tau} \tag{8}$$

is continuous, reasonable, anonymous, and satisfies $C_{i,\sigma}(x) \geq x_{i,\sigma}$ for all $i \in N$ and all $\sigma \in \Sigma$. For a deviation T we can interpret $\alpha_{i,\sigma}(T,x)$ as an indicator for the likelihood that T actually forms, given the maximum each player can obtain from T. In particular, if there is some player who cannot gain from joining T, this likelihood is 0.

We close this subsection with an example for a claim form under which consistent payoff configurations may not exist.

Example 2.8. Let v be the proper monotonic simple three player game which is defined by its minimal winning coalitions $\{1,2\}$ and $\{1,3\}$,[12] and let V the corresponding game.[13] We show that there is no payoff configuration $x \in \Delta(V)$ which is consistent with the egalitarian bargaining solution E under the claim function C from Equation (5) in Example 2.2. Assume, on the contrary, that x is such a payoff configuration (which must be efficient) and let $q^1 = x_{1,\{12,3\}}$ and $q^2 = x_{1,\{13,2\}}$.[14] Suppose $q^1, q^2 < 1$. Obviously, $13 \in \mathcal{T}^*_{1,\{12,3\}}(x)$ and therefore $C_{1,\{12,3\}}(x) \geq q^2$. For player 2 we find that $2 \in \mathcal{T}^*_{2,\{12,3\}}(x)$; $23 \in \mathcal{T}^*_{2,\{12,3\}}(x)$ only if $x_{2,\{1,23\}} < 0$; and $N \in \mathcal{T}^*_{2,\{12,3\}}(x)$ only if $x_{1,\{N\}} > q^1, x_{3,\{N\}} > 0$ and therefore $x_{2,\{N\}} < 1 - q^1$. Hence, $C_{2,\{12,3\}}(x) < 1 - q^1 = x_{2,\{12,3\}}$. The definition of E implies that in this case $q^2 \leq C_{1,\{12,3\}}(x) < x_{1,\{12,3\}} = q^1$ and for the same reasons it must hold as well that $q^1 < q^2$. As this is impossible, either q^1 or q^2 must be at least 1. Without loss of generality let $q^1 \geq 1$. In this case $N \in \mathcal{T}^*_{1,\{12,3\}}$ only if $x_{1,\{N\}} < q^1$; and $13 \in \mathcal{T}^*_{1,\{12,3\}}$ only if $q^2 < 1 \leq q^1$. We, therefore, have $C_{1,\{12,3\}}(x) < q^1 = x_{1,\{12,3\}}$. Again, by definition of E, we must have $x_{2,\{12,3\}} > C_{2,\{12,3\}}(x)$. Hence, $0 \geq x_{2,\{12,3\}} > C_{2,\{12,3\}}(x) \geq 0$ by definition C. This is impossible as well.

2.5 Solutions of games

A *solution* is a map $\mathcal{L}$ that maps each game V to a set $\mathcal{L}(V) \subseteq \Delta(V)$.[15] We propose the following properties a solutions may satisfy.

Individual Rationality. A solution $\mathcal{L}$ is called *individually rational* if $\mathcal{L}(V) \subseteq \Delta_{\text{ir}}(V)$ for all games V.

Efficiency. A solution $\mathcal{L}$ is called *efficient* if $\mathcal{L}(V) \subseteq \Delta_{\text{eff}}(V)$ for all games V.

In particular, for a bargaining solution F and a claim form C the map $V \mapsto \mathcal{K}^C_F(V)$ is a solution. We will denote this map by $\mathcal{K}^C_F$. Clearly, if F is individually rational, so is $\mathcal{K}^C_F$; and if F' is efficient, so is $\mathcal{K}^C_{F'}$.

We are usually interested in anonymous payoff configurations. Hence, the next easy lemma is useful for what follows.

Lemma 2.9. *The set $\Delta_{\text{an}}(V)$ is convex for all games V.*

Proof. Let V be a game, $x, y \in \Delta_{\text{an}}(V)$ and let $z = \lambda x + (1-\lambda)y$ for some $\lambda \in [0,1]$. Clearly, $z \in \Delta(V)$ by convexity of $\Delta(V)$. Let ρ be a permutation such that $\rho(V) = V$. Then $z_{\rho(i),\rho(\sigma)} = \lambda x_{\rho(i),\rho(\sigma)} + (1-\lambda)y_{\rho(i),\rho(\sigma)} = \lambda x_{i,\sigma} + (1-\lambda)y_{i,\sigma} = z_{i,\sigma}$ for all $i \in N$ and all $\sigma \in \Sigma$. Hence, $z \in \Delta_{\text{an}}(V)$. □

[12] A proper monotonic simple game is a function $v : \mathcal{P} \to \{0,1\}$ such that $v(N) = 1$, $v(S) \leq v(T)$ if $S \subseteq T$, and $v(S) + v(N \setminus S) \leq 1$ for all nonempty $S \subsetneq N$.

[13] That is $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$ for all $S \in \mathcal{P}$.

[14] In order to avoid brackets we use the notation jk for coalition $\{j,k\}$ and accordingly $\{i,jk\}$ for the partition $\{\{i\},\{j,k\}\}$ here and in the remainder of the paper.

[15] That is, we implicitly assume that a solution always assigns the efficient payoff to a player who stays alone.

Although we cannot expect that all consistent payoff configurations for an anonymous bargaining solution F and an anonymous claim function C are anonymous, we can at least obtain the following two existence results. They are immediate consequences of Lemma 2.9; all we shall do is sketch the proof of the first one.

Corollary 2.10. *Let V be a game, let C be a continuous and anonymous claim form, and let F be a continuous, anonymous, and individually rational bargaining solution. Then $\mathcal{K}_F^C(V) \cap \Delta_{\mathrm{an}}(V) \neq \emptyset$.*

Proof. As $\Delta_{\mathrm{an}}(V)$ is convex, so is $\Delta_{\mathrm{an,ir}}(V) = \Delta_{\mathrm{ir}}(V) \cap \Delta_{\mathrm{an}}(V)$. It is, therefore, sufficient to show that the map $\hat{F}^V$, defined as in the proof of Proposition 2.4 satisfies $\hat{F}^V(\Delta_{\mathrm{an,ir}}(V)) \subseteq \Delta_{\mathrm{an,ir}}(V)$. But this follows from anonymity of C and F. □

Corollary 2.11. *Let V be a game, let C be a continuous and anonymous claim form, and let F be a continuous, anonymous, fair, and efficient bargaining solution. Then $\mathcal{K}_F^C(V) \cap \Delta_{\mathrm{an}}(V) \neq \emptyset$.*

3 Coalition formation

In this section we describe how to transform a general game (in partition function form and with non-transferable utilities) into an ordinal game. In Subsection 3.1 we introduce ordinal coalition formation games and show how they can be derived from payoff configurations. Subsection 3.2 introduces another property of solutions for general games, namely balancedness, which is motivated by coalition formation. In Subsection 3.3 we show that the ordinal game derived from a balanced payoff configuration has a core stable partition.

3.1 From payoff configurations to ordinal games

An *ordinal game* is a pair $(N, \succeq)$ where $\succeq$ is a profile of preferences $(\succeq_i)_{i \in N}$ over partitions.[16] We call a coalition T a *deviation* of a partition σ if for all $i \in T$ we have $\tau \succeq_i \sigma$ for all $\tau \in \Sigma_T$ and $\tau \succ_i \sigma$ for some $\tau \in \Sigma_T$. In this case we also say that T *blocks* σ. If $i \in T$, we say that i has deviation T in σ. We say that σ is *blocked*, if there is a coalition T that blocks σ. We call σ *core stable* if it is not blocked.

From a payoff configuration x we canonically derive the ordinal game $(N, \succeq^x)$ where $\succeq_i^x$ is defined as

$$\tau \succeq_i^x \sigma \quad \text{if and only if} \quad x_{i,\tau} \geq x_{i,\sigma}.$$

for all $i \in N$. Let x be a payoff configuration and let σ be a partition. It can easily be checked that T is a deviation of $i \in N$ in σ in the game $(N \succeq^x)$ if

[16] We denote the symmetric part of $\succeq_i$ by $\sim_i$ and the asymmetric part by $\succ_i$.

and only if T is a positive outside option of i. We will call the collection of core stable partitions the *ordinal core* of $(N, \succeq^x)$.

The idea of ordinal games is not new. Shenoy (1979) suggested applying solutions – such as the Shapley value or the core – to a game with transferable utilities and all its subgames and deriving an ordinal game from the resulting payoff configurations. These so called *abstract games* do not exhibit externalities: that is, the preference relations $\succeq_i$ satisfy $\sigma \sim_i \pi$ whenever $\sigma(i) = \pi(i)$. Such ordinal games without externalities were introduced and studied by Drèze and Greenberg (1980) under the name *hedonic games*. It can easily be checked that if $\succeq_i$ does not exhibit externalities, our definitions of deviations and core stability are identical to those in the literature of hedonic games.

Remark 3.1. In Shenoy (1979) a solution $\mathcal{L}$ induces for each game V exactly one ordinal game. This is different here: For different payoff configurations $x \in \mathcal{L}(V)$ we may obtain different ordinal games. But this makes sense in our model of coalitional bargaining. A payoff configuration x that is consistent with a bargaining solution under a claim function will not be renegotiated and, therefore, can only be the end of a bargaining process. Hence, if x is the result of bargaining, players can only compare their payoffs with respect to this particular payoff configuration.

For general ordinal games it is not clear in which cases core stable partitions exist. In case of hedonic games several sufficient conditions have been proposed (for an overview see for instance Bogomolnaia and Jackson, 2002) and even a characterization is known (Iehlé, 2007). For ordinal games which are derived from games with transferable utilities in the spirit of Shenoy (1979) there is not much known about the existence of stable partitions (some conditions are given in Dimitrov and Haake, 2008b; Karos, 2014). But ordinal games derived from games with non-transferable utilities, with or without externalities, have not been considered in the literature at all. By the end of this section we will have given a sufficient condition for a payoff configuration x such that the ordinal game $(N, \succeq^x)$ has a nonempty core. In the next section we will show that for all games V there is $x \in \Delta(V)$ that satisfies this condition.

3.2 Balancedness

Consider a payoff configuration x and an embedded coalition (S, σ). A simple fairness condition on x would be that either no player has a positive outside option (if this is possible) or all players have. Hence, we call x *balanced* if for each embedded coalition (S, σ) there is $i \in S$ with a positive outside option in σ if and only if each $j \in S$ has a positive outside option in σ. But, although this condition seems appropriate at first sight, when combined with individual rationality, efficiency, and anonymity it might have unpleasant consequences.

Example 3.2. Consider the monotonic simple game v with winning coalitions $\{1, 2\}$ and $\{1, 2, 3\}$, let V the corresponding game, and let $x \in \Delta(V)$ be efficient and individually rational. Then $x_{1,\{13,2\}} = x_{3,\{13,2\}} = x_{2,\{1,23\}} = x_{3,\{1,23\}} = 0$.

Further $x_{1,\{12,3\}} + x_{2,\{12,3\}} = 1$ by efficiency, and thus, $x_{1,\{12,3\}} = x_{2,\{12,3\}} = \frac{1}{2}$ by anonymity. Hence, player 1 has a positive outside option in coalition 13 while player 3 does not, in contradiction to balancedness.

In order to avoid this phenomenon, we introduce the following definition.

Definition 3.3. A payoff configuration x is *constrained balanced* if for all embedded coalitions (S, σ) and all $i \in S$ one of the following holds.

1. $x_{S,\sigma} \ll d_S(x)$, or
2. $x_{S,\sigma} = d_S(x)$, or
3. $x_{S,\sigma} \geq d_S(x)$ and no $j \in S$ with $x_{j,\sigma} = d_j(x)$ has a positive outside option, and there is $i \in S$ with a positive outside option if and only if each $j \in S$ with $x_{j,\sigma} > d_j(x)$ has a positive outside option.

The somewhat complicated formulation of constrained balancedness ensures that the first priority is to satisfy all disagreement points, and only after that any further potential outside options are considered.

(Constrained) Balancedness. A solution $\mathcal{L}$ is *(constrained) balanced* if all $x \in \mathcal{L}(V)$ are (constrained) balanced for all games V.

We close this subsection with the following corollary which is an immediate consequence of our observation in Example 3.2.

Corollary 3.4. *For each solution $\mathcal{L}$ that satisfies individual rationality, efficiency, anonymity and balancedness, there is a game V such that $\mathcal{L}(V) = \emptyset$.*

3.3 Fairness, balancedness, and core stability

We have already linked individual rationality and efficiency of a bargaining solution F to the properties of solutions $\mathcal{K}_F^C$. And also the close relation between fair bargaining solutions and balanced payoff configurations seems obvious. Indeed, we can make the following statement.

Corollary 3.5. *Let C be a reasonable claim form and let F be a fair bargaining solution. Then $\mathcal{K}_F^C$ is balanced.*

Proof. Let V be a game and let $x \in \mathcal{K}_F^C(V)$. (If $\mathcal{K}_F^C(V) = \emptyset$, there is nothing to show.) Let (S, σ) be an embedded coalition and let $i \in S$. If i has a positive outside option in σ then $C_{i,\sigma}(x) > x_{i,\sigma} = \hat{F}_{i,\sigma}(x)$ by the reasonableness of C; and by the fairness of F the same is true for all $j \in S$. Hence, each $j \in S$ has a positive outside option. □

Besides satisfying a natural fairness condition, (constrained) balanced payoff configurations have another beautiful property regarding the ordinal game $(N, \succeq^x)$.

Proposition 3.6. *Let x be (constrained) balanced. Then there is a core stable partition in the ordinal game $(N, \succeq^x)$.*

Proof. We prove the claim for constrained balanced x and sketch how it would be done for balanced x afterwards. Let $j^1 \in N$ and define

$$\tilde{S}^1 = \arg\max_{S \in \mathcal{P}_{j^1}} \min_{\sigma \in \Sigma_S} x_{j^1,\sigma},$$

and let $\sigma^1 = \arg\min_{\sigma' \in \Sigma_{\tilde{S}^1}} x_{j^1,\sigma'}$. Then we have for all partitions $\sigma' \in \Sigma_{\tilde{S}^1}$

$$x_{j^1,\sigma'} \geq x_{j^1,\sigma^1} \geq \max_{T \in \mathcal{P}_{j^1}} \min_{\tau \in \Sigma_T} x_{j^1,\tau}.$$

Hence, if σ' is a partition containing $\tilde{S}^1$, j^1 has no positive outside option in σ'. In particular, $x_{j^1,\sigma'} \geq d_{j^1}$. Let

$$S^1 = \begin{cases} \{j^1\}, & \text{if } x_{j^1,\sigma^1} = d_{j^1}, \\ \tilde{S}^1, & \text{if } x_{j^1,\sigma^1} > d_{j^1}. \end{cases}$$

If $S^1 = N$, let $\sigma^* = \{N\}$. Otherwise, we recursively construct the following partition. Define $N^k = N \setminus (\bigcup_{l=1,\ldots,k-1} S^l)$ for $k \geq 2$ and, if $N^k \neq \emptyset$, choose $j^k \in N^k$. Let

$$\tilde{S}^k = \arg\max_{S \in \mathcal{P}_{j^k}(N^k)} \min_{\sigma \in \Sigma, \{S_1,\ldots,S_{k-1},S\} \subseteq \sigma} x_{i,\sigma},$$

let $\sigma^k = \arg\min_{\sigma \in \Sigma, S^1,\ldots S^{k-1}, \tilde{S}^k \in \sigma} x_{i,\sigma}$, and define

$$S^k = \begin{cases} \{j^k\}, & \text{if } x_{j^k,\sigma^k} = d_{j^k}, \\ \tilde{S}^k, & \text{if } x_{j^k,\sigma^k} > d_{j^k}. \end{cases}$$

As N is finite, there is k^* such that $N^{k^*+1} = \emptyset$, that is $\sigma^* = \{S^1, \ldots, S^{k^*}\}$ is a partition. We show that no player has a positive outside option in σ^*. We start by showing that there is no positive outside option in σ^* for any $j \in S^1$. If $S^1 = \{j^1\}$, this is clear as $\max_{T \in \mathcal{P}_i} \min_{\tau \in \Sigma_T} x_{j^1,\tau} \leq x_{j^1,\sigma^1} \leq x_{j^1,\sigma^*}$. Otherwise we have $S^1 = \tilde{S}^1$, so that j^1 cannot have a positive outside option. As x is constrained balanced and $x_{j^1,\sigma^*} \geq x_{j^1,\sigma^1} > d_i$, this means that no $j \in S^1$ has a positive outside option.

Now let $k \geq 2$ and suppose we have shown that there is no positive outside option in σ^* for any $j \in \bigcup_{l=1}^{k-1} S^l$. From the construction of S^k it is clear that j^k cannot have a positive outside option T in σ^* with $T \subseteq N^{k-1}$. Together with our hypothesis, this implies that there is no positive outside option of j^k in σ^* at all. If $S^k = \{j^k\}$, we have that there is no positive outside option for any $j \in S^k$. So, let $\left|S^k\right| \geq 2$. In this case $x_{j^k,\sigma^*} \geq x_{j^k,\sigma^k} > d_{j^k}$. By constrained balancedness of x, there cannot be a positive outside option in σ^* for any $j \in S^k$. Hence, no $j \in S^k$ has a positive outside option in σ^*. Since a player has a deviation in σ^* if and only if she has a positive outside option in σ^*, σ^* must be core stable.

If x is balanced, we simply define $S^k = \tilde{S}^k$ and keep the rest of the proof identical. □

Remark 3.7. Suppose that x is a payoff configuration without externalities, i.e. with $x_{i,\sigma} = x_{i,\pi}$ whenever $\sigma(i) = \pi(i)$. Then the ordinal game $(N, \succeq^x)$ is a hedonic game. In particular, the (constrained) balancedness of x implies that $(N, \succeq^x)$ exhibits the *weak top coalition property* introduced by Banerjee et al. (2001). The authors have shown that this property is a sufficient (but not necessary) condition for the existence of a core stable partition. Here, we have presented here a natural way for it to arise.

4 The proportional bargaining solution

We have identified four key properties of a bargaining rule, namely individual rationality, efficiency, anonymity, and fairness. But two of them are incompatible.

Lemma 4.1. *There is no bargaining rule which is individually rational and fair.*

Proof. Suppose F is a bargaining rule with both properties and let the bargaining problem (S, X, c, d) be defined as $S = \{1, 2\}$, $c = (2, 0)$, $d = (0, 0)$, and $X = \left\{x \in \mathbb{R}^2 : \; x_1 + x_2 \leq 1\right\}$. By individual rationality, $F_2(S, X, d, c) \geq 0 = c_2$ and $F_1(S, X, d, c) \leq 1 < c_1$ contradicting fairness. □

We introduce a bargaining solution which has slightly weaker properties. Let (S, X, d, c) be a bargaining problem. The *proportional solution* is defined as

$$P\left(S, X, d, c\right) = \begin{cases} d + (c - d) \max\left\{t : \; d + t\left(c - d\right) \in X\right\}, & \text{if } c > d, \\ d + \mathbb{1}_S \max\left\{t : \; d + t\mathbb{1}_S \in X\right\} & \text{otherwise.} \end{cases}$$

Clearly, P is individually rational, efficient, and anonymous. Further, P is continuous in all $c \neq d$, and is continuous in $c = d$ if $d \in \partial X$. Although P is not fair, it satisfies the following condition.

Weak Fairness. A bargaining solution F is called *weakly fair*, if for each bargaining problem (S, X, d, c) there is $i \in S$ with $F_i(S, X, d, c) > c_i$ only if there is no $j \in S$ with $F_j(S, X, d, c) < c_j$.

Chun and Thomson (1992) have already mentioned the close relation between P and the bargaining solution of Kalai and Smorodinsky (1975). Indeed, let claim form C be defined as

$$C^V_{i,\sigma}(x) = \max_{y \in V(\sigma(i), \sigma)} \{y_i : \; y \geq d^V_{\sigma(i)}\}.$$

for each game V and consider a game V. As $C^V(x)$ does not depend on x but only on V, we find that $\hat{P}^V(x)$ (defined as in the proof of Proposition 2.4) does not depend on x either. In particular, there is a unique fixed point x of $\hat{P}$ and this payoff configuration is consistent with P under C. For each embedded coalition (S, σ) we find that $x_{S,\sigma}$ coincides with the Kalai-Smorodinsky solution of the bargaining problem (without claims) $(S, V(S, \sigma), d^V_S)$. To this extent, we can consider solutions of the form $\mathcal{K}_C^P$ as variations of the Kalai-Smorodinsky solution with general claim forms and for general games.

We show that consistent payoff configurations also exist for more general claim functions.

Proposition 4.2. *Let V be a non-leveled game and C be a continuous claim form with $C^V(x) \geq x$. Then $\mathcal{K}_C^P(V) \neq \emptyset$ and $\mathcal{K}_C^P$ is individually rational and efficient. If C is, additionally, anonymous, then $\mathcal{K}_C^P(V) \cap \Delta_{\mathrm{an}}(V) \neq \emptyset$ for all games V.*

Proof. Define $\hat{P}^V$ on $\mathrm{convh}(\Delta_{\mathrm{ir,eff}}(V))$, the convex hull of $\Delta_{\mathrm{ir}}(V) \cap \Delta_{\mathrm{eff}}(V)$, by

$$\hat{P}^V_{i,\sigma}(x) = \begin{cases} P_i\left(\sigma(i), V(\sigma(i),\sigma), d^V_{\sigma(i)}, C_{\sigma(i),\sigma}(x)\right), & \text{if } |\sigma(i)| \geq 2, \\ x_{i,\sigma}, & \text{otherwise.} \end{cases}$$

We show that $\hat{P}^V$ is a continuous map from $\mathrm{convh}(\Delta_{\mathrm{ir,eff}}(V))$ into itself. It is clear that $\hat{P}^V(x) \in \Delta_{\mathrm{ir,eff}}(V)$ for all x, as P is individually rational and efficient. We show that $\hat{P}^V$ is, indeed, continuous. Let $x \in \mathrm{convh}(\Delta_{\mathrm{ir,eff}}(V))$ and let $(S,\sigma) \in \mathcal{E}$ with $|S| \geq 2$. If $x_{S,\sigma} > d^V_S$, we find that $C^V_{S,\sigma}(x) \geq x_{S,\sigma} > d^V_S$, so that P is continuous in $C^V_{S,\sigma}(x)$. In this case, $\hat{P}^V_{i,\sigma}$ is continuous in x for all $i \in S$. Suppose that $x_{S,\sigma} = d^V_S$. As $x_{S,\sigma}$ is a convex combination of points $y \in \partial V(S,\sigma)$ with $y \geq d^V_S$, this is possible only if $x_{S,\sigma}$ is an extreme point of the convex hull. This means $d^V_S = x_{S,\sigma} \in \partial V(S,\sigma)$. As V is non-leveled, we have that d^V_S is the only efficient and individually rational element of $V(S,\sigma)$. In particular, for any sequence (x^n) in $\mathrm{convh}\,(\Delta_{\mathrm{ir,eff}}(V))$ approaching x we have that $x^n_{S,\sigma} = d^V_S$. Hence, $\lim_{n\to\infty} \hat{P}^V_{S,\sigma}(x^n) = d^V_S = \hat{P}^V_{S,\sigma}(x) = \hat{P}^V_{S,\sigma}(\lim_{n\to\infty} x^n)$, so that $\hat{P}^V_{i,\sigma}$ is continuous in x for all $i \in S$.

Therefore, $\hat{P}^V$ is a continuous map from $\mathrm{convh}(\Delta_{\mathrm{ir,eff}}(V))$ into itself and, by Brouwer's fixed point theorem, must have a fixed point. Clearly, this fixed point is individually rational, efficient in V, and consistent with P under C.

If C is, additionally, anonymous, we see that $\hat{P}^V$ also maps $\mathrm{convh}(\Delta_{\mathrm{an,ir,eff}}(V))$ into itself. Hence, $\hat{P}^V$ has a fixed point in $\Delta_{\mathrm{an,ir,eff}}(V)$ which is consistent with P under C. □

In the context of bargaining the condition $C(x) \geq x$ is easily justified. Assuming that a payoff configuration is the result of previous negotiations, there is no reason for players to claim less than they had agreed to before. This does not necessarily mean that these claims are satisfied in the end.

We have already seen in Corollary 3.4 that there is no solution which is individually rational, efficient, anonymous, balanced, and nonempty. If we replace balancedness by constrained balancedness, this changes. We start with the following lemma.

Lemma 4.3. *Let C be reasonable. Then $\mathcal{K}_C^P$ is constrained balanced.*

Proof. Let $x \in \mathcal{K}_C^P(V)$. Let $(S,\sigma) \in \mathcal{E}$ with $|S| \geq 2$ and let $i \in S$. Clearly, $x_{S,\sigma} \geq d_S^V$. If $x_{S,\sigma} = d_S^V$, there is nothing to show, so let $x_{S,\sigma} > d_S^V$. Suppose that there is $i \in S$ with $x_{i,\sigma} = d_i^V$ such that i has a positive outside option. Then $C_{i,\sigma}^V(x) > x_{i,\sigma}$ by the reasonableness of C. We have $d_i^V = x_{i,\sigma} = \hat{P}_{i,\sigma}(x)$. Hence,

$$d_i^V = P_i\left(S, V(S,\sigma), d_S^V, C_{S,\sigma}^V(x)\right) = d_i^V + t_{S,\sigma}\left(C_{i,\sigma}^V(x) - d_i^V\right) \tag{9}$$

for some $t_{S,\sigma} \geq 0$. As $C_{i,\sigma}^V(x) > x_{i,\sigma} \geq d_i^V$, we have $t_{S,\sigma} = 0$. Hence, $x_{S,\sigma} = P\left(S, V(S,\sigma), d_S^V, C_{S,\sigma}^V(x)\right) = d_S^V$, in contradiction to $x_{S,\sigma} > d_S^V$. Therefore, no i with $x_{i,\sigma} = d_i^V$ can have a deviation.

Suppose now that there is $i \in S$ with a positive outside option. In this case we find

$$C_{i,\sigma}^V(x) > x_{j,\sigma} = d_i + t_{S,\sigma}\left(C_{i,\sigma}^V(x) - d_i\right)$$

as C is reasonable, and therefore, $0 < t_{S,\sigma} < 1$. In particular, for any $j \in S$ with $x_{j,\sigma} > d_j^V$ we have that $C_{j,\sigma}^V > d_j^V$ and, hence, $x_{j,\sigma} = d_j + t_{S,\sigma}\left(C_{j,\sigma}^V(x) - d_i\right) < C_{j,\sigma}^V(x)$. By reasonableness of C, j must have a positive outside option. $\square$

As mentioned earlier, a variety of models deriving ordinal games from games with transferable utilities have been introduced without answering the question under what condition the ordinal core is nonempty. Propositions 3.6, 4.2, and Lemma 4.3 together imply the following Corollary which closes this gap in the literature.

Corollary 4.4. *Let V be a game and C be a reasonable claim form. Then there is an individually rational, efficient, and anonymous payoff configuration $x \in \Delta(V)$ that is consistent with P under C, and for each such x the ordinal game $(N, \succeq^x)$ has a core stable partition.*

5 Other solutions

For games without externalities many solutions are known and well studied (Peters, 2003; Peleg and Sudhölter, 2007). We will present only some of the most prominent and show how they fit into our framework. Throughout this section let V be a game without externalities, that is $V(S,\sigma) = V(S)$ for all $\sigma \in \Sigma_S$.

Shapley Value. There a many generalizations of the Shapley value (Shapley, 1953) for games with non-transferable utilities. The *monotonic solution* of Kalai and Samet (1985) relies on the idea of *dividends* (Harsanyi, 1963). It uses the egalitarian bargaining solution to distribute dividends of all coalitions among its members. *Dividends* are defined recursively by $\delta^V(\emptyset) = 0$, $z^V(S) = \sum_{T \subsetneq S} \delta^V(S)$, and $\delta^V(S) = \mathbb{1}_S \cdot \max\left\{t : z^V(S) + t\mathbb{1}_S \in V(S)\right\}$.

Corollary 5.1. *Let $C^V_{i,\sigma}(x) = z^V_i(\sigma(i))$. Then for each game V there is a unique $x \in \mathcal{K}^E_C(V)$ and $x_{N,\{N\}}$ coincides with the Kalai-Samet monotonic solution.*

Proof. See Kalai and Samet (1985). □

Note that the functions C^V do not define a claim form as $C^V_{i,\sigma}(x) \geq d_i(x)$ is not guaranteed. In particular, the quadruple $(S, V(S), d^V_S, C^V_{S,\sigma}(x))$ is not necessarily a bargaining problem with claims. However, as the egalitarian bargaining solution E does not depend on disagreement points d^V, it is still well defined on such quadruples. In the same way that we can interpret solutions of the form $\mathcal{K}^P_C$ as generalizations of the Kalai-Smorodinsky solution, we can interpret solutions of the form $\mathcal{K}^E_C$ as generalizations of the Kalai-Samet solution.

Bargaining Sets. The *bargaining set* (Davis and Maschler, 1963; Peleg, 1963) $\mathcal{M}(\sigma)$ for a partition σ consists of all those $x \in \prod_{S\in\sigma} V(S)$ for which no player has a *justified objection* against any other player. That is, if there is a player i, a coalition $S \in \mathcal{P}_i$ and $y \in V(S)$ such that $y_j > x_j$ for all $j \in S$, then for each $k \in \sigma(i) \setminus S$ there is $T \in \mathcal{P}_k(N \setminus \{i\})$ and $z \in V(T)$ such that $z_l \geq y_l$ for all $l \in S \cap T$ and $z_l \geq x_l$ for all $l \in T \setminus S$.

Consider the claim function from Equation (4) in Example 2.2. If V does not exhibit externalities, C^V takes the form

$$C^V_{i,\sigma}(x) = \max_{T\in\mathcal{P}_i} \max_{y\in V(T)} \{y_i : y_j \geq x_{j,\sigma} \text{ for all } j \in T \setminus \{i\}\}. \tag{10}$$

Player i has a (not necessarily justified) objection if and only if $C_{i,\sigma}(x) > x_{i,\sigma}$. The following corollary is an immediate consequence of this observation.

Corollary 5.2. *Let V be a game and let $x \in \mathcal{K}^P_C(V)$ where C^V is defined as in Equation (10). If σ is such that $x_{i,\sigma} \geq C_{i,\sigma}(x)$ for all $i \in N$ then $x_{N,\sigma} \in \mathcal{M}(\sigma)$.*

Peleg (1963) has shown that for games with non-transferable utilities the bargaining set can be empty. However, since the claim form C in Equation (10) is not only continuous but also satisfies $C(x) \geq x$, we have that $\mathcal{K}^P_C(V) \neq \emptyset$ for all non-leveled games V by Proposition 4.2.

Core. Let $V_0(S) = V(S) \times \mathbb{R}^{N\setminus S}$. The *core* of game V (Gillies, 1959; Aumann, 1961) is defined as $\mathcal{C}(V) = V(N) \setminus \bigcup_{S\in\mathcal{P}} \text{int}V_0(S)$. Sufficient conditions for the nonemptyness of $\mathcal{C}(V)$ have been given by Scarf (1967) or Billera (1970). Necessary and sufficient conditions can be found for instance in Keiding and Thorlund-Petersen (1987) or Predtetchinski and Herings (2004). We give with a necessary and sufficient condition for a consistent payoff configuration to deliver a core stable allocation.

Corollary 5.3. *Let V be a non-leveled game. Let $x \in \mathcal{K}^P_C(V)$ where C^V is defined as in Equation (10). The vector $x_{N,\{N\}}$ lies in the core of V if and only if $x_{N,\{N\}} \geq C_{N,\{N\}}(x)$.*

Proof. Let $x_{N,\{N\}} \in \mathcal{C}(V)$ and suppose there is $i \in N$ with $x_{i,\{N\}} < C_{i,\{N\}}(x)$. Then there is a coalition S such that

$$x_{i,\{N\}} < \max_{y \in V(S)} \left\{ y_i : \; y_j \geq x_{j,\{N\}} \text{ for all } j \in S \setminus \{i\} \right\}.$$

Since V is non-leveled there must be $y \in V(S)$ such that $y > x_{S,\{N\}}$. Hence, $x_{S,\{N\}} \in \mathrm{int}V(S)$ and therefore, $x_{N,\{N\}} \in \mathrm{int}V_0(S)$.

On the other hand let $x_{N,\{N\}} \geq C_{N,\{N\}}(x)$ and suppose there is a coalition S such that $x_{S,\{N\}} \in \mathrm{int}V(S)$. Then there is $y \in V(S)$ such that $y \gg x_{S,\{N\}}$. In particular, $C_{S,\{N\}}(x) \geq y > x_{S,\{N\}}$ – contradicting our initial condition. □

We close this section with two further examples of proper monotonic simple games. It is well known that a simple game has a nonempty core if and only if there is at least one *veto player*, that is a player $i \in N$ such that $v(N \setminus \{i\}) = 0$ (Bondareva, 1962; Shapley, 1967).

Example 5.4. Let C be the claim function defined in Equation (10).

1. Let v be the game defined in Example 2.8. The only core element is $(1, 0, 0) \in \mathbb{R}^3$. Let the payoff configuration x be defined as

$$x_{i,\sigma} = \begin{cases} 1, & \text{if } v(\sigma(i)) = 1 \text{ and } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Then $x \in \mathcal{K}_C^F(V)$ for each fair and efficient bargaining solution F.

2. Let $N = \{1, 2, 3, 4\}$ and let v be the apex game with apex player 1, i.e. $v(S) = 1$ if and only if $\{2, 3, 4\} \subseteq S$ or $1 \in S$ and $|S| \geq 2$. Let $x_{i,\sigma} = 0$ whenever $v(\sigma(i)) = 0$. Let $x_{i,\sigma} = \frac{1}{3}$ if $\sigma(i) = \{1, 2, 3\}$. Let $x_1^\sigma = \frac{1}{2}$ if $|\sigma(1)| = 2$ or $|\sigma(1)| = 3$, and let $x_1^{\{N\}} = \frac{2}{5}$. Let the remaining $x_{i,\sigma}$ be determined by efficiency and symmetry. Then $x \in \mathcal{K}_C^E(V)$. Moreover, each partition containing a winning coalition of two players is core stable in the ordinal game $(N, \succeq^x)$.

6 Claim forms and opportunity costs

So far, we have imposed properties on claim forms (such as continuity and reasonableness) and investigated what the corresponding solutions look like. Another way to motivate claims is to ask: What drives players' expectations when bargaining? In this section, we will focus on two quantities, namely opportunity costs and marginal contributions. Let x be a payoff configuration, let σ be a partition of N and let $i \in N$. Player i's *opportunity costs* of joining $\sigma(i)$ rather than any of the other coalitions are given by

$$O_{i,\sigma}(x) = \max_{T \in \sigma \setminus \{\sigma(i)\}} x_i^{\sigma_{T+i}} - d_i(x), \tag{11}$$

where $\sigma_{T+i} = (\sigma \setminus \{\sigma(i), T\}) \cup \{\sigma(i) \setminus \{i\}, T \cup \{i\}\}$. The opportunity costs of i in σ are the surplus (compared to her disagreement point) she would have received if she had joined her best coalition in σ apart from $\sigma(i)$.

The definition of opportunity costs is straightforward, but things become complicated when dealing with marginal contributions in the presence of non-transferable utilities or externalities (or both). A suggestion how to deal with externalities has been given by de Clippel and Serrano (2008), where vectors of marginal contributions are defined. Otten et al. (1998) proposed a definition of marginal contributions for games with non-transferable utilities and without externalities. A discussion how marginal contributions should be defined in either case lies beyond the scope of this paper, and we will only assume that $M^V_{i,\sigma}$ is a suitable measure of player i's marginal contribution to coalition $\sigma(i)$ in partition σ in the game V.

Opportunity costs are independent of the game V and only depend on a specified payoff configuration x, whereas marginal contributions are independent of x and depend only on the game V. Marginal contributions are well established in cooperative game theory, but opportunity costs appear only implicitly, for instance as a motivation for the bargaining set. Nonetheless, both seem to be important for players' expectations of and agreement on their payoffs. From a rational point of view, marginal contributions should be irrelevant for a player's decision to accept or reject a payoff allocation. And yet, experimental studies (for instance of the ultimatum game, see Oosterbeek et al., 2004) have shown that outside options alone cannot explain people's behavior. We, therefore, consider a "hybrid" claim form which mixes opportunity costs and marginal contributions. More specifically, we are interested in claim forms of type

$$C^{\mu,V}_{i,\sigma}(x) = d_i(x) + \mu \max\left\{0, M^V_{i,\sigma(i)}\right\} + (1-\mu) O_{i,\sigma}(x) \tag{12}$$

for $\mu \in [0,1]$. A few comments on this claim functions are in order. First, it is normalized in such a way that $C^{\mu,V}_{i,\sigma}(x) \geq d_i(x)$. Even without any marginal contributions, there is no good argument for a claim to lie below the payoff a player could realize by simply leaving the game. Second, marginal contributions might be negative. But it seems unreasonable to include a negative marginal contribution in one's claim. We, therefore, assume that marginal contributions are relevant only if they are positive. Third, while opportunity costs are an easy-to-interpret absolute value, the scale of $M^V_{i,\sigma(i)}$ is not clear. However, this shall not cause too much trouble if we choose $M^V_{i,\sigma(i)}$ as an absolute measure of the damage player i can cause when leaving $\sigma(i)$. Fourth, the weight μ between marginal contribution and opportunity costs can be interpreted as a measure of cooperation: While opportunity costs are a rational and justified claim, as they can be realized when leaving a coalition, marginal contributions are a rather destructive bargaining chip. By leaving a coalition, player i cannot gain $M^V_{i,\sigma(i)}$; she can just cause a damage of $M^V_{i,\sigma(i)}$ for the remaining players.

The aim of this Section is to prove the following theorem.

Theorem 6.1. *Let $\mu > \frac{1}{2}$. Then $\mathcal{K}^E_{C^\mu}$ is nonempty and single valued.*

The theorem states that for each game V there is exactly one payoff configuration which is consistent with E under C^μ; and, obviously, this payoff configuration is efficient. The following lemma plays a key role in the proof of Theorem 6.1. We use the maximum norm $\|.\|_\infty$ throughout this section, that is for $x \in \mathbb{R}^S$ we have $\|x\|_\infty = \max_{i\in S} |x_i|$.

Lemma 6.2. *Let (S, X, d, b) and (S, X, d, c) be two bargaining problems. Then*

$$\|E(S, X, d, b) - E(S, X, d, c)\|_\infty \leq 2\,\|b - c\|_\infty . \tag{13}$$

Proof. Let $b \neq c$ and assume without loss of generality $b \in \partial X$, that is $b = E(S, X, d, b)$. We can further assume that $|b_i - c_i| = |b_j - c_j|$ for all $i, j \in S$. If this is not the case, there is c' which satisfies this condition such that $c' = c + t\mathbb{1}_S$ for some $t \in \mathbb{R}$. Then

$$\|E(S, X, d, b) - E(S, X, d, c')\|_\infty = \|E(S, X, d, b) - E(S, X, d, c)\|_\infty$$

and $\|b - c'\|_\infty \leq \|b - c\|_\infty$, so that it is sufficient to show the claim for c'. Hence, let b, c be as described and let $t = |b_i - c_i| = \|b - c\|_\infty$. There are two disjoint sets $S^1, S^2 \subseteq S$ such that $b_i < c_i$ for all $i \in S^1$ and $b_i > c_i$ for all $i \in S^2$, i.e. $c = b + t\,(\mathbb{1}_{S^1} - \mathbb{1}_{S^2})$. Define

$$\begin{aligned} a^1 &= \sup\{c + t'\mathbb{1}_S : t' \in \mathbb{R} \text{ and } c + t'\mathbb{1}_S \not\gg b\} \\ a^2 &= \inf\{c - t'\mathbb{1}_S : t' \in \mathbb{R} \text{ and } c + t'\mathbb{1}_S \not\ll b\}. \end{aligned}$$

It is easy to see that $a^1 = c + t\mathbb{1}_S = b + 2t\mathbb{1}_{S^1}$ and $a^2 = c - t\mathbb{1}_S = b - 2t\mathbb{1}_{S^2}$. By definition of E and comprehensiveness of X, we must have $E(S, X, d, c) = \lambda a^1 + (1 - \lambda)a^2$ for some $\lambda \in [0, 1]$. Hence, we find

$$\begin{aligned} \|E(S, X, d, b) - E(S, X, d, c)\|_\infty &= \left\|b - \lambda a^1 - (1 - \lambda)a^2\right\|_\infty \\ &= \|b - \lambda(b + 2t\mathbb{1}_{S^1}) - (1 - \lambda)(b - 2t\mathbb{1}_{S^2})\|_\infty \\ &= 2t\,\|(1 - \lambda)\mathbb{1}_{S^2} - \lambda\mathbb{1}_{S^1}\|_\infty \\ &= 2t \max\{\lambda, 1 - \lambda\}. \end{aligned}$$

Since $t = \|b - c\|_\infty$ and $\max\{\lambda, 1 - \lambda\} \leq 1$, the claim is proved. □

The power of Lemma 6.2 lies in the fact that Inequality (13) does not depend on the set X. However, it can be considerably sharpened if the structure of X is known. In particular, if $X = \left\{x \in \mathbb{R}^S : \sum_{i\in S} x_i \leq y\right\}$ for some $y \in \mathbb{R}$ then E is simply an orthogonal projection of c on the surface of X. Hence, in this case it holds that

$$\|E(S, X, d, b) - E(S, X, d, c)\|_\infty \leq \|b - c\|_\infty .$$

We are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let V be a game. We show that $\hat{E}^V$ as defined in the proofs of Subsection 2.4 is a contraction from $\Delta_{\text{eff}}(V)$ into itself. For this purpose let $x, y \in \Delta_{\text{eff}}(V)$, let $(S, \sigma) \in \mathcal{E}$, and let $i \in S$. Let $T, T' \in \{\emptyset\} \cup \sigma \setminus \{S\}$

be such that $O_{i,\sigma}(x) = x_{i,\sigma_{T+i}} - d_i(x)$ and $O_{i,\sigma}(y) = y_{i,\sigma_{T'+i}} - d_i(x)$. Without loss of generality let $x_{i,\sigma_{T+i}} \geq y_{i,\sigma_{T'+i}}$. Clearly, $y_{i,\sigma_{T'+i}} \geq y_{i,\sigma_{T+i}}$ by definition of $O_{i,\sigma}(y)$. Hence,

$$\begin{aligned}\left|C^{\mu,V}_{i,\sigma}(x) - C^{\mu,V}_{i,\sigma}(y)\right| &= (1-\mu)\left|O_{i,\sigma}(x) - O_{i,\sigma}(y)\right| \\ &= (1-\mu)\left|x_{i,\sigma_{T+i}} - y_{i,\sigma_{T'+i}}\right| \\ &\leq (1-\mu)\left|x_{i,\sigma_{T+i}} - y_{i,\sigma_{T+i}}\right| \\ &\leq (1-\mu)\left\|x-y\right\|_\infty .\end{aligned}$$

This implies $\left\|C^{\mu,V}(x) - C^{\mu,V}(y)\right\|_\infty \leq (1-\mu)\left\|x-y\right\|_\infty$, and by Lemma 6.2 we have

$$\begin{aligned}\left|\hat{E}_{i,\sigma}(x) - \hat{E}_{i,\sigma}(y)\right| &= \Big|E_i\left(S, V(S,\sigma), d^V_S, C^{\mu,V}_{S,\sigma}(x)\right) \\ &\qquad -E_i\left(S, V(S,\sigma), d^V_S, C^{\mu,V}_{S,\sigma}(y)\right)\Big| \\ &\leq 2\left\|C^{\mu,V}(x) - C^{\mu,V}(y)\right\|_\infty \\ &\leq 2(1-\mu)\left\|x-y\right\|_\infty .\end{aligned}$$

As $\mu > \frac{1}{2}$, we see that $\hat{E}$ is a contraction from $\Delta_{\text{eff}}(V)$ into itself. Hence, by Banach's fixed point theorem, $\hat{E}$ has a unique fixed point in $\Delta_{\text{eff}}(V)$. □

Theorem 6.1 shows that $\mathcal{K}^E_{C^\mu}$ is, in fact, a single-valued, efficient solution for all games, if $\mu > \frac{1}{2}$. Moreover, it can easily be motivated in terms of marginal contributions and opportunity costs, two basic concepts in economics. If we apply $\mathcal{K}^E_{C^\mu}$ on a game with transferable utility, it is single-valued actually for all $\mu > 0$. A deeper investigation of this solution in the context of proper monotonic simple games can be found in Karos (2013).

7 A non-cooperative game

Hart and Kurz (1983, 1984) introduced the following non-cooperative coalition formation game. Let x be a payoff configuration. Let the strategy set for each player i be the set $\mathcal{P}_i$ of coalitions containing i, and for any strategy vector $(S^1, \ldots, S^n)$ let the outcome of the game be the payoff vector $x_{N,\sigma}$, where $\sigma = \{\sigma(1), \ldots, \sigma(N)\}$ is such that

$$\sigma(i) = \begin{cases} S^i, & \text{if } S^i = S^j \text{ for all } j \in S^i; \\ \{i\}, & \text{otherwise.}\end{cases}$$

Hence, a coalition S forms if and only if all members of S choose S as their strategy; if the coalition a player chooses does not form, she stays alone.[17]

[17] The authors also investigate a version in which coalitions of players with the same strategy form. Our results in this section could be applied to this model in the same way.

Given a payoff configuration x, we will refer to this game as game Γ_x. The authors observe that a Nash equilibrium in this game always exists, but that it is not a meaningful equilibrium in this context. Instead they use the following equilibrium concept from Aumann (1967).

Definition 7.1. Let $(N, \mathcal{S}, u)$ be a game in normal form with player set N, strategy set $\mathcal{S}$, and payoff function $u : \mathcal{S} \to \mathbb{R}^N$. A *strong Nash equilibrium* is a strategy profile $(s^1, \ldots, s^N) \in \mathcal{S}$ such that for each group of players T and each strategy profile $\tilde{s}$, where $\tilde{s}^j = s^j$ for all $j \in N \setminus T$, there is $i \in T$ with

$$u_i(s) \geq u_i(\tilde{s}).$$

Hart and Kurz (1984) apply the Owen value (Owen, 1977) to games with transferable utility in order to construct Γ_x and observe that there are games for which such a strong equilibrium does not exist. We will use the tools from the previous sections to derive, for each game, a payoff configuration for which the aforementioned non-cooperative game possesses a strong equilibrium. More precisely, we will prove the following theorem.

Theorem 7.2. *Let V be a game without externalities. Then there is an efficient payoff configuration x with $x_{i,\sigma} = x_{i,\pi}$ for all $\sigma, \pi \in \Sigma$ with $\sigma(i) = \pi(i)$, such that the game Γ_x possesses a strong Nash equilibrium.*

There have been many approaches to non-cooperative formation of coalitions in a cooperative game. Most of them follow the tradition of Rubinstein (1982), where players make proposals on how to share payoffs. Chatterjee et al. (1993) generalized this theory to a model in which players propose not only how to share payoffs but also how to form coalitions. A variety of models in this spirit can be found in Ray (2007).

The theory outlined here, although resulting in a non-cooperative equilibrium, is different: The bargaining process itself does not play a key role, rather we fix a certain payoff configuration and do not leave any bargaining option to players. Nonetheless, this payoff configuration itself can be motivated by consistency conditions with respect to bargaining rules and claim functions using the previous sections. The big advantage compared to the model of Chatterjee et al. (1993) is that the strategies and payoffs of Γ_x are easily defined, once an appropriate vector x has been found. The game Γ_x is a one-shot game with complete information. In particular, no assumptions on players' preferences over time, and hence on their discounting factors, are required.

Before we prove Theorem 7.2, we make the following easy observation about the connection between the game Γ_x and the hedonic coalition formation game derived from x.

Lemma 7.3. *Let x be a payoff configuration without externalities. The game Γ_x possesses a strong Nash equilibrium if and only if the hedonic game $(N, \succeq^x)$ has a nonempty core.*

Proof. Suppose there is a core stable partition σ and assume that the strategies $s_i = \sigma(i)$ do not constitute a strong Nash equilibrium. Then there is a group T

and strategies t_i for all $i \in T$ which yield higher payoffs for all $i \in T$, provided that no $j \in N \setminus T$ changes her strategies. We can assume without loss of generality that $t_i = T$ for all $i \in T$. Indeed, if this were not the case, either there would have to be a coalition $T' \subseteq T$ such that $t_i = T'$ for all $i \in T'$, or all players would end up as singletons $\{i\}$. As a player's payoff depends only on the coalition she is a member of, we have in the first case that T' is a deviation with the desired property; and in the latter case that each player could have deviated by choosing $t_i = \{i\}$. Since the payoff configuration x does not exhibit externalities, i's payoff when choosing T does not depend on the strategies of players $j \in N \setminus T$. Hence, as $x_{i,\tau} > x_{i,\sigma}$ for all $i \in T$ and all $\tau \in \Sigma_T$, coalition T is a deviation of σ, which contradicts σ's being core stable.

On the other hand, suppose that there is a strong Nash equilibrium s and that σ is the corresponding partition of the player set. Assume that there is a deviation T of σ. Then s cannot be a strong equilibrium. Indeed, if all $i \in T$ chose strategy $t_i = T$, they would deviate from the strong equilibrium and be better of. □

The article of Shenoy (1979), and also those of Dimitrov and Haake (2008a) and Karos (2014), considered games with transferable utilities, calculated payoff configurations by applying fixed rules such as the Shapley value, and derived hedonic games from the resulting payoff configurations. One question in all of these papers was which conditions a game needs to satisfy in order to guarantee the existence of a core stable partition in the derived hedonic game. Unfortunately, none of these articles stated conditions which would be both sufficient and necessary; even in the case of proper monotonic simple games it remained an open question for which games a core stable partition exists.

In our model, however, the situation is different. Reasonable claim functions do not only guarantee the existence of consistent payoff configurations, but also non-emptiness of the ordinal core in the derived ordinal game.

Lemma 7.4. *Let V be a game without externalities and let C be a continuous and reasonable claim function. Then there is $x \in \mathcal{K}_C^E(V) \neq \emptyset$ such that $x_{i,\sigma} = x_{i,\pi}$ for all $\sigma, \pi \in \Sigma$ with $\sigma(i) = \pi(i)$. Moreover, there is a core stable partition in the hedonic game $(N, \succeq^x)$.*

Proof. Let $\Delta^*(V) \subseteq \Delta(V)$ be the collections of payoff configurations in $\Delta(V)$ with $x_{i,\sigma} = x_{i,\pi}$ for all $i \in N$ and all $\sigma, \pi \in \Sigma$ with $\sigma(i) = \pi(i)$. Clearly, $\Delta^*(V)$ is convex. As in the proof of Proposition 2.5 we can show that there is $x \in \Delta^*(V)$ which is consistent with E under C. Since E is fair and C is reasonable, x is balanced. Hence, the existence of a core stable partition in $(N \succeq^x)$ follows from Proposition 3.6. □

We can now complete the proof of Theorem 7.2 by recalling that the claim function in Equation (8) in Example 2.7 satisfies the conditions of Lemma 7.4. So far, it remains an open question whether the idea of this Section generalizes to games with externalities. Nonetheless, Theorem 7.2 is a useful result. It enables us to construct for each cooperative game without externalities a non-cooperative coalition formation game with a strong Nash equilibrium. Players do not have the option to negotiate payoffs or to make any proposals in this game. They are only confronted with the choice which coalition to form, given the payoffs they see.

8 Conclusion

Since the seminal book of von Neumann and Morgenstern (1944) the theory of cooperative games has made significant developments. A variety of solutions for games with and without transferable utilities has been proposed and motivated by different axioms in different contexts. The special cases of bargaining and bankruptcy problems have lead to insights which have been incorporated into this theory. In this paper we propose an approach which applies to very general games and yet can be refined in ways such that we obtain classical solutions. The idea of bargaining and claims seems, therefore, to give a common framework to many solutions that seemed otherwise unrelated.

But it is not only its generality that makes this model appealing: it delivers a framework in which other questions in the theory of coalition formation can be solved. We have seen that we can transform every game into an ordinal game for which a core stable partition exists, and that we can find, for each game without externalities, a payoff configuration which delivers a strong equilibrium in a non-cooperative coalition formation game. Hence, our model provides some answers to classical questions.

It will be important to investigate which bargaining rules and claim forms are the right choice, either to depict reality or to derive normative results. The research on axiomatic bargaining, which would be used for the latter, is already very advanced (see for instance Peters, 1992). In the context of bargaining with claims (or rationing with constraints) it seems that the proportional solution has many useful properties, both in our model and in previous articles (Chun and Thomson, 1992; Moulin, 2000). But it remains an open question what other efficient, individually rational and weakly fair bargaining solutions may look like.

From a descriptive perspective, the results of Nydegger and Owen (1974) also suggest that the proportional bargaining rule is appropriate (see for instance Kalai, 1977). Findings from experiments in the context of the ultimatum game (see Oosterbeek et al., 2004, for a meta-study) suggest that fairness is an important property of division rules. It is an established fact that responders often reject proposals which they consider too low, and this observation can be explained in a model of bargaining with claims where a bargaining rule is accepted only if it is fair. The fairness property of the proportional solution makes it therefore very appealing.

With the research into claim functions we enter unknown territory. In the context of bankruptcy problems, claims were assumed to be externally given, for instance in the form of unpaid debts. If we reinterpret the bargaining solution of

Kalai and Smorodinsky (1975) as a proportional solution in a bargaining problem with claims, a player's claim would be the highest payoff she could receive while her opponent's payoff does not fall below her disagreement point. Unfortunately, experimental evidence in Nydegger and Owen (1974) does not support this solution. This means that, under the assumption that the proportional rule is appropriate, this claim function is not.

A last point that is worth mentioning in the context of claim functions is that they enable us to solve a game even if players do not know the surpluses of all coalitions. If we turn back to our initial example of Alstom, General Electric, and the consortium of Siemens and Mitsubishi, we see that Siemens required full access to Alsom's data room prior to any offer. The potential surpluses of a collaboration were not known to Siemens, let alone to General Electric. And even after Siemens made an offer, General Electric only observed this offer, but not how much Alstom could gain from a collaboration with Siemens and Mitsubishi in an optimal agreement. The latter was not relevant for General Electric's revision of the initial proposal. Hence, in order to apply our model on a game V, this game does not need to be known to all players. If the claim form does not depend on the game itself, it is sufficient that each $i \in N$ knows $V(S, \sigma)$ for all embedded coalitions (S, σ) with $i \in S$ in order to find a consistent payoff configuration.

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