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WHICH DEMAND SYSTEMS CAN BE GENERATED BY DISCRETE CHOICE?

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Abstract

We provide a simple necessary and sufficient condition for when a multiproduct demand system can be generated from a discrete choice model with unit demands.

Keywords: Discrete choice, unit demand, multiproduct demand functions.

1 Introduction

In a variety of economic settings the decision problem facing agents is one of discrete choice. For example, in markets for durable goods such as cars or refrigerators, each consumer who makes a purchase typically buys one unit of one of the products on offer (or buys nothing). If v_i is a consumer's valuation for product i and p_i is its price, then the rational consumer will buy the product with the best value for money given her preferences, i.e., the highest $(v_i - p_i)$ if that is positive, and will otherwise buy nothing. By specifying an underlying probability distribution for the vector of valuations within the population of consumers, one can derive aggregate multi-product demand as a function of the vector of prices.

In this paper we ask the converse question: which aggregate demand functions have discrete choice micro-foundations? When there is just a single product, *any* (bounded) downward-sloping aggregate demand function can be generated by a population of unit-demand consumers—the demand function can simply be interpreted as the fraction of consumers who are willing to pay the specified price for their unit. With more than one

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product, though, the answer is less obvious. We show that discrete choice foundations for an aggregate demand system (which is bounded and exhibits the usual Slutsky symmetry property) exist if and only if all mixed partial derivatives (with respect to prices) of the *total* quantity demanded are negative. Thus there is a simple test for whether a given demand system is consistent with discrete choice.

Early contributions to the theory and econometrics of discrete choice are surveyed by McFadden (1980), who developed the modern economics of discrete choice analysis in a variety of applications including choices of education and residential location. Relationships between discrete choice models and demand systems for differentiated products are explored in chapter 3 (and elsewhere) of the classic analysis by Anderson, de Palma and Thisse (1992). In particular, their Theorem 3.1 states necessary and sufficient properties of demand functions that ensure these demands are consistent with discrete choice. Their result presumes that consumers must buy one option or another, so that total demand always sums to one. In most situations of interest, however, consumers have, and use, the option to buy nothing, and we provide a result in the same spirit as Anderson *et al.*, but which allows for this. Indeed, the way that total demand varies with prices is the key to our analysis.

More recently, Jaffe and Weyl (2010) show how a linear demand system cannot be generated from (continuous) discrete choice foundations when there are at least two products and buyers can consume an outside option.[1] Jaffe and Kominers (2012) extend this analysis to show how (continuous) discrete choice cannot induce a demand system where demand for each product is additively separable in its own price. The analysis in the present paper sets those contributions in a wider context.

The next section states a preliminary result, which is not specific to discrete choice, that individual product demands can be derived from the total demand function. The main section then derives necessary and sufficient conditions for the total demand function to be consistent with discrete choice, and illustrates by way of some examples.

[1]Strictly speaking, they show that linear demand does not have discrete choice foundations where the valuations are continuously distributed (so a density exists). In section 3.2 we show how linear demand is often consistent with a discrete choice model in which the support of valuations does not have full dimension.

2 A preliminary result

Suppose there are n products, with associated price vector $p = (p_1, ..., p_n)$ and aggregate demand for product $i = 1, ..., n$ given by $q^i(p) \geq 0$. We only consider prices in the non-negative orthant $\mathbb{R}^n_+$, and we assume quasi-linear preferences, so that demand q_i is the derivative of an indirect utility function $CS(p)$: $q^i(p) \equiv -\partial CS(p)/\partial p_i$, where $CS(\cdot)$ is convex and decreasing in p. For simplicity, suppose that demand functions are differentiable, in which case we have Slutsky symmetry:

$$\frac{\partial q^i(p)}{\partial p_j} \equiv \frac{\partial q^j(p)}{\partial p_i} \text{ for } i \neq j \ . \tag{1}$$

Given the demand system $q(p)$, define $Q(p) \equiv \sum_{i=1}^n q^i(p)$ to be the total quantity of all products demanded with the price vector p. We make the innocuous assumptions that $Q(0) > 0$ and that $Q(p) \to 0$ as all prices p_i simultaneously tend to infinity.

A result which is useful in the "sufficiency" part of the following analysis, and perhaps of interest in its own right, is:

Lemma 1 *Suppose the demand system satisfies (1). Then the demand for product i, $q^i(p)$, satisfies*

$$q^i(p) = -\int_0^\infty \frac{\partial}{\partial p_i} Q(p_1 + t, ..., p_n + t) dt \ , \tag{2}$$

where $Q \equiv \sum_i q^i$ is total demand.

Proof. We need to show that

$$q^i(p) = -\int_0^\infty \frac{\partial}{\partial p_i} Q(p_1 + t, ..., p_n + t) dt = -\int_0^\infty \sum_{j=1}^n \frac{\partial q^j}{\partial p_i}(p_1 + t, ..., p_n + t) dt \ .$$

But (1) implies that the right-hand side above is equal to

$$-\int_0^\infty \sum_{j=1}^n \frac{\partial q^i}{\partial p_j}(p_1 + t, ..., p_n + t) dt = -\int_0^\infty \frac{d}{dt} q^i(p_1 + t, ..., p_n + t) dt = q^i(p)$$

as required. ∎

Lemma 1, which is true regardless of whether demand is consistent with discrete choice, implies that the total demand function $Q(\cdot)$ summarises all information about the demands for individual products, and demand for a particular product can be recovered from total demand via the procedure (2).[2]

3 Which demand systems are consistent with discrete choice?

We wish to understand which restrictions on $q(p)$ are implied if this demand system can be generated by the simplest discrete choice model. By "discrete choice model" we mean, first, that any individual consumer wishes to buy a single unit of one product (or to buy nothing). In particular, a consumer gains no extra utility from buying more than one product or from buying more than one unit of a product. Specifically and furthermore[3], the discrete choice model assumes that a consumer has a valuation v_i for a unit of product i (where valuations can be negative), where the vector of valuations $v = (v_1, ..., v_n)$ is drawn from a joint cumulative distribution function (CDF), denoted $G(v)$, for the vector of valuations v, and if she makes a purchase she buys the product offering the best greatest next surplus $v_i - p_i$. If she buys nothing she obtains a deterministic payoff of zero.[4] Faced with price vector p, the type-v consumer in this discrete choice problem will therefore

$$\text{choose product } i \text{ if } v_i - p_i \geq \max_{j \neq i}\{0, v_j - p_j\} \ . \tag{3}$$

The demand for product i, $q^i(p)$, is then the measure of consumers who satisfy (3). For most of our discussion we suppose that the distribution for v is continuous—i.e., there is a density function $g(\cdot)$ which generates $G(\cdot)$—which ensures that only a measure-zero set of consumers have a "tie" in (3) and the demand system is well-defined and continuous in

[2] For instance, if total demand is additively separable in prices, it follows from (2) that demand for a particular product depends only on its own price. If total demand depends only on the sum of prices, so does the demand for each product. If total demand depends symmetrically on prices (i.e., $Q(p)$ is unchanged if prices are permuted), then demands for products are symmetric (i.e., each $q^i(p)$ is the same function of own price p_i and the list of other prices $\{p_j\}_{j \neq i}$).

[3] As we discuss and illustrate in section 3.3 there are settings where consumers buy one unit of one product if they buy at all, but where (3) is not satsified (e.g., because of search or transactions costs). Such settings do not come within the discrete choice model as we have defined it.

[4] The following analysis applies equally to the situation where the consumer's outside option, say v_0, is stochastic, and a consumer buys product with the highest value of $(v_i - p_i)$ provided this is above v_0. However, one can just subtract v_0 from each v_i to return to our set-up with a deterministic outside option of zero.

prices p. (At various points we also discuss situations where the support of valuations does not have full dimensional support in $\mathbb{R}^n$, although in such cases demand is still continuous.) With the choice procedure (3) a consumer buys nothing if and only if $v \leq p$, and so the proportion of consumers who buy nothing with price vector p is just $G(p)$. Figure 1 depicts the pattern of demand with two products, where consumers are partitioned into three regions: those who buy product 1, those who buy product 2, and those who buy neither.

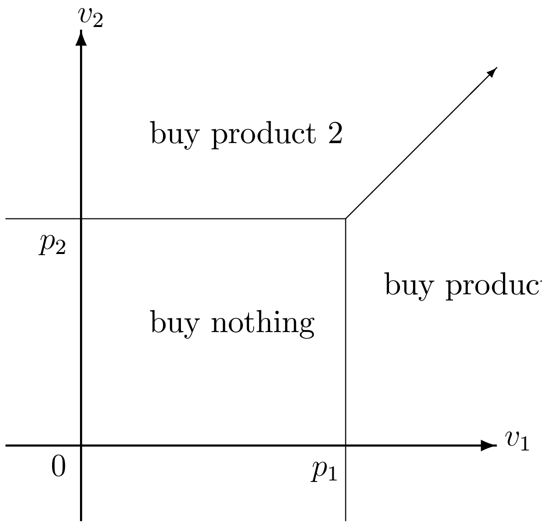

Figure 1: Pattern of demand in discrete choice model

3.1 Necessity

Any demand system arising out of the procedure (3) involves gross substitutes (i.e., cross-price effects are non-negative), since the right-hand side of (3) decreases with p_j. (This is easily seen from Figure 1 in the case with two products.) That is to say, a necessary condition for the demand system to be consistent with discrete choice is that $q^i(p)$ weakly increases with p_j for all $j \neq i$.

A second restriction on the demand system $q(\cdot)$ if it is to be consistent with discrete choice is that demand q^i must weakly decrease if all prices increase by the same amount. Intuitively, if the price vector increases from $(p_1, ..., p_2)$ to $(p_1 + t, ..., p_n + t)$, no consumer will switch from buying one product to buying another, but some may switch from buying product i to buying nothing. (Again, this is clear from Figure 1.) Regardless of whether it is consistent with discrete choice, as shown in the proof of Lemma 1, any demand system which satisfies (1) satisfies

$$\left.\frac{d}{dt}q^i(p_1+t,...,p_n+t)\right|_{t=0} = \frac{\partial}{\partial p_i}Q(p) \ ,$$

where $Q \equiv \sum_j q^j$ is total demand. Therefore, a necessary condition for the demand system to be consistent with discrete choice is that total demand Q decrease with each price p_i.

More generally, for a demand system consistent with discrete choice it must be that total demand Q and the CDF G are related by

$$G(p) \equiv 1 - Q(p) \ , \tag{4}$$

so that $1-Q$ has the properties of a joint CDF.[5] This is the crucial step in our argument. If a demand system is generated by a discrete choice framework with CDF for valuations G, then $G(p)$, which is the proportion of consumers who buy nothing at price vector p in the discrete choice framework, must be equal to 1 minus the proportion of consumers who buy something, i.e., $1 - Q(p)$. Thus, given any demand system $q(p)$, one can derive the *unique* underlying distribution of valuations which could generate this demand—if such a distribution exists—using (4).[6]

If G is a CDF with density function g, then

$$G(p) \equiv \int_{-\infty}^{p_1} \cdots \int_{-\infty}^{p_n} g(v)dv_1...dv_n \ . \tag{5}$$

Expression (5) implies that all mixed partial derivatives of G (i.e., which do not involve any "∂p_i" more than once), if they exist, must be non-negative, and the density g can be recovered from G via the partial derivative

$$g(p) \equiv \frac{\partial^n}{\partial p_1 ... \partial p_n} G(p) \ . \tag{6}$$

Since total demand Q satisfies (4), the following necessary conditions on Q are immediate:

[5] The "1" in (4) simply reflects a normalization of the measure of all consumers to be 1. The analysis could trivially be extended to allow the total measure of consumers to be N, say, in which case total demand Q is bounded by N rather than 1.

[6] More precisely, the CDF for valuations is uniquely determined for $p \geq 0$. As discussed in the proof of Lemma 2, there is some freedom to choose the distribution when some valuations are negative.

Proposition 1 *Suppose that the demand system $q(p)$ is consistent with discrete choice. Then:*
(i) total demand $Q(p) \equiv \sum_{i=1}^{n} q_i(p)$ is continuous at $p = 0$;
(ii) at any price where Q is sufficiently differentiable, for any $1 \leq k \leq n$ and collection of k distinct elements from $\{1, ..., n\}$ denoted $i_1, ..., i_k$ we have

$$\frac{\partial^k}{\partial p_{i_1}...\partial p_{i_k}} Q(p) \leq 0 \ ,$$

and the corresponding density function for valuations is

$$g(p) = -\frac{\partial^n}{\partial p_1...\partial p_n} Q(p) \ .$$

Proposition 1(ii) implies results derived in earlier papers. If $n \geq 2$ then in any region where total demand Q is linear in prices the valuation density must vanish, confirming the result in Jaffe and Weyl (2010). More generally, when $n \geq 2$ consider any region where demand for each product is additively separable in its own price, so that $\partial^2 q^i / \partial p_i \partial p_j \equiv 0$ for $j \neq i$. It follows that the full cross-derivative $\partial^n q^i / \partial p_1...\partial p_n$ is zero for each demand function q^i, and so the same is true for total demand Q. Again, the density g must vanish in this region, confirming the result derived by Jaffe and Kominers (2012).[7]

However, the fact that the implied density for valuations is zero in all (or almost all) of $\mathbb{R}^n_+$ does not mean that the demand system cannot arise from discrete choice. For instance, in section 3.2 we will see that any (smooth and bounded) demand system without cross-price effects, so that q^i is a function only of its own price p_i, is consistent with discrete choice, although the density will be zero within $\mathbb{R}^n_+$ and no consumer has positive valuation for all n products. We will also see that a linear demand system can be consistent with discrete choice if we allow the support of valuations not to have full dimension in $\mathbb{R}^n$.

While any total demand function Q which is additively separable in prices (i.e., which arises from a demand system without cross-price effects) can be implemented with a discrete choice model, a total demand function which *multiplicatively* separable in prices—even

[7]Indeed, using a similar argument, Proposition 1(ii) implies that the density vanishes in any region in which demand functions are additively separable in any non-trivial partition of prices have . If $n \geq 2$ and each demand function q^i can be written in the form $A_i(\cdot) + B_i(\cdot)$, where A_i is a function of some non-empty strict subset of prices and B_i is a function of the remaining prices, then again $\frac{\partial^n}{\partial p_1...\partial p_n} Q(p) = 0$ and the density vanishes.

locally so—is never consistent with discrete choice. If $Q(p) = \prod_{i=1}^{n} A_i(p_i)$ then each $A_i(\cdot)$ must be decreasing if Q is to decrease with each price, and so any mixed second derivative $\partial^2 Q/\partial p_i \partial p_j$ is positive, which from Proposition 1(ii) is inconsistent with discrete choice.

Part (i) of Proposition 1 rules out commonly used demand functions which have a discontinuity at $p = 0$. For example, demand which results from homothetic preferences (such as CES preferences) is inconsistent with a discrete choice model. In more detail, suppose the gross utility of the "representative consumer" is homothetic in quantities. It follows that net consumer surplus, $CS(p)$, takes the form $CS(p) = V(P(p))$, where $P(p)$ is a concave and homogeneous degree 1 function of prices and $V(P)$ is a decreasing convex function of the scalar price index P. Then the demand functions are

$$q^i(p) = X(P(p))\frac{\partial P(p)}{\partial p_i} \ ,$$

where $X(P) \equiv -V'(P)$. However, unlike any demands induced by a discrete choice model, this demand system is not continuous at $p = 0$. This might be because $\partial P/\partial p_i \to \infty$ when $p \to 0$, and so demand is unbounded for small prices. Alternatively, even if $\partial P/\partial p_i$ is bounded, q^i is not continuous at $p = 0$. To see this, let p^* be any price vector. Then as $\lambda \to 0$ we have

$$q^i(\lambda p^*) = X(P(\lambda p^*))\frac{\partial P(\lambda p^*)}{\partial p_i} = X(\lambda P(p^*))\frac{\partial P(p^*)}{\partial p_i} \to X(0)\frac{\partial P(p^*)}{\partial p_i} \ ,$$

and so this limit depends on the reference price vector p^*. (The second equality follows from the fact that $\partial P/\partial p_i$ is homogeneous degree zero.) In sum, any demand system based on a representative consumer with homothetic preferences is not consistent with discrete choice, due to demand behaviour when prices are small.

Remark on the interpretation of Proposition 1: In the context of discrete choice $1 - Q(p)$ can be interpreted as demand for the outside option of buying nothing – which we may label as notional 'product 0', which by assumption always gives zero consumer surplus – as a function of the prices $p_1, ..., p_n$ of the n actual products. In those terms Proposition 1 is a statement about demand for product 0, and Lemma 1 shows how demand for each product can be derived from demand for product 0. Given that the sum of demands for products 0 to n is by construction equal to one in the discrete choice setting, the method used to derive Lemma 1 also yields that, for any i, demand for each product (including notional product 0) can be derived from demand $q^i(p)$ for product i. In particular, when $\sum_{j=0}^n q^j(p) \equiv 1$, demand for product i can be expressed in terms of the demand function $q^1(p)$ for (say) product $1 \neq i$ by

$$\begin{aligned} q^i(p_0, p_1, ..., p_n) &= -\int_0^\infty \frac{d}{dt} q^i(p_0 + t, p_1, p_2 + t, ..., p_n + t) dt \\ &= \int_0^\infty \frac{\partial}{\partial p_1} q^i(p_0 + t, p_1, p_2 + t, ..., p_n + t) dt \\ &= \int_0^\infty \frac{\partial}{\partial p_i} q^1(p_0 + t, p_1, p_2 + t, ..., p_n + t) dt \ . \end{aligned}$$

The second equality uses the fact that $0 = \sum_{j=0}^n \partial q^j(p)/\partial p_i = \sum_{j=0}^n \partial q^i(p)/\partial p_j$ when demands sum to one. So for any demand system consistent with discrete choice, knowing the demand function for any one product implies the demand functions for all products.

This observation is useful in relating Proposition 1 to Theorem 3.1 of Anderson *et al.* (1992), which was highlighted in the Introduction. For a setting where product demands sum to one, that theorem states, among other things, that consistency with discrete choice requires that all mixed partial derivatives of demand for each product $q^i(p)$ which do not involve its own price p_i be non-negative. Proposition 1 accords with this, but is simpler to state, being just about total demand (equivalently demand for notional product 0) rather than demand for each of n products. Thus it would appear that, with demands by assumption always adding to one, Theorem 3.1 in Anderson *et al.* (1992) could likewise be stated in terms of demand for a single product rather than all.[8]

3.2 Sufficiency

In this section we show, in broad terms, how the necessary conditions outlined in Proposition 1 are also sufficient for the demand system to be consistent with a discrete choice framework. Since we consider only non-negative prices, formula (4) for the candidate CDF for underlying valuations is also defined only on $\mathbb{R}^n_+$. Because of this, and since we wish to allow for negative valuations, we need to understand when a function G defined only on

[8] Our setting differs from that of Anderson *et al* (1992, chapter 3) not only in that total demand $Q(p)$ for actual products varies with prices p. In our setting it is natural to require that all prices $p_i \geq 0$ for $i = 1, ...n$. This is equivalent to requiring all prices for actual products to exceed the price of notional product 0.

$\mathbb{R}^n_+$ can be extended to create a valid CDF defined on the whole space $\mathbb{R}^n$.[9]

Lemma 2 *Suppose G is a sufficiently differentiable function defined on $\mathbb{R}^n_+$ which satisfies $G(0, ..., 0) = 0$, $G(\infty, ..., \infty) = 1$, and for any $1 \le k \le n$ and collection of k distinct elements from $\{1, ..., n\}$ denoted $i_1, ..., i_k$ we have*

$$\frac{\partial^k}{\partial p_{i_1} ... \partial p_{i_k}} G(p) \ge 0 \ . \tag{7}$$

Then $G(\cdot)$ is part of a valid CDF for a continuous distribution on $\mathbb{R}^n$.

Proof. Setting $k = 1$ in (7) implies that G is increasing in each argument, and so G lies in the interval $[0, 1]$ throughout $\mathbb{R}^n_+$. The density g in the region $\mathbb{R}^n_+$ must be given by (6), which from (7) is non-negative.

There are many ways to choose a distribution for v outside $\mathbb{R}^n_+$ which yield the same CDF G when restricted to $\mathbb{R}^n_+$. One particular way to extend the function G beyond $\mathbb{R}^n_+$ to a function $\hat{G}$ defined throughout $\mathbb{R}^n$ is as follows:
(i) If $v \in \mathbb{R}^n_+$, set $\hat{G}(v) = G(v)$.
(ii) If any component of v is strictly below -1, set $\hat{G}(v) = 0$.
(iii) The remaining case is where v is such that a non-empty subset $S \subset \{1, ..., n\}$ of products have valuations in the interval $[-1, 0)$, while remaining products have valuations in $[0, \infty)$. In this case we define

$$\hat{G}(v) = \left(\prod_{i \in S} (1 + v_i) \right) G(v_+) \ , \tag{8}$$

where v_+ is the vector v with all negative components replaced by zero (i.e., the ith component of v_+ is v_i if $v_i \ge 0$ and 0 otherwise).

One can check that $\hat{G}$ lies in the interval $[0, 1]$ throughout $\mathbb{R}^n$, is zero when any v_i is below -1, is continuous throughout $\mathbb{R}^n$, and is weakly increasing throughout $\mathbb{R}^n$. By differentiating (8), one sees that the density corresponding to $\hat{G}$ at a point v such that $k < n$

[9] Note that in the following construction the extended density is discontinuous as we cross a plane $v_i \equiv 0$, but that doesn't matter for the argument. One could adjust the argument to make the extended density continuous, if desired.

components of v labelled $i_1, ..., i_k$ are non-negative, while all the remaining components lie in $[-1, 0)$, is

$$\hat{g}(v) = \frac{\partial^n}{\partial p_1 ... \partial p_n} \hat{G}(p) = \frac{\partial^k}{\partial p_{i_1} ... \partial p_{i_k}} G(v_+) \ . \tag{9}$$

From (7), this is non-negative as required.

Define the extended density $\hat{g}$ by (i) $\hat{g}(v) = \frac{\partial^n}{\partial p_1 ... \partial p_n} G(v)$ if each $v_i \geq 0$, (ii) $\hat{g}(v) = 0$ if any $v_i < -1$, and (iii) $\hat{g}(v)$ is given by (9) otherwise. Since $\hat{G}(v) = 0$ if any component $v_i = -1$, it follows that

$$\hat{G}(p) = \int_{-1}^{p_1} \cdots \int_{-1}^{p_n} \frac{\partial^n}{\partial p_1 ... \partial p_n} \hat{G}(p) dv_1 ... dv_n = \int_{-1}^{p_1} \cdots \int_{-1}^{p_n} \hat{g}(v) dv_1 ... dv_n \ .$$

In particular for $p \in \mathbb{R}^n_+$ we have

$$G(p) = \int_{-1}^{p_1} \cdots \int_{-1}^{p_n} \hat{g}(v) dv_1 ... dv_n \ ,$$

and so G defined on $\mathbb{R}^n_+$ is indeed part of a valid CDF. (In particular, the extended density $\hat{g}$ integrates to 1.) ■

Now consider a demand system $q(p)$ which satisfies the required Slutsky symmetry condition (1) such that total demand Q is differentiable throughout $\mathbb{R}^n_+$. It follows that Q is bounded in the neighborhood of $p = 0$, and without loss of generality we can therefore normalize demand so that $Q(0) = 1$. Suppose that $G(p) \equiv 1 - Q(p)$ satisfies the conditions in Lemma 2, i.e., that all the mixed partial derivatives of Q are non-positive. It follows that G is part of a valid CDF for valuations v. By construction, the total demand function which results from the discrete choice model with CDF G is precisely Q. Because the two demand systems—our original $q(p)$ and the demand system implemented by the discrete choice model with CDF G—have the same total demand, Lemma 1 implies that the two demand systems are the same. In particular, $q(p)$ has discrete choice micro-foundations.

We summarise this discussion in the following:

Proposition 2 *Suppose $q(p)$ is a demand system which satisfies (1) such that total demand $Q(p) \equiv \sum_{i=1}^n q^i(p)$ is sufficiently differentiable throughout $\mathbb{R}^n_+$, and for any $1 \leq k \leq n$ and collection of k distinct elements from $\{1, ..., n\}$ denoted $i_1, ..., i_k$ we have*

$$\frac{\partial^k}{\partial p_{i_1} ... \partial p_{i_k}} Q(p) \leq 0 \ .$$

Then this demand system can be generated by discrete choice.

A demand system which satisfies the conditions for Proposition 2 must therefore involve gross substitutes, since demands from a discrete choice model do so. This can be seen directly as follows. Since Q is differentiable, we can differentiate both sides of (2) with respect to p_j, where $j \neq i$. This implies that a condition which ensures $\partial q^i/\partial p_j \geq 0$ is that the total demand satisfies $\partial^2 Q/\partial p_i \partial p_j \leq 0$ as required by Proposition 2. Likewise, a demand system which satisfies the conditions for Proposition 2 must have a negative semi-definite matrix of derivatives, since demands from a discrete choice model (or, indeed, any demands resulting from quasi-linear preferences) do so.[10]

Note that any smooth demand system which has no cross-price effects satisfies the conditions of Proposition 2, although the corresponding density g is zero throughout the positive orthant $\mathbb{R}^n_+$. The construction used in Lemma 2 finds a density for valuations which is only positive if only one valuation v_i is positive. To illustrate, suppose that there are two products with independent linear demand functions $q^i(p_i) = \frac{1}{2}(1 - p_i)$ (and $q_i = 0$ if $p_i \geq 1$). Then one can check this demand system results from a discrete choice model with density $g(v) = \frac{1}{2}$ if $0 \leq v_i \leq 1$ and $-1 \leq v_j \leq 0$ and $j \neq i$ (and $g(v) = 0$ otherwise).

Proposition 2 applies to demand systems which are differentiable, and characterized valid total demand functions in terms of the mixed partial derivatives. However, the more fundamental property is that total demand Q is such that $1-Q$ is a valid, but not necessarily differentiable, CDF. In the two-product case, the condition for G for to be a valid CDF is that it is weakly increasing in v_1 and v_2 and the difference $G(v_1^H, v_2) - G(v_1^L, v_2)$ is weakly increasing in v_2 (where $v_1^H > v_2^L$), so that G is increasing and supermodular, i.e., that $Q = 1 - G$ is decreasing and submodular.

To illustrate, consider the continuous and piecewise-linear demand system depicted on Figure 2.[11] Total demand can be calculated and the candidate CDF $G = 1-Q$ then derived,

[10] This can be seen directly as follows. To ease notation, write $a_{ij} = \partial q^i/\partial p_j$, so that we need to show that the symmetric matrix A with entries (a_{ij}) is negative semi-definite, i.e., that $x^T Ax \leq 0$ for any vector x. Since $\partial^2 Q/\partial p_i \partial p_j \leq 0$ we know that the demand system involves gross substitutes, so that $a_{ij} \geq 0$ if $j \neq i$. Since by assumption Q decreases with p_i, we know that $\sum_{j=1}^n a_{ij} \leq 0$ for each i. It follows that $x^T Ax \leq \sum_{j \neq i} a_{ij} x_i (x_j - x_i) = \sum_{i=1}^n \sum_{j=i+1}^n a_{ij} \left(2x_i x_j - x_i^2 - x_j^2\right)$. However, each term in $(\cdot)$ is weakly negative and each a_{ij} is weakly positive, and the claim is proved.

[11] This demand system corresponds to a representative consumer with quadratic gross utility given by $u(q_1, q_2) = \frac{3}{4}(q_1+q_2) - \frac{3}{8}(q_1^2+q_2^2+q_1 q_2)$. Although this specification is consistent with discrete choice, not all all linear demand systems are. To see this, consider a two-product system where $q^i(p_1, p_2) = a_i - b_i p_i + c p_j$. To be consistent with a concave utility function, we require that $b_1 b_2 > c^2$. However, it is still possible that

as shown on the figure. One can check that this G is increasing and supermodular, and so this demand system is consistent with discrete choice. Indeed, the required distribution of valuations is that v is equally likely to lie on any of the four bold line segments which make up the boundary of the "kite" shape on the figure, and on any line segment valuations are uniformly distributed.[12] With more than two products, the condition for G to be a valid CDF is that all "mixed differences" in G are increasing in the remaining parameters.[13] This condition will likely be hard to verify in practice, which is why we focus on the differentiable case.

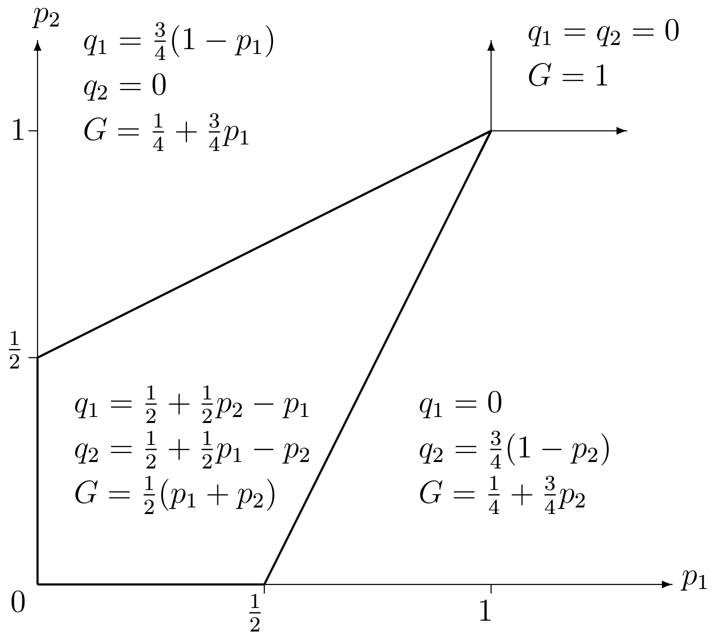

Figure 2: A linear demand system

$b_i < c$ for one product, in which case total output $q^1 + q^2$ increases with p_i, which cannot be consistent with discrete choice.

[12] In a similar manner, consider the piecewise linear demand system where $q^i = \frac{1}{2}(1-p_i+p_j)$ if $p_1+p_2 \leq 1$, $q^i = 1 - p_i$ if $p_i \leq 1$ and $p_1 + p_2 \geq 1$, and otherwise $q^i = 0$. Again, the implied G is increasing and supermodular, and corresponds to the example where (v_1, v_2) is uniformly distributed on the single line segment $v_1 + v_2 = 1$ and $v_1, v_2 \geq 0$. This is the demand system for the standard Hotelling model, in which the two products are located at each end of the unit interval, consumers are willing to pay 1 in total for either product, are uniformly located within this interval and incur a "transport cost" equal to their distance from the product they buy.

[13] For instance, with three products G should be increasing in (v_1, v_2, v_3), first differences of the form $[G(v_1^H, v_2, v_3) - G(v_1^L, v_2, v_3)]$ should be increasing in (v_2, v_3), and the "difference in difference" $[G(v_1^H, v_2^H, v_3) - G(v_1^L, v_2^H, v_3)] - [G(v_1^H, v_2^L, v_3) - G(v_1^L, v_2^L, v_3)]$ should be increasing in v_3.

4 Applications and extensions

We now consider some examples and extensions of the discrete choice model, and related examples that do not accord with it.

Total demand is a completely monotonic function of an additively separable function of prices: A rich class of demand systems consistent with discrete choice has $1 - Q(p) = Z(\sigma(p))$ as a completely monotonic function of a sum $\sigma(p) \equiv \sum_{i=1}^{n} \alpha_i(p_i)$ of positive, decreasing functions of price, one for each product.[14] (A function $Z : (0, \infty) \to \mathbb{R}$ is said to be *completely monotonic* if for all k the k^{th} derivative has the sign of $(-1)^k$. For our purposes it suffices that this condition holds for $k \leq n$.) Then

$$\frac{\partial^k}{\partial p_1 ... \partial p_k} Q(p) = -Z^{(k)}(\sigma) \prod_{i=1}^{k} \alpha_i'(p_i) < 0 \tag{10}$$

because $Z^{(k)}$ and the product of the α_i' terms both have the sign of $(-1)^k$ so (10) has the sign of $-(-1)^{2k} = -1$. So Proposition 2 implies that a distribution of valuations can be found which generates this total demand via discrete choice.

The Logit demand system, perhaps the most familiar model of discrete choice, belongs to this class.[15] This demand system has

$$q^i(p) = \frac{1+n}{n} \cdot \frac{e^{-p_i/\mu}}{1 + \sum_j e^{-p_j/\mu}}$$

for some parameter $\mu > 0$. (Demands are normalized by the factor $\frac{1+n}{n}$ to satisfy our convention that $Q = 1$ when $p = 0$.) Here, $1 - Q(p) = Z(\sigma(p))$, where $Z(\sigma) = 1 - \frac{1+n}{n} \frac{\sigma}{1+\sigma}$ and $\sigma(p) = \sum_i e^{-p_i/\mu}$. Also in this class is the case of discrete choice where valuations v_i are independently distributed and non-negative. With $G_i(v_i)$ as the CDF of v_i we can write $1 - Q(p) = \prod_{i=1}^{n} G_i(p_i)$ as $Z(\sigma(p))$, where $Z(\sigma) = e^{-\sigma}$ and $\sigma(p) = -\sum_{i=1}^{n} \log G_i(p_i)$. (In either of these special cases, one can check that $Z(\sigma)$ is completely monotonic.)

[14] So that $Q(0) = 1$ and $Q(\infty) = 0$, suppose that each α_i satisfies $\alpha_i(\infty) = 0$, while $Z(\Sigma\alpha_i(0)) = 0$ and $Z(0) = 1$.

[15] See, for example, Anderson *et al.* (1992, section 7.4). The usual micro-foundations for this demand system has consumer valuations—including the value of the outside option—being independent extreme value variables. In particular, the value of the outside option is stochastic. Anderson *et al.* (1992, section 7.4) also present the demand system when product valuations are independent extreme value variables but the outside option has a deterministic value of zero, but this is algebraically messier.

Completely monotonic functions can be used more generally to extend a given discrete choice model to a wider family. For if $Q(p)$ satisfies the conditions of Proposition 2, then so does $\hat{Q}(p) = 1 - \zeta(Q(p))$ where ζ is a completely monotonic function with $\zeta(0) = 1$ and $\zeta(1) = 0$. (One can check that any k^{th} order mixed partial derivative of $\hat{Q}(p)$ is a sum of negative terms.)

Homothetic total demand: In section 3.1 we noted that demand which results from homothetic preferences is inconsistent with the discrete choice model. However, as we now show, total demand that is itself homothetic in prices can accord with discrete choice, depending on demand parameters. Suppose then that

$$Q(p) = X(P(p)) \tag{11}$$

where $P(p)$ is concave and homogeneous of degree 1 in prices, and $X(P)$ is decreasing with $X(0) = 1$ and $X(\infty) = 0$.[16] Consider the 2-product case. It is immediately clear that total output is decreasing in prices because $\partial Q/\partial p_1 = X'(P)P_1 < 0$. (Here P_i denotes the partial derivative $\partial P/\partial p_i$.) It is straightforward to show that the cross-partial derivative is

$$\frac{\partial^2 Q}{\partial p_1 \partial p_2} = X'' P_1 P_2 + X' P_{12} = \frac{-X' P_1 P_2}{P} [\theta(P) - \eta(p_1/p_2)] \ , \tag{12}$$

where $\theta(P) \equiv -PX''(P)/X'(P)$ measures demand curvature and $\eta(p_1/p_2) > 0$ is the elasticity of substitution in demand between products. From (12) we see that the cross-partial is negative if and only if $\theta < \eta$, which always holds if $X(P)$ is (weakly) convex. So total demand function (11) is consistent with discrete choice if and only if the elasticity of substitution always exceeds demand curvature.

Consumer search: We have defined the discrete choice model by condition (3) that the consumer will buy the product with the highest $v_i - p_i \geq 0$. That accords with consumers being able to learn their valuations costlessly. However, the discrete choice model can be used also to analyze some (but not all) settings with search costs, as the following 2-product example illustrates. Suppose that the valuation for product i has independent CDF $G_i(v_i)$ (where both valuations are always non-negative), that the consumer knows both prices

[16] An example is $Q(p) = 1 - p_1 p_2$ with the domain of prices being the unit square, which corresponds to a discrete choice model with valuations v_1 and v_2 being uniformly distributed on $[0, 1]^2$.

and can observe v_1 costlessly but that she has to pay search cost s, with independent CDF $F(s)$, to learn v_2.

Assume first that there is *free recall* in that the consumer can costlessly return to buy product 1 if she investigates but doesn't end up buying product 2. In this case, the consumer will buy nothing if both (a) $v_1 < p_1$ and (b) either $v_2 < p_2$ or $s > V(p_2) \equiv \int_{p_2}^{\infty}(v_2 - p_2)dG_2(v_2)$. Therefore the proportion who buy nothing is

$$\begin{aligned} 1 - Q(p) &= G_1(p_1)[F(V(p_2))G_2(p_2) + 1 - F(V(p_2))] \\ &= G_1(p_1)[1 - F(V(p_2))(1 - G_2(p_2))] \ . \end{aligned} \tag{13}$$

Denoting the square-bracketed term in the above by $\tilde{G}_2(p_2)$, we have $\tilde{G}_2'(p_2) > 0$ and $1 - Q(p) = G_1(p_1)\tilde{G}_2(p_2)$ satisfies the conditions of the discrete choice model. In short, this model with search costs has a counterpart without them that is consistent with discrete choice.

But that is not the case with costly recall. Suppose that a consumer who searches product 2 must pay the search cost s again if she revisits product 1. Then $Q = 1$ when $p_1 = 0$ and $p_2 = \infty$ because all consumers buy product 1 without searching further. But $Q < 1$ when $p_1 = 0$ and $p_2 > 0$ but is small enough that $F(V(p_2)) > 0$. This is because consumers with low s and low v_1 will search product 2 only to find that $v_2 < p_2$, and when $v_1 < s$ they will not wish to return to buy product 1 either. Therefore, Q is not monotonic in p_2 and the discrete choice model does not apply.

Extending discrete choice to allow consumers to buy several products: An extension of the standard discrete choice model allows consumers to buy several options, rather than having to choose just one. The question then arises when this extended notion of discrete choice conforms with the basic one described at the start of section 3. To examine this issue briefly, suppose for simplicity there are two products, that v_i is a consumer's valuation for product $i = 1, 2$ on its own, while her valuation for the bundle of both products is $v_1 + v_2 - z$ for some constant $z \geq 0$. Here, z reflects an intrinsic "disutility" from joint consumption, reflecting an assumption that the products are partial substitutes. (The usual model of discrete choice is the limiting case of this when $z \to \infty$.) The pattern of demand given the pair of prices (p_1, p_2) is shown in this figure.[17]

[17] This figure is taken from Armstrong (2013). Gentzkow (2007) empirically investigates a related discrete

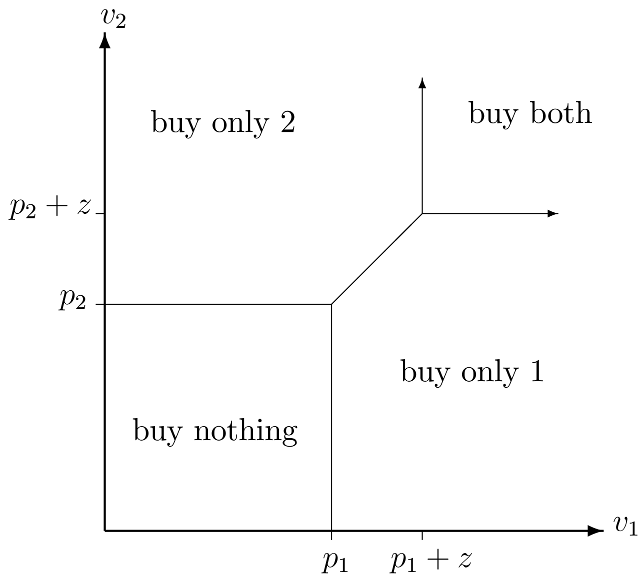

Figure 3: Pattern of demand when products are partial substitutes

If $F(v_1, v_2)$ is the CDF for (v_1, v_2), then total demand with prices (p_1, p_2) is

$$Q = 1 - F(p_1, p_2) + \{1 - F(p_1 + z, \infty) - F(\infty, p_2 + z) + F(p_1 + z, p_2 + z)\} \ .$$

(Here, the term in brackets $\{\cdot\}$ is the fraction of consumers who buy both products.) Then Q decreases with each price p_i, as needed to be consistent with the usual discrete choice model with single-product demand. The cross-partial derivative is

$$\frac{\partial^2 Q}{\partial p_1 \partial p_2} = f(p_1 + z, p_2 + z) - f(p_1, p_2) \ ,$$

where f is the density function for valuations (v_1, v_2). Thus, if the above expression is always negative—which is the case, for instance, if f decreases with (v_1, v_2)—the demand system induced by this extended discrete choice model is consistent with another basic discrete choice model in which all consumers buy at most one product.

Extending discrete choice to allow consumers to buy multiple units of their chosen product: The final extension we examine allows consumers to buy their chosen product in continuous quantities, although as in the basic discrete choice model each consumer buys at most one product.[18] Specifically, suppose that all consumers have the same demand for a given

choice model in which some consumers purchase two items.

[18]See Hanemann (1984) for an early investigation of this demand model.

product, and each consumer has demand $x_i(p_i)$ if she buys product i with price p_i. Let $s_i(p_i)$ be the surplus function which corresponds to $x_i(p_i)$. Consumers incur idiosyncratic additive shocks to their surplus vector (e.g., "transport costs" to reach a product), denoted $\tau = (\tau_1, ..., \tau_n)$, and the type-$\tau$ consumer chooses the buy the product with the highest value of $s_i(p_i) - \tau_i$ (or buys nothing if $\tau_i \geq s_i(p_i)$ is negative for all products). Let $X^i(s)$ be the fraction of consumers who choose product i when the surplus vector is $s = (s_1, ..., s_n)$.

As in any discrete choice problem of this form, X^i increases with s_i and decreases with any other s_j. Aggregate demand for product i is

$$q^i(p) = X^i(s(p))x_i(p_i) ,$$

and so total demand is

$$Q(p) = \sum_{i=1}^{n} X^i(s(p))x_i(p_i) .$$

It follows that

$$\frac{1}{x_i}\frac{\partial Q}{\partial p_i} = \frac{X^i}{x_i}x_i' - \frac{\partial X^i}{\partial s_i}x_i - \sum_{j \neq i} \frac{\partial X^j}{\partial s_i}x_j . \tag{14}$$

We claim that under plausible conditions expression (14) is positive when p_i approaches the choke price for product i, say $\bar{p}_i$ (where we might have $\bar{p}_i = \infty$). Specifically, suppose that $X^i(s) = 0$ if $s_i = 0$, i.e., if $p_i = \bar{p}_i$. (This is the case when τ is interpreted as transport costs, so that each $\tau_i \geq 0$.) Then from l'Hôpital's rule, $X^i(s_1(p_1), ..., s_n(p_n))x_i'(p_i)/x_i(p_i) \to 0$ as $p_i \to \bar{p}_i$. Therefore, the right-hand side of (14) is approximately equal to $-\sum_{j \neq i} \frac{\partial X^j}{\partial s_i} x_j$, which is positive.

We deduce that a demand system of this form is not consistent with the basic discrete choice framework with unit demands. Intuitively, when one product has a high price, increasing that price still further has a first-order positive impact on demand for the other products but only a second-order negative impact on the demand for that product.

5 Conclusion

Propositions 1 and 2 together show that, assuming that total demand is bounded, the necessary and sufficient condition for consistency with the discrete choice model is that all mixed partial derivatives of total demand be non-positive. This is a strong form of product substitutability.

We have focused mainly on the basic discrete choice model where each consumer buys one unit of one product, specifically the product with highest $(v_i - p_i)$, or else nothing. But our first example with search costs, where this condition does not always hold, was shown to be equivalent to a basic discrete choice model that by definition meets the condition. We also showed that an example in which consumers could buy more than one product was equivalent to the basic unit-choice setting. So the analysis of the basic discrete choice model has more general application.

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