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EXISTENCE OF SPE IN DISCOUNTED STOCHASTIC GAMES; REVISITED AND SIMPLIFIED

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Existence of SPE in Discounted Stochastic Games; Revisited and Simplified

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Abstract

Mertens and Parthasarathy (1987) proved the existence of sub-game perfect equilibria in discounted stochastic games. Their method involved new techniques in dynamic programming, which were presented in a very general framework, with no expense spared in highlighting versatility and scope. This paper presents the fundamentals of their technique which are necessary, as well as identifies and elaborates on the components of their method, hence giving the core of the proof in a much more concise, direct, and illuminating manner.

Keywords: Stochastic Games; Equilibrium Existence; Subgame-Perfect Equilibrium
JEL Classifications: C62, C65, C73

1 Introduction

Mertens and Parthasarathy (1987), [14], proved the existence of sub-game perfect equilibrium in discounted stochastic games. Their proof, however, is set upon a very general backdrop; in addition, they strived to build all the tools they require in the broadest sense possible. While their work represented a brilliant tour-de-force of sophisticated mathematical tools, the generality for which they strive has unfortunately left their proof somewhat shrouded in mystery.

Stochastic games were introduced by Shapley (1953), [20]. In a stochastic game, players play in discrete stages, with stochastic transitions between states chosen using distributions determined by the state and action. In the β-discounted game, each player receives the β-discounted sum of the stream of his stage-by-stage payoffs.

When the state space is finite or discrete, it is in fact possible to deduce the existence of equilibrium in a particular simple class of strategies, those of *stationary strategies*, in which players makes their decisions based only on the

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current state. [20], which deals with zero-sum games, already establishes the existence of stationary optimal strategies; existence of stationary equilibria for finite state games was proven independently in [5] and [22], and for the countable state case in [16].

In fact, it had been unknown for a long time whether for general stochastic games - those with general state spaces - stationary equilibria must exist. A negative answer was only recently given by the author in [10] and [11].[1]

When the state space is general, even proving equilibrium existence in general (behavioural) strategies is no easy feat, and as we said it was given by [14], relying on a slew of heavy technical tools, particular a measurable "measurable selection" theorem given in [13]; subsequent proofs were given in [21] and [12] also relying on the theorem given in [13].

As we have mentioned, the proof given in [14] is quite lengthy and cumbersome. The tools that are developed are done so with the utmost generality in mind, which unfortunately makes it difficult to follow the fundamental ideas. [21] and [12] take an approach which, although with similarities to the approach of [14], has a substantial difference. [14] constructs the candidate set of SPE games in two stages: First a large set of payoffs which are feasible in the game is built up, and then the set is winnowed down. Once this has been accomplished, and one has the theorem given in [13] in hand, one can show that the resulting payoffs can be supported by an SPE. The approach taken in [21] and [12] instead constructs the candidate payoffs via a technique of approximation: First by establishing the existence of a sequence of sets of approximate equilibria, and then by taking certain limits of these.

The purpose of this paper is not to offer an original proof, but to present largely the proof from [14] in a concise and illuminating manner, hence both giving tribute to the novel and elegant techniques of that paper and making those techniques as accessible as possible. As such, with these intents in mind, the layout of the proof (following the presentation of the model in Section 2) takes the following form:

- First we isolate the measure-theoretical preliminaries on correspondences in Section 3.1, concentrated in Proposition 3.3. Although this proposition is of course important, the reader can skip the proof without hurting the comprehension of the later sections. This isolation allows the essential parts of the proof to then proceed significantly more smoothly.

- In Section 3.2, we present the result [13] and derive Theorem 3.2, which is the selection theorem we will need. In this section in particular, care has been taken to help the reader understand the content and implication of these theorems, which otherwise appear rather cryptic. Also, unlike previous applications of the selection theorem, which only identify payoff and strategy functions, our Theorem 3.2 also identifies a 'game function'.

[1] [10] presents two counter-examples of different types; the second has a serious flaw, which is pointed out and corrected in [11].

- Following these, the proof of the existence of a SPE follows in three short steps: First, in Section 4 we construct an appropriate sub-class of payoffs (those given by the 'oblivious strategies' - strategies which ignore previous actions); in Section 5 we whittle away at these to get those payoffs which are candidates for SPE payoffs; then in Section 6 we easily apply Theorem 3.2 to construct an equilibrium. In particular in the first step, we are able to identify the payoffs derived as those supported by a particular class of strategies, hence further shedding light the construction.

2 The Discounted Stochastic Games Model

The components of a discounted stochastic game with a continuum of states and compact action spaces are the following:

- A standard Borel[2] space Ω of states.
- A finite set $\mathcal{P}$ of players.
- A compact metric set of actions A^p for each $p \in \mathcal{P}$. Denote $A = \prod_{j\in\mathcal{P}} A^j$ and $X = \prod_{j\in\mathcal{P}} \Delta(A^j)$.[3]
- A discount factor $\beta \in (0, 1)$.
- A bounded payoff function $r : \Omega \times A \to \mathbb{R}^{\mathcal{P}}$, which is Borel-measurable and such that for each $\omega \in \Omega$, $r(\omega, \cdot)$ is continuous.
- A transition function $q : \Omega \times A \to \Delta(\Omega)$, which is Borel-measurable in the sense that for each Borel $B \subseteq \Omega$, the mapping $\Omega \times A \to \mathbb{R}$ given by $(z, a) \to q(B \mid z, a)$ is Borel, and such that for each $z \in \Omega$, $q(z, \cdot)$ is continuous in total-variation norm.

The game is played in discrete time. If $z \in \Omega$ is the state at some stage of the game and the players select an action profile $a \in A$, then $q(z, a)$ is the conditional (given the past) probability distribution of the next state of the game. For any profile of behavioral strategies $\sigma = (\sigma^p)_{p\in\mathcal{P}}$ of the players and every initial state $z_1 = z \in \Omega$, a probability measure P^σ_z and a stochastic process $(z_n, a_n)_{n\in\mathbb{N}}$ are defined on $H^\infty := (\Omega\times A)^{\mathbb{N}}$ in a canonical way, where the random variables z_n, a_n describe the state and the action profile chosen by the players, respectively, in the n-th stage of the game (see, e.g., [3]). The payoff is given by

$$r_\infty(z_1, a_1, z_2, \ldots) = \sum_{n=1}^{\infty} \beta^{n-1} r(z_n, a_n)$$

Hence, the expected payoff vector under σ in the game starting from state z is $E^\sigma_z[r_\infty]$.

[2] That is, a space that is homeomorphic to a Borel subset of a complete, metrizable space.
[3] For a Borel set B, $\Delta(B)$ denotes the set of regular Borel probability measures on B.

Denote $H_n = (\Omega \times A)^{n-1} \times \Omega$ for $n \in \mathbb{N}$, and $H_* = \cup_n H_n$ is the space of finite histories. For each $p = (\zeta_1, \alpha_1, \ldots, \zeta_n) \in H_*$ and each behavioural strategy σ, we have the probability measure P_p^σ defined by $z_k = \zeta_k$ and $a_l = \alpha_l$ for $k \leq n$, $l < n$, and henceforth choices are made according to σ. Hence, the expected payoff - interpreted as the expected payoff *given that* p *has occurred* - is $E_p^\sigma[r_\infty]$.

Let Σ^p denote the set of behavioral strategies for Player $p \in \mathcal{P}$, and $\Sigma = \prod_{p \in \mathcal{P}} \Sigma^p$. A profile $\sigma \in \Sigma$ will be called a subgame-perfect Nash equilibrium (SPE) if

$$E_p^\sigma[r_\infty^j] \geq E_p^{(\tau, \sigma^{-j})}[r_\infty^j] \qquad \forall j \in \mathcal{P}, \forall p \in H_*, \forall \tau \in \Sigma^j \tag{2.1}$$

The following is a well-known dynamic programming result:

Proposition 2.1. *$\sigma \in \Sigma$ is an SPE if and only if for each $n \in \mathbb{N}$ and each $p = (\cdots, z) \in H_n$, $\sigma(p)$ is an equilibrium of the game X_p^σ defined by*

$$X_p^\sigma(a) = r(z, a) + \beta \int_\Omega E_{(p,a,z')}^\sigma [r_{\geq n+1}] q(dz' \mid z, a) \tag{2.2}$$

where

$$r_{\geq k}(z_k, a_k, z_{k+1}, \ldots) = \sum_{t \geq k} \beta^{t-k} r(z_t, a_t) \tag{2.3}$$

The result of [14], which we re-prove in this paper, is:

Theorem 2.1. *Every discounted stochastic games possesses an SPE.*

We remark that more general models, e.g., [14] or [15], allow for action spaces which depend in a measurable way on the state. Such generalisations do not present particular difficulties (largely due to the Castaing representation of measurable correspondences, see Proposition 3.2 below), but makes the notations much more cumbersome and the proofs more technical.

3 Preliminaries

3.1 Correspondences

Throughout, fix some norm $||\cdot||$ on Euclidean spaces. The Banach space of bounded Borel (resp. continuous) functions, with supremum norm $||\cdot||_\infty$, from a space X with appropriate Borel (resp. topological) structure to a Banach space Y is denoted $B(X, Y)$ (resp. $C(X, Y)$). A *correspondence* N from X to Y, denoted $N : X \Longrightarrow Y$, assigns to each $x \in X$ a subset $N(x) \subseteq Y$. The $(N(x))_{x \in X}$ are the *values* of N. The graph of N is $Gr(N) := \{(x, y) \mid y \in N(x)\}$. Denote $||N||_\infty = \sup_{(x,y) \in Gr(N)} ||y||$. By *Borel correspondence*, we mean one with a Borel graph; by a *bounded* correspondence, we mean that $||N||_\infty < \infty$. Given a correspondence $N : X \Longrightarrow Y$, denote the collection of *Borel selectors* of N by:

$$\mathcal{S}_N = \{f : X \to Y \mid f \text{ is Borel and } \forall x \in X, f(x) \in N(x)\}$$

Definition 3.1. *In this paper, we will call a correspondence well-behaved if it is bounded, Borel, with non-empty compact values.*

By the Kuratowski-Ryll Nardzewski selection theorem, if N is a well-behaved correspondence,[4] $\mathcal{S}_N \neq \emptyset$; we will make repeated implicit use of this fact.

For a Polish space X with metric d, let 2^X denote the space of compact subsets of X with the Hausdorff topology given by the metric[5]

$$d_H(A,B) = \max[\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)] \tag{3.1}$$

where for a point x and set A, $d(x,A) = \inf_{y\in A} d(x,y)$. See, e.g., [3, Appendix C] or [1, Sec. 3.16] for more on this topology.

Proposition 3.2. *For a correspondence $N : X \Longrightarrow Y$ between Borel spaces with non-empty compact values, the following are equivalent:*

- *N is Borel (i.e., $Gr(N)$ is Borel).*
- *The mapping $x \to N(x)$ is a Borel mapping $X \to 2^Y$.*
- *(Castaing Representation) There is a countable collection $D \subseteq \mathcal{S}_N$ such that for all $x \in X$, $N(x) = \overline{\{f(x) \mid f \in D\}}$.*

See [7], also [6] or [4, Ch. 3], for equivalence between the first two conditions, and [6, Thm. 5.6] for equivalence with the third.

For each $V \in B(\Omega, \mathbb{R}^{\mathcal{P}})$, let

$$\Phi(V)(z)(a) = r(z,a) + \beta \int_\Omega V(z')q(dz' \mid z,a)$$

We view $\Phi(V)$ as a map from Ω to $\mathfrak{G} := C(A, \mathbb{R}^{\mathcal{P}})$ - the space of continuous functions from A to $\mathbb{R}^{\mathcal{P}}$, identified with the set of games - given by $z \to \Phi(V)(z)(\cdot)$. For each $G \in \mathfrak{G}$, let $E(G) \subseteq X$ denote the set of Nash equilibria of G.

Define for each $V \in B(\Omega, \mathbb{R}^{\mathcal{P}})$ correspondences $\Pi_X(V), \Pi_E(V) : \Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$ by

$$\Pi_X(V)(z) = \{\Phi(V)(z)(x) \mid x \in X\} \text{ and } \Pi_E(V)(z) = \{\Phi(V)(z)(x) \mid x \in E(\Phi(V)(z))\}$$

Φ generalises to correspondences: if $N : \Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$ is a correspondence, then

$$\Phi(N) = \{\Phi(V) \mid V \in \mathcal{S}_N\}$$

and let $\overline{\Phi} : \Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$ be defined by $\overline{\Phi}(N)(z) = \overline{\Phi(N)(z)}$. Then define

[4] The boundedness is not required for this theorem.

[5] The topology induced by d_H can be shown to be independent of the specific metric d on X.

$$\overline{\Pi}_X(N)(z) = \{G(x) \mid G \in \overline{\Phi}(N)(z),\ x \in X\}$$

$$\overline{\Pi}_E(N)(z) = \{G(x) \mid G \in \overline{\Phi}(N)(z),\ x \in E(G)\}$$

The following proposition contains the technical machinery we will need, save for the selection theorem given in Section 3.2.

Proposition 3.3. *Suppose $N : \Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$ is well-behaved.*

1. *For $\sigma \in \Sigma$, $n \in \mathbb{N}$ and any bounded Borel function $f : H_\infty \to \mathbb{R}^n$, the mapping $H_* \to \mathbb{R}^n$ is given by $p \to E^\sigma_p(f)$ is Borel.*
2. *For each $V \in B(\Omega, \mathbb{R}^{\mathcal{P}})$, $\Phi(V)$ is a Borel mapping from Ω to $\mathfrak{G}$.*
3. *If $z \in \Omega$, and $(V_n)_{n=1}^\infty$ and $(V'_n)_{n=1}^\infty$ are bounded sequences in $B(\Omega, \mathbb{R}^{\mathcal{P}})$ such that $V_n - V'_n \to 0$ weak-$*$ in $L^\infty(\mu)$, where $\mu \in \Delta(\Omega)$ is such that*[6] *$q(z,a) \ll \mu$ for all $a \in A$, then $\Phi(V_n)(z) - \Phi(V'_n)(z) \to 0$ in $\mathfrak{G}$. In particular, if $V_n \to V$ weak-$*$ in $L^\infty(\mu)$, $\Phi(V_n)(z) \to \Phi(V)(z)$ in $\mathfrak{G}$.*
4. *For all $z \in \Omega$ and $a \in A$, $\Phi(N)(z)(a) = \overline{\Phi}(N)(z)(a)$,*[7] *and in particular,*

$$\overline{\Pi}_X(N)(z) = \{\Pi_X(V)(z) \mid V \in \mathcal{S}_N\}$$

5. *$\overline{\Phi}(N), \overline{\Pi}_X(N)$, and $\overline{\Pi}_E(N)$ are well-behaved; in particular, if $V \in B(\Omega, \mathbb{R}^{\mathcal{P}})$, then $\Pi_X(V)$, $\Pi_E(V)$ are well-behaved.*
6. *If $(N_k)_{k=1}^\infty$ are a decreasing sequence of well-behaved correspondence $\Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$ (that is, $N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots$ point-wise), and if $N = \cap_{n\in\mathbb{N}} N_n$ point-wise, then N is well-behaved and $N \subseteq \overline{\Pi}_E(N)$.*

The reader can skip the proof of these technical results without any loss in comprehension in the continuation of the paper.

Proof. 1. Since any bounded Borel function can be uniformly approximated by linear combinations of indicator functions, and any Borel function on an infinite product space can be approximated point-wise by a sequence of Borel functions which depend only on finitely many coordinates, it suffices to show that for each $N \in \mathbb{N}$ the transition kernel from H_* to H_N, which assigns to each $p \in H_*$ the marginal P^σ_p on H_N, is Borel,[8] and this is a standard result, e.g., [8, Lemma 1.38].

[6] $\nu \ll \mu$ means that $\mu(A) = 0$ impies $\nu(A) = 0$ for Borel $A \subseteq \Omega$.

[7] Note that this does not imply that $\Phi(N)(z) = \overline{\Phi}(N)(z)$; it is possible for two collections of games $\mathfrak{G}_1$ and $\mathfrak{G}_2$ to be different and yet satisfy $\{G(a) \mid G \in \mathfrak{G}_1\} = \{G(a) \mid G \in \mathfrak{G}_2\}$ for each $a \in A$.

[8] I.e., for each Borel $E \subseteq H_N$, the mapping $p \to P^\sigma_p(E)$ is Borel.

2. For each $x \in \Delta(A)$, the mapping $z \to \Phi(V)(z)(x)$ is Borel.[9] Hence, since the dual space of $\mathfrak{G}$ is $\Delta(A)$, the mapping $z \to \Phi(V)(z)$ is weakly measurable (i.e., its composition with any linear functional on $\mathfrak{G}$ is measurable), and $\mathfrak{G} = C(A, \mathbb{R}^{\mathcal{P}})$ is separable; by the Pettis measurability theorem, [17, Thm. 1.1], in separable Banach spaces, weak measurability implies measurability.

3. It suffices to show that for all $z \in \Omega$, $\int_\Omega (V_n(z') - V'_n(z'))q(dz' \mid z, a) \to 0$ uniformly in a. The convergence point-wise in A follows by assumption; hence, it converges uniformly on any finite set $B \subseteq A$; since the set $\{q(z,a) \mid a \in A\}$ is norm-compact and the (V_n) and (V'_n) are uniformly bounded, the uniform convergence then follows by a typical "3ε"-type argument using (3.2) below.

4. This follows from Lemma 3.4, which is established in the course of the proof of Theorem 3.2.

5. It's enough to show that $\overline{\Phi}$ is well-behaved; the well-behavedness of $\overline{\Pi}_E(N)$ (resp. $\overline{\Pi}_X(N)$) then follows, since it is a composition of the map $\overline{\Phi}(N) : \Omega \to 2^{\mathfrak{G}}$ and the map $\mathfrak{G} \to 2^{\mathbb{R}^{\mathcal{P}}}$ given by $G \to \{G(x) \mid x \in E(G)\}$ (resp. $G \to \{G(x) \mid x \in X\}$), both of which are well-behaved; see Lemma 7.3. We just need to show that it has a Borel graph; the non-emptiness of values and the boundedness are immediate and compactness of values follows from the Arzela-Ascoli theorem, since one can easily check that for each $z \in \Omega$, $\Phi(N)(z)$ is an equicontinuous family of functions; indeed, for any $V \in \mathcal{S}_N$ and any $a, b \in A$,

$$\begin{aligned} ||\Phi(V)(z)(a) - \Phi(V)(z)(b)|| &= || \int_\Omega V(z')q(dz' \mid z, a) - \int_\Omega V(z')q(dz' \mid z, b)|| \\ &\leq ||N||_\infty ||q(z,a) - q(z,b)|| \end{aligned} \tag{3.2}$$

Let $D \subseteq \mathcal{S}_N$ be a countable set such that for any $\mu \in \Delta(\Omega)$, D is weak-$*$ dense in $\mathcal{S}_N \subseteq L^\infty(\Omega, \mu)$; such exists by Proposition 7.2. Let $F : \Omega \Longrightarrow \mathfrak{G}$ be defined by $F(z) = \{\Phi(V)(z) \mid V \in D\}$. By part (3), for all $z \in \Omega$, $\Phi(N)(z) \subseteq \overline{\{F(z)\}} \subseteq \overline{\Phi}(N)(z)$; hence $\overline{\Phi}(N)(z) = \overline{\{F(z)\}}$. Hence, $\Phi(N)$ is well-behaved by Proposition 3.2.

6. The well-behavedness of N follows easily, noticing $Gr(N) = \cap_{n=1}^\infty Gr(N_n)$. Let $z \in \Omega$ and $q \in N(z)$; hence, for each $n \in \mathbb{N}$, there is $G_n \in \overline{\Phi}(N_n)(z)$ and $x_n \in E(G_n)$ with $q = G_n(x_n)$. W.l.o.g.,[10] $(G_n), (x_n)$ converge to G, x respectively; then $x \in E(G)$ (by the upper-semicontinuity of the Nash correspondence) and $q = G(x)$. We will show that $G \in \overline{\Phi}(N)(z)$; this will

[9] This follows by definition of the measurability of the transition kernel if V is an indicator function, and one can use an approximation argument to show that it holds for all bounded Borel V.

[10] Recall that $\overline{\Phi}(N_1)(z)$ is compact by part (5).

show that $q \in \overline{\Pi}_E(N)(z)$, as required.

Let $\varepsilon > 0$, let $Y \subseteq X$ be finite such that $\{q(z,y) \mid y \in Y\}$ is $\frac{\varepsilon}{3||N||_\infty}$-dense in $\{q(z,x) \mid x \in X\}$ in total variation norm,[11] and fix $\mu \in \Delta(\Omega)$ such that $q(z,a) \ll \mu$ for each $a \in A$.[12] Choose a sequence (W_n) with $W_n \in \mathcal{S}_{N_n}$, such that $\Phi(W_n)(z) - G_n \to 0$, and hence $\Phi(W_n)(z) \to G$; such exists by the definition of $\overline{\Phi}$. For each $\omega \in Z$, it's standard to see that $d_H(N_n(\omega), N(\omega)) \to 0$.[13] Now for each n, let V_n correspond to W_n and the correspondence N as in Proposition 7.1; i.e., for each $\omega \in \Omega$,

$$||W_n(\omega) - V_n(\omega)|| = d(W_n(\omega), N(\omega)) \leq d_H(N_n(\omega), N(\omega)) \to 0$$

and in particular $W_n - V_n \to 0$ weak-$*$ in $L^\infty(\mu)$. Hence $\Phi(V_n)(z) - \Phi(W_n)(z) \to 0$ by part (3), and in particular $\Phi(V_n)(z) \to G$. By the Banach-Alaoglu theorem we can further assume that $V_n \to W$ weak-$*$ in $L^\infty(\mu)$ for some $W \in \mathfrak{G}$, and hence again by part (3), $G = \Phi(W)(z)$. By Lemma 7.4 and Proposition 7.5, there is $V \in \mathcal{S}_N$ such that for each $y \in Y$, $\int_\Omega W(z')q(dz' \mid z,y) = \int_\Omega V(z')q(dz' \mid z,y)$. From our density assumption on Y and from (3.2), $||\Phi(V)(z) - G||_\infty = ||\Phi(V)(z) - \Phi(W)(z)||_\infty < \varepsilon$. Since $\varepsilon > 0$ was arbitrary and $V \in \mathcal{S}_N$, $G \in \overline{\Phi}(N)(z)$.

□

3.2 A Selection Theorem

The following is the fundamental result from [13], adapted to a bounded Borel setting:[14]

Theorem 3.1. *Let Y, Z be Borel spaces, and let F be a well-behaved correspondence from $Y \times Z$ to $\mathbb{R}^n$. Let q be a Borel transition kernel from Y to Z. Define a correspondence from $F^\diamond$ from Y to $\mathbb{R}^n$ by:*

$$F^\diamond(y) = \{\int_Z f(y,z)q(dz \mid y) \mid f \in \mathcal{S}_F\}$$

Then:

- *$F^\diamond$ is a well-behaved correspondence.*
- *There is a Borel mapping $g : Gr(F^\diamond) \times Z \to \mathbb{R}^n$ such that for each $(y,u) \in Gr(F^\diamond)$ and $z \in Z$, $g(y,u,z) \in F(y,z)$ and*

$$u = \int_Z g(y,u,s)q(ds \mid y)$$

[11] The latter set is norm-compact.

[12] E.g., let $(a_n)_{n=1}$ be dense in A, and let $\mu = \sum_{n=1}^\infty 2^{-n} q(z,a_n)$; by the norm-continuity of the transitions, this suffices.

[13] Recall the definition of Hausdorff metric given in (3.1).

[14] A very similar but weaker result is proven in [2]; the essential difference that is that [2] only gives an "almost everywhere" type of selection.

The theorem actually says something quite simple: For each $(y,u) \in Gr(F^\diamond)$, it follows by the definition of $F^\diamond$ there is a Borel function $g_{y,u}(\cdot) \in \mathcal{S}_{F(y,\cdot)}$ - i.e., $g_{y,u}(z) \in F(y,z)$ for all $z \in Z$ - such that $u = \int_Z g_{y,u}(s)q(ds \mid y)$. The theorem says that this assignment can be done in a measurable way - jointly measurable in all variables.

We will use this to prove:

Theorem 3.2. *Let N be a well-behaved correspondence $\Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$. Then there exists:*

- *a Borel mapping $W : Gr(\overline{\Pi}_E(N)) \times A \times \Omega \to \mathbb{R}^{\mathcal{P}}$,*
- *a Borel mapping $G : Gr(\overline{\Pi}_E(N)) \to \mathfrak{G}$,*
- *a Borel map $S : Gr(\overline{\Pi}_E(N)) \to X$,*

such that for all $(z,q) \in Gr(\overline{\Pi}_E(N))$, all $a \in A$, and all $z' \in \Omega$:

- *$G(z,q)(a) = \Phi(W(z,q,a,\cdot))(z)(a)$, or explicitly,*

$$G(z,q)(a) = r(z,a) + \beta \int_\Omega W(z,q,a,s)q(ds \mid z,a)$$

- *$W(z,q,a,z') \in N(z')$.*
- *$S(z,q) \in E(G(z,q))$.*
- *$q = G(z,q)(S(z,q))$.*

This theorem too says something quite simple: First, if we have $(z,q) \in \overline{\Phi}_E(N)$, then by definition, there is $G_{z,q} \in \overline{\Phi}(N)(z) \subseteq \mathfrak{G}$ such that q is an equilibrium payoff of $G_{z,q}$ under some equilibrium profile $x_{z,q}$; the existence of such G, S as guaranteed by the theorem allow these selections to be done measurably. (This part does not require yet any use of Theorem 3.1.) If we further knew that $G_{z,q}$ could be chosen to be in $\Phi(N)(z)$, then we would deduce that there is $V_{z,q}(\cdot) \in \mathcal{S}_N$ such that $G_{z,q} = \Phi(V_{z,q})(z)$; however, we don't know this. We overcome the difference between $\Phi(N)$ and $\overline{\Phi}(N)$ by using the closed-ness of values of the integral correspondence that the first conclusion of Theorem 3.1 guarantees, but to do this we may need to require that the selection $V_{z,q} \in \mathcal{S}_N$, which represents "tomorrow's payoff" be dependent not only on (z,q) but also on "today's action profile" a; hence, the selection is of $V_{z,q,a} \in \mathcal{S}_N$ such that $G_{z,q}(a) = \Phi(V_{z,q,a})(z)(a)$.[15] The function W produced by Theorem 3.2 precisely implies that this selection could be done measurably.

[15] If the action spaces were finite, we would not need this step and the dependence of W on the actions could be done away with, by using a version of Theorem 3.1 as given in [14] which allows for vector-valued transition kernels. Hence, when action spaces are finite, we can obtain equilibria which are independent of previous actions.

Proof. Define a correspondence $\Psi : Gr(\overline{\Pi}_E(N)) \Longrightarrow \mathfrak{G} \times X$ given by

$$\Psi(z, q) = \{(G, x) \in \mathfrak{G} \times X \mid G \in \overline{\Phi}(N)(z), x \in E(G), q = G(x)\}$$

We contend that Ψ is well-behaved. The non-emptiness and boundedness of values follows by definition, the compactness of values follows from the well-behavedness of $\overline{\Phi}(N)$ and E. That $Gr(\Psi)$ is Borel follows since Ψ can be written point-wise as the intersection of two correspondences which have Borel graphs, and $Gr(\Psi)$ is then then the intersection of their graphs:

$$\Psi(z, q) = \overline{\Phi}(N)(z) \times X \cap \{(G, x) \in \mathfrak{G} \times X \mid x \in E(G), q = G(x)\}$$

Let $\psi \in \mathcal{S}_\Psi$, and write $\psi = (G, S)$. Clearly the final two properties required of G, S in the theorem hold. Now, denote $Y = \Omega \times A$, $Z = \Omega$, and define the correspondence $M : Y \times Z \Longrightarrow \mathbb{R}^{\mathcal{P}}$ by $M(y, z) = r(y) + \beta N(z)$. M is clearly well-behaved. Define

$$\begin{aligned} M^\diamond(y) &= \{\int_Z f(y, z')q(dz' \mid y) \mid f \in \mathcal{S}_M\} \\ &= \{r(y) + \beta \int_Z f(z')q(dz' \mid y) \mid f \in \mathcal{S}_N\} \end{aligned}$$

Since it is used in the proof of Proposition 3.3.4, we remark as a separate lemma:

Lemma 3.4. *For all $z \in \Omega, a \in A$, $M^\diamond(z, a) = \Phi(N)(z)(a)$, and this set is closed.*

The equality is immediate, while the closed-ness of $M^\diamond(z, a)$ follows from the first conclusion of Theorem 3.1.

By the second conclusion of Theorem 3.1, there is Borel $g : Gr(M^\diamond) \times Z \to \mathbb{R}^{\mathcal{P}}$ such that for $(y, v, z') \in Gr(M^\diamond) \times Z$, $g(y, v, z') \in r(y) + \beta N(z')$ and $v = \int_\Omega g(y, v, z')q(dz' \mid y)$.

Now define $W : Gr(\overline{\Pi}_E(N)) \times A \times \Omega \to \mathbb{R}^{\mathcal{P}}$ by

$$W(z, q, a, z') = \frac{1}{\beta}[g(z, a, G(z, q)(a), z') - r(z, a)]$$

We contend indeed that this is well-defined, i.e., for each $(z, q, a) \in Gr(\overline{\Pi}_E(N)) \times A$, $G(z, q)(a) \in M^\diamond(z, a)$. Indeed, for any $G \in \Phi(N)(z)$, we clearly have $G(a) \in \Phi(N)(z)(a) = M^\diamond(z, a)$. But $G(z, q) \in \overline{\Phi}(N)(z)$ and therefore,

$$G(z, q)(a) \in \overline{\{G(a) \mid G \in \Phi(N)(z)\}} \subseteq \overline{M^\diamond(z, a)} \subseteq M^\diamond(z, a)$$

Clearly $W(z, q, a, z') \in \frac{1}{\beta}[M((z, a), z') - r(z, a)] \in N(z')$, and by definition

$$\begin{aligned} \Phi(W(z, q, a, \cdot))(z)(a) &= r(z, a) + \beta \int_\Omega W(z, q, a, z')q(dz' \mid z, a) \\ &= r(z, a) + \int_\Omega [g(z, a, G(z, q)(a), z') - r(z, a)]q(dz' \mid z, a) \\ &= \int_\Omega g(z, a, G(z, q)(a), z')q(dz' \mid z, a) = G(z, q)(a) \end{aligned}$$

as required.

∎

3.3 Oblivious and Parameterized Oblivious Strategies

Let Σ_0^i be the space of behavioural strategy profiles of Player i which do not condition on previous actions, only on previous states; hence, they can be viewed as Borel mappings from $\cup_{n=1}^{\infty}\Omega^n$ to $\Delta(A^i)$. These will be known as *oblivious strategies*. Let $\Sigma_0 = \prod_i \Sigma_0^i$.

Given a well-behaved correspondence $N : \Omega \Longrightarrow \mathbb{R}^{\mathcal{P}}$, an N-parametrized oblivious strategy of Player k is an oblivious strategy on the enlarged space in which the initial state is from the graph of N; formally, for each $n \in \mathbb{N}$, denote $H_n^N = Gr(N) \times \Omega^{n-1}$, $H_*^N = \cup_{n=1}^{\infty} H_n^N$, and $H_\infty^N = Gr(N) \times \Omega^{\mathbb{N}}$. An N-parametrized oblivious strategy of Player k is then a Borel mapping $\sigma^k : H_*^N \to \Delta(A^k)$. Similarly, N-parameterized oblivious strategy profiles are defined, and an N-parametrized strategy profile σ and an $(z,q) \in Gr(N)$ induce a distribution $P_{(z,q)}^{\sigma}$ on H_∞^N.

4 Step #1: Oblivious Payoffs

Define, iteratively,

$$Q_1 = \overline{\Pi}_X(0), Q_{n+1} = \overline{\Pi}_X(Q_n)\ \forall n \in \mathbb{N} \tag{4.1}$$

Lemma 4.1. *For each $n \in \mathbb{N}$, Q_n is a well-behaved correspondence.*

Proof. This follows inductively by Proposition 3.3.5. ∎

Define $r_n : H_\infty \to \mathbb{R}^{\mathcal{P}}$ by $r_n(z_1, a_1, \ldots) = \sum_{k=1}^{n} \beta^{k-1} r(z_n, a_n)$.

Lemma 4.2. *For each $z \in Z$,*[16] *$Q_n(z) = \overline{\{E_z^{\sigma}[r_n] \mid \sigma \in \Sigma_0\}}$, and for each $\varepsilon > 0$, there is a Q_n-parameterized oblivious strategy profile σ such that for each $(z,q) \in Gr(Q_n)$, $||q - E_{(z,q)}^{\sigma}[r_n]|| < \varepsilon$.*

Proof. First, we show the second part: When $n = 1$, one merely chooses a selector from the well-behaved correspondence $Gr(Q_1) \Longrightarrow X$ given by $(z,q) \to \{x \in X \mid r(z,x) = q\}$. We give the inductive step from n to $n+1$. Suppose we have the Q_n-parameterized strategy τ corresponding to $\frac{\varepsilon}{2}$; we will define σ. By Proposition 7.2, there is a countable set $D = \{V_k\}_{k\in\mathbb{N}} \subseteq \mathcal{S}_{Q_n}$ such that for any $\mu \in \Delta(\Omega)$, D is weak-$*$ dense in $\mathcal{S}_{Q_n}$ in $L^\infty(\mu)$. By part (4) of Proposition 3.3, $q \in \{\Pi_X(V)(z) \mid V \in \mathcal{S}_{Q_n}\}$. By part (3), for any $(z,q) \in Gr(Q_{n+1})$, there is k such that $||q - \Pi_X(V_k)(z)|| \leq \frac{\varepsilon}{2}$; since $\Pi_X(V_k)$ is Borel by part (5), it's possible to partition $Gr(Q_n)$ by a countable Borel partition $(\Omega_k)_{k\in\mathbb{N}}$ such that

[16] Recall the notations and definitions from Section 3.3.

for $(z,q) \in \Omega_k$, $||q - \Pi_X(V_k)(z)|| \le \frac{\varepsilon}{2}$. The correspondence $C : Gr(Q_n) \Longrightarrow X$ defined by

$$\begin{aligned} C(z,q) &= \{x \in X \mid ||q - \Phi(V_k)(z)(x)|| \le \frac{\varepsilon}{2}\}, \text{ if } (z,q) \in \Omega_k \\ &= \{x \in X \mid ||q - r(z,x) - \beta \int_A \int_\Omega V_k(z')q(dz' \mid a)x(da)|| \le \frac{\varepsilon}{2}\}, \text{ if } (z,q) \in \Omega_k \end{aligned} \tag{4.2}$$

is easily seen to be well-behaved;[17] let $S \in \mathcal{S}_C$. Define $\sigma(z_1,q) = S(z_1,q)$ and $\sigma(z_1,q,z_2,\ldots,z_k) = \tau(z_2,V_k(z_2),z_3,\ldots,z_k)$ for $k \ge 2$, $z_1,\ldots,z_k \in \Omega$, $q \in Q_{n+1}(z_1)$. It is standard to see that,

$$E^\sigma_{(z_1,q)}[r_{n+1}] = r(z_1,S(z_1)) + \beta \int_A \int_\Omega E^\tau_{(z_2,V_k(z_2))}[r_n]q(dz_2 \mid z_1,a_1)S(z)(da_1), \text{ if } (z_1,q) \in \Omega_k$$

and by assumption, $||E^\tau_{(z_2,V_k(z_2))}[r_n] - V_k(z_2)|| < \frac{\varepsilon}{2}$; hence, by comparison with (4.2) and the definition of S as a selector of C, $||q - E^\sigma_z[r_{n+1}]|| < \frac{\varepsilon}{2} + \beta\frac{\varepsilon}{2} < \varepsilon$.

To prove the first part, it now suffices to show that $\{E^\sigma_z[r_n] \mid \sigma \in \Sigma_0\} \subseteq Q_n(z)$ (and to recall that Q_n has closed values). This is clear for $n = 1$ by definition. We do the induction step from n to $n+1$. Let $\sigma \in \Sigma_0$ and $z \in \Omega$, let $\tau \in \Sigma_0$ be defined by $\tau(z_1,z_2,\ldots,,z_k) = \sigma(z,z_1,\ldots,z_k)$. Then clearly,

$$E^\sigma_z[r_{n+1}] = r(z,\sigma(z)) + \beta \int_\Omega E^\tau_{z'}[r_n]q(dz' \mid z,a)\sigma(z)(da)$$

and, by Proposition 3.3.1 and the induction hypothesis, the mapping $z' \to E^\tau_{z'}[r_n]$ is a selector of Q_n; hence $E^\sigma_z[r_{n+1}] \in Q_n(z)$. □

5 Step #2: Equilibrium Payoffs

Lemma 5.1. *For each $z \in \Omega$, $N_0(z) = \overline{\{E^\sigma_z[r_\infty] \mid \sigma \in \Sigma_0\}}$ is a well-behaved correspondence.*

Proof. Clearly, for each $z \in Z$, since $Q_n(z) = \overline{\{E^\sigma_z[r_n] \mid \sigma \in \Sigma_0\}}$ by Lemma 4.2 and

$$||r_n - r_\infty||_\infty \le \frac{\beta^n}{1-\beta} \cdot ||r||_\infty \tag{5.1}$$

we have that for each $z \in Z$, $N(z) = \lim_{n\to\infty} Q_n(z)$, the convergence being in the Hausdorff topology; the measurability of the correspondence is now clear from Proposition 3.2; also, $||N||_\infty \le \frac{||r||_\infty}{1-\beta}$, and the non-emptiness of values is trivial. □

Lemma 5.2. *For each $\varepsilon > 0$, there is an N_0-parameterized oblivious strategy profile such that if $(z,q) \in Gr(N_0)$, $||q - E^\sigma_{(z,q)}[r_\infty]|| < \varepsilon$.*

[17] It is similar to the proof of the well-behavedness of Ψ in the proof of Theorem 3.2.

Proof. Apply Lemma 4.2 with $\frac{\varepsilon}{2}$ and n large enough such that $\frac{\beta^n}{1-\beta} \cdot ||r||_\infty < \frac{\varepsilon}{2}$, and use (5.1). $\square$

Now, define for $n \in \mathbb{N}$, $N_n(z) = \overline{\Pi}_E(N_{n-1})(z)$.

Lemma 5.3. *For each $n \in \mathbb{N}$, N_n is a well-behaved correspondence.*

Proof. This follows by Lemma 5.1 and inductive use of Proposition 3.3.5. $\square$

Lemma 5.4. *For all $z \in \Omega$ and each $n \in \mathbb{N}$, $N_n(z) \subseteq N_{n-1}(z)$.*

Proof. It suffices to show $N_1 \subseteq N_0$ point-wise, and the result follows inductively by applying $\overline{\Pi}_E$. Let $z \in \Omega$, $q \in N_1(z)$. By definition, there is $G \in \overline{\Phi}(N_0)$ and $x \in E(G)$ such that $q = G(x)$. Fix $\varepsilon > 0$, and let $V \in \mathcal{S}_{N_0}$ with $||q - \Phi(V)(z)(x)|| < \varepsilon$. Let $\hat{\sigma}$ be as in Lemma 5.2. Define an oblivious strategy by

$$\sigma(z_1) = x, \ \forall z_1 \in \Omega$$

and

$$\sigma(z_1, z_2, \dots, z_k) = \hat{\sigma}(z_2, V(z_2), z_3, \dots, z_k), \ \forall k \geq 2, \ z_1, \dots, z_k \in \Omega$$

Recalling the notation $r_{\geq k}$ from (2.3), it is standard to see that

$$E_z^\sigma[r_\infty] = r(z, x) + \beta \int_A \int_\Omega E^\sigma_{(z,a,z')}[r_{\geq 2}] q(dz' \mid z, a) x(da) \tag{5.2}$$

where the mapping $z' \to E^\sigma_{(z,a,z')}[r_{\geq 2}]$ is Borel by Proposition 3.3.1. By assumption,

$$||E^\sigma_{(z,a,z')}[r_{\geq 2}] - V(z')|| = ||E^{\hat{\sigma}}_{(z', V(z'))}[r_{\geq 2}] - V(z')|| < \varepsilon \tag{5.3}$$

Hence, comparing (5.2), (5.3), and the definition of $\Phi(V)(z)(x)$ gives

$$||q - E_z^\sigma[r_\infty]|| \leq ||q - \Phi(V)(z)(x)|| + ||\Phi(V)(z)(x) - E_z^\sigma[r_\infty]|| < \varepsilon + \beta\varepsilon$$

Clearly, $E_z^\sigma[r_\infty] \in N_0(z)$, and this could be repeated for any $\varepsilon > 0$. Since $N_0(z)$ is closed, we see that $q \in N_0(z)$. $\square$

For $z \in \Omega$, let $N(z) = \cap_{n=1}^\infty N_n(z)$.

Lemma 5.5. *N is well-behaved and $N = \overline{\Pi}_E(N)$.*

Proof. By Proposition 3.3.6, N is well-behaved and $N \subseteq \overline{\Pi}_E(N)$ point-wise. Furthermore, we have the opposite inclusion as well: if $z \in Z$ and $q \in \overline{\Pi}_E(N)(z)$, then $q \in \overline{\Pi}_E(N_n)(z) = N_{n+1}(z)$ for all $n \in \mathbb{N}$; hence, $q \in N(z)$. $\square$

6 Step #3: Equilibrium

Let W, G, S be as in Theorem 3.2 for the correspondence N defined before Lemma 5.5. Observe that $N = \overline{\Pi}_E(N)$, and hence $W : Gr(N) \times A \times \Omega \to \mathbb{R}^{\mathcal{P}}$, $G : Gr(N) \to \mathfrak{G}$, and $S : Gr(N) \to X$, and the properties given there hold for each $(z, q) \in Gr(N)$, $a \in A$, $z \in \Omega$.

Let $V_1 \in \mathcal{S}_N$; i.e., V_1 is any Borel selector of N, and then inductively define for $n \geq 2$,

$$V_n(z_1, a_1, \ldots, z_n) = W(z_{n-1}, a_{n-1}, V_{n-1}(z_1, a_1, \ldots, z_{n-1}), z_n) \tag{6.1}$$

The idea is that the equilibrium strategy will be constructed such that for each $n \in \mathbb{N}$ and $p \in H_n$, recalling the notation $r_{\geq n}$ from (2.3), we will have

$$V_n(p) = E_p^\sigma[r_{\geq n}] \tag{6.2}$$

Indeed, simply define for each $n \in \mathbb{N}$,

$$\sigma(z_1, a_1, \ldots, z_n) = S(z_n, V_n(z_1, a_1, \ldots, z_n)) \tag{6.3}$$

Let $n \in \mathbb{N}$, $p = (z_1, \ldots, z_n) \in H_n$. By assumption, $\sigma(p)$ is an equilibrium of $G(z_n, V_n(p))$ with payoff $V_n(p)$. But since $G(z_n, q)(a) = \Phi(W(z_n, a, q, \cdot))(z_n)(a)$ and $V_{n+1}(p, a, z) = W(z_n, a, V_n(p), z')$ for each $q \in N(z_n)$, $a \in A$, $z' \in Z$, we have

$$V_n(p) = r(z_n, \sigma(p)) + \beta \int_A \int_\Omega V_{n+1}(p, a, z') q(dz' \mid z_n, a) \sigma(p)(da)$$

Hence, by standard iterations, (6.2) holds. Hence, for such $p \in H_n$ and $a \in A$, and recalling (2.2),

$$\begin{aligned} G(z_n, V_n(p))(a) &= r(z_n, a) + \beta \int_\Omega V_{n+1}(p, a, z') q(dz' \mid z_n, a) \\ &= r(z_n, a) + \beta \int_\Omega E^\sigma_{(p,a,z')}[r_{\geq n+1}] q(dz' \mid z_n, a) = X_p^\sigma(a) \end{aligned}$$

and as remarked above, $\sigma(p) = S(z_n, V_n(p))$ is an equilibrium of $G(z_n, V_n(p)) = X_p^\sigma(\cdot)$; therefore, the criterion of Proposition 2.1 holds, and hence σ is an SPE.

We make a few remarks:

- In particular, for any $V_1 \in \mathcal{S}_N$, we can choose the SPE to give payoffs (for the entire game) $V_1(z)$ when beginning at state z. In particular, for any $z \in Z$ and any $q \in N(z)$, we have an equilibrium which gives payoff q if beginning from state z.
- In general, there may be SPE which give payoffs not in N.
- The strategies are not directly functions of the past, rather of the current state and a parameter V, which evolves in a stationary manner as a function of the states visited and action played via (6.1).

7 Appendix: Auxilliary Results

This section brings some auxiliary results, most of which are standard (except Proposition 7.2). The following is [2, Prop. 3.2]:[18]

Proposition 7.1. *Let* $N : \Omega \Longrightarrow \mathbb{R}^m$ *be a well-behaved correspondence,*[19] *and let* $W : \Omega \to \mathbb{R}^m$ *be Borel. Then there exists* $V \in \mathcal{S}_N$ *such that for all* $z \in \Omega$, $d(W(z), N(z)) = ||W(z) - V(z)||$.

Proposition 7.2. *Let* $F : \Omega \Longrightarrow \mathbb{R}^m$ *be a well-behaved correspondence.*[20] *There is a uniformly bounded countable collection* $D \subseteq \mathcal{S}_F$*, such that for every* $\mu \in \Delta(\Omega)$ *and each* $W \in \mathcal{S}_F$*, there is a sequence* (V_n) *in* D *with* $V_n \to W$ μ*-a.e., and in particular* $V_n \to W$ *weak-*$*$ *in* $L^\infty(\mu)$.

Proof. Let $(P_n)_{n=1}^{\infty}$ be an increasing sequence of finite Borel partitions of Ω which generate the Borel σ-algebra. Let E_n denote the collection of $(\mathbb{Q} \cap [-||F||_\infty, ||F||_\infty])^m$-valued functions which are $\mathcal{A}(P_n)$-measurable.[21] Let $E = \cup_n E_n$. Clearly E is countable. Let $\mu \in \Delta(\Omega)$ and let $W \in \mathcal{S}_F$; since if $W_n = E_\mu[W \mid \mathcal{A}(P_n)]$, then W_n is $\mathcal{A}(P_n)$-measurable and $W_N \to W$ μ-a.s. by the martingale convergence theorem.[22] Clearly for any n any $\mathcal{A}(P_n)$-measurable function can be uniformly approximated by functions in E, and therefore we have a sequence in E converging μ-a.e. to W.

For each $\psi \in E$, fix $V_\psi \in \mathcal{S}_F$ satisfying $d(\psi(z), F(z)) = ||\psi(z) - V_\psi(z)||$; such exists by Proposition 7.1. In particular, for each $z \in \Omega$ and any $W \in \mathcal{S}_F$,

$$\begin{aligned} ||W(z) - V_\psi(z)|| &\leq ||V_\psi(z) - \psi(z)|| + ||W(z) - \psi(z)|| \\ &= d(\psi(z), F(z)) + ||W(z) - \psi(z)|| \leq 2||W(z) - \psi(z)|| \end{aligned}$$

the last inequality because $W(z) \in F(z)$. Take $D = (V_\psi)_{\psi \in E}$. $D \subseteq \mathcal{S}_F$ is countable and if (ψ_n) is a sequence in E converging μ-a.e. to $W \in \mathcal{S}_F$ - such exists by the above argument - then denoting $V_n = V_{\psi_n}$, (V_n) is a sequence in D converging μ-a.e. to W.

□

[18] For convenience, we sketch a proof: Since the metric $d_H : 2^{\mathbb{R}^m} \times 2^{\mathbb{R}^m} \to \mathbb{R}$ is continuous, as is the embedding $\mathbb{R}^m \to 2^{\mathbb{R}^m}$ given by $x \to \{x\}$, the mapping $e : \Omega \to \mathbb{R}$ given by $e(z) = d(W(z), N(z)) = d_H(\{W(z)\}, N(z))$ is Borel. The correspondence $B : \mathbb{R}^m \times (0, \infty) \to \mathbb{R}$ given by $B(x, r) = \{y \in \mathbb{R}^m \mid ||y - x|| \leq r\} \Longrightarrow \mathbb{R}^m$ is Borel with non-empty compact values, and hence so is the correspondence $F : \Omega \Longrightarrow \mathbb{R}^m$ given by

$$F(z) = B(W(z), e(z)) \cap N(z)$$

(In particular, the non-emptiness of values follows from the definition of e, and $Gr(F)$ is the intersection of two Borel graphs.) Let $V \in \mathcal{S}_F$; such V suffices.

[19] The bounded-ness is not necessary, the proof we sketch goes through without it.

[20] The bounded-ness is not necessary, since it is not necessary for Proposition 7.1, and the set E_n in the proof could be defined as the collection of $\mathbb{Q}^m$-valued functions which are $\mathcal{A}(P_n)$-measurable.

[21] $\mathcal{A}(P_n)$ denotes the algebra generated by P_n; $\mathbb{Q}$ denotes the rationals.

[22] W is bounded.

Lemma 7.3. *Let X, Y, Z be Borel spaces, let $F : X \Longrightarrow Y$ and $G : Y \Longrightarrow Z$ be correspondences with Borel graphs, and let F have compact values. Then the composition $G \circ F : X \Longrightarrow Z$ defined by $G \circ F(x) = \underset{y \in F(x)}{\cup} G(y)$ has a Borel graph.*

Proof. $Gr(G \circ F)$ is the projection of the Borel set

$$\Xi = \{(x, y, z) \mid y \in F(x), z \in G(y)\} = (Gr(F) \times Z) \cap (X \times Gr(G))$$

to $X \times Z$, and since each fiber of the form $\Xi_{x,z} = \{y \in Y \mid (x, y, z) \in \Xi\}$ is compact, the projection sends Ξ to a Borel set by the Arsenin-Kunugni theorem, e.g., [9, Thm. 35.46]. □

The following is the bounded case of Lemma 1 of [13]; our proof is simpler.

Lemma 7.4. *Let $N : \Omega \Longrightarrow \mathbb{R}^m$ be a well-behaved correspondence, $\mu \in \Delta(\Omega)$, and let $(V_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{S}_N$ which converges weak-$*$ in $L^{\infty}(\mu)$ to V. Then for μ-a.e. $z \in \Omega$, $V(z) \in co(N(z))$,*[23] *and if z is an atom of μ, then $V(z) \in N(z)$.*

Proof. The weak-$*$ convergence in $L^{\infty}(\Omega, \mu)$ implies weak convergence in $L^1(\Omega, \mu)$. Hence, there is a sequence $(W_n)_{n=1}^{\infty}$ of convex combinations of the $(V_n)_{n=1}^{\infty}$ which converges strongly in $L^1(\Omega, \mu)$ to V (e.g., [18, Thm. 3.13]), and by passing to a subsequence of the $(W_n)_{n=1}^{\infty}$, we may assume that it converges μ-a.e. (e.g., [19, Ch. 3]). This establishes the first part. Now let z be an atom of μ; there's some subsequence of $(V_n(z))_{n=1}^{\infty}$ which converges to a limit $x \in N(z)$; by the weak-$*$ convergence, $\langle V_k(p), p\rangle \to \langle V(z), p\rangle$ for all $p \in \mathbb{R}^m$; hence $\langle x, p\rangle = \langle V(z), p\rangle$ for all $p \in \mathbb{R}^m$, and hence $V(z) = x \in N(z)$. □

The following is a generalisation of Lyapunov's theorem (or, actually, the Dvoretzky-Wald-Wolfowitz theorem); see [4, Thm IV.17].[24]

Proposition 7.5. *Let $N : \Omega \Longrightarrow \mathbb{R}^m$ be a well-behaved correspondence and $\mu_1, \ldots, \mu_k \in \Delta(\Omega)$, and let $V : \Omega \to \mathbb{R}^m$ be Borel such that for each $j = 1, \ldots, k$, $V(z) \in co(N(z))$ μ_j-a.e and if z is an atom of μ_j then $V(z) \in N(z)$. Then there is $g \in \mathcal{S}_N$ such that $\int_{\Omega} g d\mu_j = \int_{\Omega} f d\mu_j$ for each $j = 1, \ldots, k$.*

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[23] $co(A)$ denotes the convex hull of A.

[24] [4] deals with the non-atomic case, but the general case follows trivially, since we assume that if if z is an atom of some μ_j, then $V(z) \in N(z)$.

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