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Dominik Karos

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Manor Road Building, Manor Road, Oxford OX1 3UQ

Coalition Formation in General Apex Games Under Monotonic Power Indices*

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Abstract

We generalize the class of apex game by combining a winning coalition of symmetric minor players with a collection of apex sets which can form winning coalitions only together with a fixed quota of minor players. By applying power indices to these games and their subgames we generate players' preferences over coalitions which we use to define a coalition formation game. We focus on strongly monotonic power indices and investigate under which conditions on the initial general apex game there are core stable coalitions in the resulting coalition formation game. Besides several general results, we develop condition for the Shapley-Shubik index, the Banzhaf index, and the normalized Banzhaf index in particular. It turns out that many statements can be easily verified for arbitrary collections of apex sets. Nevertheless, we give some relations between the collection of apex sets and the set of core stable coalitions.

Keywords: Apex Games, Core Stability, Hedonic Games, Strong monotony
JEL classification: C71, G34

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†Department of Economics, University of Oxford, Manor Road, Oxford OX13UQ. Email: `dominik.karos@seh.ox.ac.uk`.

1 Introduction

This paper considers three concepts of cooperative game theory, namely coalition formation in simple games, apex games, and strongly monotonic power indices. The aim is to find stable outcomes of a hedonic coalition formation game which is derived from a generalized apex game and a strongly monotonic power index.

If a group of players has to make a decision, there may be subgroups (coalitions) which are able to impose the will of their members. Usually, the grand coalition should be able to unanimously decide, whereas the empty coalition should not be able to do so. Coalitions which are able to impose their will are called winning. A situation that is fully described by the collection of winning coalitions is a simple game. An important question which arises in those games is, how powerful various players are. There are different ways how to measure this power. In particular, it is interesting not only to ask what the power of a player in the overall game is, but also: How powerful is a player within a certain coalition? Players' power might be used to distribute the worth of a coalition between its members (think of the number of cabinet seats a party gets within a political coalition). Hence, a power measure provides each player with a tool to compare different coalitions he might belong to. Preferences over coalitions thus derived induce a coalition formation game (Shenoy, 1979). The subsequent question is: Which coalition shall form? Or: Which coalition is stable, in the sense that no members will leave it? We assume that the power a player has in a coalition does not depend on the behaviour of players outside of this coalition, that is that there are no externalities. Such coalition formation games are called hedonic games. They have been introduced in Drèze and Greenberg (1980) and despite the absence of externalities, their analysis is quite complex. In particular, it is not even clear under which circumstances a stable coalition exists. Although there are some general conditions which guarantee existence (see for instance Banerjee et al., 2001; Bogomolnaia and Jackson, 2002; Iehlé, 2007), an analysis of hedonic games which are derived from a simple game and a power measure has not led to sufficiently general results. A good basis of this topic can be found for instance in Dimitrov and Haake (2006, 2008b).

A special subclass of simple games is the class of apex games - our second concept. They have already been studied in von Neumann and Morgenstern (1944). These games with one major player (originally called chief player in von Neumann and Morgenstern (1944), later called apex player) and a set of minor players have been investigated in many articles (see for instance Aumann and Myerson, 1988; Hart and Kurz, 1983, 1984; Montero, 2002). An apex player can form a winning coalition with each of the minor players. But the set of all minor players together is winning as well. We generalize this idea in the following sense. Let there be several coalitions I_k in a simple game and a set J with an empty intersection with each I_k. A coalition is winning if it contains either at least one coalition I_k and at least r players from J, or at least q players from J, where $q > r$. We call these games general apex games with apex sets I_k, minor players J, apex quota r and minor quota q. General apex games can occur in various situations. A decision process in a company containing two boards (Example 2.1) might be considered, as well as the relation between a parliament and a constitutional court (Example 2.2), games with symmetric players who might be bribed (Example 4.1), or voting games with two different kinds of voting rights.

The third concept we consider is that of monotonic power indices. Let u and v be two different proper simple games on the same set of players such that a player is pivotal in all coalitions with respect to v in which he is pivotal with respect to u. If there is at least one coalition in which he is pivotal with respect to v but not with respect to u then his power in v should be strictly greater than in u. This monotony property appears for example in Sagonti (1991), where it is shown to be satisfied by the Shapley-Shubik index (see Shapley, 1953), and the Banzhaf index (see Banzhaf, 1965; Coleman, 1971). We show that the normalized Banzhaf index is in general not monotonic in this sense, but that its behaviour on general apex games is quite similar to that of a monotonic power index.

Shenoy (1979) considered hedonic games which are derived from proper monotonic simple games with at most four players together with the Shapley-Shubik index. He showed that the four player apex game is the only such game for which no stable coalitions exist. In our paper we will derive necessary and sufficient conditions for the existence of stable coalitions in hedonic games which are induced by a general apex game and a strongly monotonic solution. In particular, we give necessary and sufficient conditions for the stability of coalitions. We refine these results for the special cases of Shapley-Shubik index, Bazhaf index, and normalized Banzhaf index. Although the existence of a stable coalition in such a hedonic game highly depends on the structure of the family of apex sets, it will be shown that many insights can be derived if simple conditions hold true. In particular, it will be shown that the players of J are an important part of any core stable coalition.

Section 2 develops the basics of simple games and hedonic games. We introduce the class of general apex games; and we define power indices as well as some properties they might have, in particular strong monotony. Further, we show how to induce a hedonic coalition formation game from a simple game and a power index. In Section 3 we state three main results presenting necessary and sufficient conditions for the existence of stable coalitions. Section 4 refines the results of Section 3 by considering the Shapley-Shubik index, the Banzhaf index, and the normalized Banzhaf index. Section 5 briefly summarizes our results and gives some conclusions. All proofs can be found in the appendix.

2 The Model

2.1 Preliminaries

Throughout the paper let N be a finite set of players. A *coalition* is a subset $S \subseteq N$ and the set $\mathcal{P} = \mathcal{P}(N)$ is the collection of all coalitions. For $i \in N$, let $\mathcal{P}_i = \mathcal{P}_i(N)$ be the collection of all coalitions which contain i. A *partition* is a set of nonempty coalitions $\pi = \{S_1, \ldots, S_m\}$ such that $S_k \cap S_l = \emptyset$ for all $k \neq l$ and $\bigcup_{k=1}^{m} S_k = N$. The collection of all partitions of N is denoted by $\Pi = \Pi(N)$. For a partition $\pi \in \Pi$ and a player $i \in N$, let $\pi(i)$ denote the unique coalition in π which contains i. For a coalition $S \subseteq N$, let $\Pi_S = \Pi_S(N)$ be the collection of all partitions containing S.

A *simple game* is a function $v : \mathcal{P}(N) \to \{0,1\}$ with $v(\emptyset) = 0$ satisfying $v(S) \leq v(T)$ for all $S, T \in \mathcal{P}$ with $S \subseteq T$. In particular, the *zero game* ω which assigns 0 to each coalition $S \subseteq N$ is a simple game. A simple game v is called *proper* if $v(S) + v(N \setminus S) \leq 1$ for all $S \subseteq N$. The set of all proper simple games is denoted by $\mathcal{V}$. A coalition $S \subseteq N$ with $v(S) = 1$ is called *winning*. A winning coalition S which does not contain any proper, winning subcoalition is called *minimal winning*. If S is a winning coalition and $i \in S$ is such that $v(S \setminus \{i\}) = 0$ then i is called *pivotal* in S with respect to v. If $i \in N$ is such that $v(S) - v(S \setminus \{i\}) = 0$ for all coalitions S then i is called a *null player* (with respect to v). If $i \in N$ is such that $v(S) - v(S \setminus \{i\}) = 1$ for all winning coalitions $S \subseteq N$ then i is called a *veto player* (with respect to v). Two players $i, j \in N$ are called *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

For each coalition S it can be asked how powerful a player $i \in S$ is. Therefore, it is reasonable to consider proper simple games on subcoalitions of N (see for instance Dimitrov and Haake, 2008b). For $v \in \mathcal{V}$ and a set $S \subseteq N$ the *subgame* v_S is defined by

$$v_S(T) = v(S \cap T)$$

for all $T \subseteq N$. In particular, v_S is a game on N. In this new game all players from $N \setminus S$ are null players with respect to v_S. For each coalition S it is clear that if v is proper then v_S is proper, that is, for any $v \in \mathcal{V}$ and any $S \subseteq N$ we have $v_S \in \mathcal{V}$. If S is not a winning coalition then $v_S = \omega$; if $S = N$ then $v_S = v$. As $v_S \in \mathcal{V}$ for each $S \subseteq N$, veto and null players are well defined with respect to v_S. Given v, a *veto player in* S is a veto player with respect to v_S and a *null player in* S is a null player with respect to v_S. In particular, a veto player (null player) $i \in N$ with respect to v is also a veto player (null player) in v_S for all winning coalitions $S \in \mathcal{P}_i$. We say that S is a *null player free coalition* if there is no $i \in S$ who is a null player with respect to v_S.[1]

2.2 Apex Games

Let N be set of players and let a be a proper simple game on N such that there are $i \in N$ and $J \subseteq N \setminus \{i\}$ with

$$a(S) = \begin{cases} 1, & \text{if } J \subseteq S \text{ or } (i \in S \text{ and } S \cap J \neq \emptyset), \\ 0, & \text{else.} \end{cases}$$

[1] In the literature a coalition S is often called null player free if there is no null player with respect to v in S rather than if there is no null player with respect to v_S in S as it is in our paper.

Then $a = a_{iJ}$ is called an *apex game* with *apex player* i and *minor players* $j \in J$. Apex games have already been introduced by von Neumann and Morgenstern (1944); a broad overview about the existing literature on apex games can be found in Montero (2002). In this subsection we generalize this class of games. The following example provides a motivation for this generalization.

Example 2.1. In a stock corporation under German law there is a two-tier board system. An executive board is in charge of managing the company; and a supervisory board has to ensure that the interests of the stakeholders are not violated. In particular, there are decisions which the supervisory board has to approve. We assume that there is a proper and monotonic voting system in the executive board to make a decision, and such a decision is accepted by the supervisory board if at least r members agree. On the other hand, if at least $q > r$ members of the supervisory board vote for a decision, they are able to force the executive board to follow this decision. Let I be the executive board and $\mathcal{I}$ be the collection of those coalitions in the executive board which are able enforce a decision (within the executive board). Let further J be the supervisory board. Then the game

$$v(S) = \begin{cases} 1, & \text{if } (|J \cap S| \geq q) \text{ or } (S \cap I \in \mathcal{I} \text{ and } |S \cap J| \geq r), \\ 0, & \text{else.} \end{cases}$$

describes the situation.

Example 2.1 already reveals the structure of a general apex game and motivates the following definition.

Definition 2.1. Let N be a set of players and let a be a simple game on N such that there are

1. a coalition $J \subseteq N$ with $|J| \geq 3$,
2. a nonempty collection $\mathcal{I}$ of nonempty subsets of $N \setminus J$ with pairwise nonempty intersection such that $I' \in \mathcal{I}$ for all $I' \subseteq N \setminus J$ with $I \subseteq I'$ for some $I \in \mathcal{I}$,
3. two integers $0 < r < q \leq |J|$ with $q > \frac{1}{2}|J|$ and $r + q \geq |J| + 1$,

such that

$$a(S) = \begin{cases} 1, & \text{if } (|J \cap S| \geq q) \text{ or } (S \setminus J \in \mathcal{I} \text{ and } |S \cap J| \geq r), \\ 0, & \text{else.} \end{cases}$$

Then $a = a_{\mathcal{I}J}^{q,r}$ is a *general apex game* with *apex sets* $I \in \mathcal{I}$, *minor players* $j \in J$, *minor quota* q, and *apex quota* r.

We will write $I = \bigcup_{I' \in \mathcal{I}} I' = N \setminus J$ for the union of all apex sets. For the sake of simplicity we will sometimes refer to members of I as *apex players* if it is clear that we not talk about a classic apex game with a single apex player.

Consider the special case with $\mathcal{I} = \mathcal{P}_i$ for $i \in N \setminus J$, $r = 1$, and $q = |J|$. Then the resulting game is an apex game in the classical sense. Before we close this subsection with another example, we briefly show that general apex games are indeed proper.

Lemma 2.2. *Let $\mathcal{I}$ and J satisfy conditions 1 and 2 in Definition 2.1 and q, r integers with $0 < r < q \leq |J|$. Then $a_{\mathcal{I}J}^{q,r}$ is proper if and only if $q > \frac{1}{2}|J|$ and $r + q \geq |J| + 1$.*

The game v in Example 2.1 is a general apex game: The supervisory board consists of the minor players, each $\mathcal{I}$ is the collection of apex sets, r is the apex quota and q is the minor quota. We close this section with another example of a general apex game.

Example 2.2. Consider a parliament with 120 members which can pass laws with absolute majority, and which can change the constitution with a majority of two thirds. All members of the parliament have the same voting rights. The constitutional court can forbid a law which is not in line with the constitution (according to the interpretation of the constitutional court). Assume that the parliament would like to pass a law which, however, would be blocked by the constitutional court. Then the parliament could change the constitution such that the controversial law is in line with the new constitution. This situation is a general apex game: J is the set of members of the parliament, $\mathcal{I}$ is the collection of coalitions of judges at the constitutional court which are able to impose a decision, and $r = 61$, $q = 81$ are the respective quotas.

2.3 Power Indices

A *power index* is a mapping $\varphi : \mathcal{V}(N) \to \mathbb{R}^N$. A power index φ

1. is *efficient* if $\sum_{i \in N} \varphi_i(v) = v(N)$ for all $v \in \mathcal{V}$;

2. satisfies the *null player property* if $\varphi_i(v) = 0$ for all $v \in \mathcal{V}$ and all null players $i \in N$ with respect to v;

3. satisfies *anonymity* if for each permutation $\rho : N \to N$ and every game $v \in \mathcal{V}$, we have $\varphi_{\rho(i)}(\rho v)\, \varphi_i(v)$ where $\rho v(S) = v\left(\rho^{-1}(S)\right)$.

Note that anonymity implies $\varphi_i(v) = \varphi_j(v)$ for symmetric players i, j with respect to v. Although these three properties are standard, we prove our results under maximal generality and do not assume that these properties are always satisfied. The following definition is due to Sagonti (1991), where the monotony properties of several power indices are analyzed.

Definition 2.3. A power index φ is called *strongly monotonic* if $\varphi_i(v) > \varphi_i(u)$ for all $i \in N$ and all $u, v \in \mathcal{V}$ with

$$\begin{aligned} & v(S) - v(S \setminus \{i\}) \geq u(S) - u(S \setminus \{i\}) \quad \text{for all } S \in \mathcal{P}_i, \qquad (1) \\ \text{and} \quad & v(S) - v(S \setminus \{i\}) > u(S) - u(S \setminus \{i\}) \quad \text{for some } S \in \mathcal{P}_i. \end{aligned}$$

This property is quite natural: If a player is pivotal with respect to v in each coalition S in which he is pivotal with respect to u and in at least one more then he is more powerful in v than in u according to a strongly monotonic power index. A similar definition of strong monotony can be found in Young (1985). There, a power index φ is called strongly monotonic if $\varphi_i(v) \geq \varphi_i(u)$ for all players $i \in N$ and all $u, v \in \mathcal{V}$ satisfying (1). Note that none of these properties implies the other. Clearly, Young's notion does not imply Sagonti's. On the other hand, if a player i is pivotal in exactly the same coalitions with respect to u and v then Definition 2.3 does not make any statement about the relation between $\varphi_i(u)$ and $\varphi_i(v)$, whereas monotony à la Young implies $\varphi_i(v) = \varphi_i(u)$.

We close this subsection with the following brief remark on the role of strong monotony for non null players.

Remark 2.4. Let $v \in \mathcal{V}$, let ω be the zero game, let φ be a strongly monotonic power index, and let i in N. If i is not a null player with respect to v, application of the definition of strong monotony on v and ω shows that $\varphi_i(v) > \varphi_i(\omega)$. Hence, under strong monotony a player always has more power in a winning coalition in which he is not a null player than he has in the zero game ω.

2.4 Hedonic Coalition Formation Games

A *hedonic game* is a set N together with a profile of preferences[2] $(\succeq_i)_{i \in N}$, where $\succeq_i$ is defined on $\mathcal{P}_i(N)$ for all $i \in N$. Hedonic games belong to a special class of coalition formation games and have been introduced by Drèze and Greenberg (1980). The main characteristic is that for each player $i \in N$, the preference relation $\succeq_i$ depends only on the coalitions to which player i belongs, and not on the partiton of N. The crucial question is, whether there is a partition of the player set N such that no group of players would leave their coalitions and form a new one together (see Definition 2.5). There are several answers to this question; the probably best known are sufficient conditions as *ordinal balancedness* and *consecutiveness* in Bogomolnaia and Jackson (2002) and the *weak top coalition property* in Banerjee et al. (2001). Unfortunately, they are not necessary. A characterization of hedonic games for which stable partitions of the player set exist is given in Iehlé (2007). The author defines a property, called *pivot balancedness*, which is necessary and sufficient for the existence of core stable partitions. Unfortunately, neither can this property be efficiently verified nor is it constructive, in the sense that a core stable partition can easily be found.

Definition 2.5. Let $(N, \succeq)$ be a hedonic game and let $\pi \in \Pi$.

1. A *deviation* of π is a coalition $S \subseteq N$ such that $S \succ_i \pi(i)$ for all $i \in S$.
2. π is called *core stable* if it has no deviations.

A partition π is said to be *blocked by* S if S is a deviation of π. A partition π is said to *blocked* if it is blocked by some $S \subseteq N$. The *core* of a hedonic game $(N, \succeq)$ is the set of all core stable partitions and is denoted by $\mathcal{C}(N, \succeq)$.

[2] A preference relation is a complete, transitive, and reflexive binary relation.

Given a proper simple game v and a power index φ we can define a preference profile $(\succeq_i)_{i\in N}$ by

$$S \succeq_i T \qquad \text{if and only if} \qquad \varphi_i\left(v_S\right) \geq \varphi_i\left(v_T\right)$$

for all $S,T \in \mathcal{P}_i$. Thus, a proper simple game v and a power index φ together induce a hedonic game. In this hedonic game each player prefers a coalition S over T if he is more powerful in S than in T. For convenience we write from here $\varphi_i\left(S\right) = \varphi_i\left(v_S\right)$ if the game v is known.

In general, a coalition might be part of one partition which is core stable and part of another one which is not. This is different in our case. Let $v \in \mathcal{V}$ and let φ be a power index. Let S be winning and let $\pi, \sigma \in \Pi_S$. Then $\varphi_i\left(\pi\left(i\right)\right) = \varphi_i\left(S\right) = \varphi_i\left(\sigma\left(i\right)\right)$ for each $i \in S$. Further $v\left(T\right) = 0$ for each $T \subseteq N \setminus S$, so that $v_T = \omega$. Hence, for each $j \in N \setminus S$ we have $\varphi_j\left(\pi\left(j\right)\right) = \varphi_j\left(\omega\right) = \varphi_j\left(\sigma\left(j\right)\right)$. Consequently, either both π and σ are core stable or none of them is. We therefore call a coalition $S \subseteq N$ *core stable* if each partition $\pi \in \Pi_S$ is core stable. We denote the set of core stable coalitions by $\mathcal{C}\left(N, v, \varphi\right)$.

3 General Results

As mentioned before, the question of existence of core stable partitions is difficult to answer. Although we are working with core stable coalitions, the problem is not much easier, as previous works have shown (see for instance Dimitrov and Haake, 2006, 2008b). However, we establish necessary and sufficient conditions for the existence of core stable coalitions in hedonic games derived from general apex games together with strongly monotonic power indices. The first theorem considers the case in which the apex quota r is at least the absolute majority of the minor players.

Theorem 3.1. *Let $a^{q,r}_{\mathcal{I}J}$ be a general apex game with $r > \frac{1}{2}\left|J\right|$ and let φ be a strongly monotonic and anonymous power index. Then $\mathcal{C}\left(N, a^{q,r}_{\mathcal{I}J}, \varphi\right) \neq \emptyset$.*

This result is strong. We do not need, efficiency or the null player property and we do not make any assumptions with respect to the collection of apex sets $\mathcal{I}$.

Nevertheless, we are also interested in games with a lower quota. Before we start to consider the hedonic coalition formation game, we analyze how strong monotony affects the power of players within coalitions. The natural intuition is the following: Apex players are more powerful in a coalition S if S contains many minor players, who are substitutable, but not so many that they could form a coalition on their own. For the minor players the situation is similar: As long as there are less than q minor players in a coalition, they are more powerful if there are less other (competing) minor players. This is the statement of Lemma 3.2.

Lemma 3.2. *Let $a^{q,r}_{\mathcal{I}J}$ be a general apex game on N and let φ be a strongly monotonic power index which satisfies anonymity. Let $J_1, J_2 \subseteq J$ with $r \leq \left|J_2\right| < \left|J_1\right| < q$ and let $I' \in \mathcal{I}$.*

1. *If* $i \in I'$ *is not a null player in* $I' \cup J_2$ *with respect to* $a_{\mathcal{I}J}^{q,r}$ *then* $\varphi_i (I' \cup J_1) > \varphi_i (I' \cup J_2)$*.*

2. *Let* φ *satisfy, additionally, efficiency and the null player property. Then* $\sum_{j \in J_1} \varphi_j (I' \cup J_1) < \sum_{j \in J_2} \varphi_j (I' \cup J_2)$*.*

The condition $|J_1| < q$ in Lemma 3.2 is necessary. Indeed, if $|J_2| < q$ and $|J_1| \geq q$ then J_1 is winning in the subgame of $a_{\mathcal{I}J}^{q,r}$ on $I' \cup J_1$ while J_2 is not winning in the subgame of $a_{\mathcal{I}J}^{q,r}$ on $I' \cup J_2$. Hence, we can construct coalitions S such that members of I' are pivotal in S with respect to the subgame of $a_{\mathcal{I}J}^{q,r}$ on $I' \cup J_2$ but not with respect to the subgame on $I' \cup J_2$. Hence, strong monotonicity does not apply.

We can now prove our second main theorem. This theorem gives a lower bound for the number of minor players which must be contained in a core stable coalition.

Theorem 3.3. *Let* $a_{\mathcal{I}J}^{q,r}$ *be a general apex game and let* φ *be a strongly monotonic power index satisfying anonymity. Then* $|S \cap J| \geq \frac{1}{2} |J|$ *for each* $S \in \mathcal{C} (N, a_{\mathcal{I}J}^{q,r}, \varphi)$*.*

This result is surprising at first sight: Although the minor players may replace each other in each winning coalition and therefore seem quite weak, each core stable coalition must contain at least half of them.

In the same way apex players are more powerful in coalitions with many minor players, the minor players are more powerful in coalitions containing many apex sets. In particular, it is easy to see in which coalition the power of minor players is maximal.

Lemma 3.4. *Let* $a_{\mathcal{I}J}^{q,r}$ *be a general apex game and let* φ *be a strongly monotonic power index. Then* $\varphi_j (I \cup J') \geq \varphi_j (I' \cup J')$ *for each* $I' \in \mathcal{I}$*, each* $J' \subseteq J$ *with* $r \leq |J'| < q$*, and each* $j \in J'$*. If* $I' \in \mathcal{I}$ *is such that there is* $i \in I \setminus I'$ *who is not a null player with respect to* $a_{\mathcal{I}J}^{q,r}$ *then the inequality is strict.*

An immediate consequence of Lemmas 3.2 and 3.4 is that $\varphi_i (S) \leq \varphi_i (I \cup J')$ for all $S \subseteq N$ with $|S \cap J| < q$, where $J' \subseteq J$ with $|J'| = r$.

Theorems 3.3 and 3.1 hold true without any efficiency or null player assumption. Nevertheless, both are in most cases desirable properties of a power index. With the additional assumption of efficiency the result of Theorem 3.3 can be sharpened considerably. In part 1 of the Theorem 3.5 we give a sufficient condition for the nonemptyness of the core and characterize all core stable coalitions if, additionally, $r < \frac{1}{2} |J|$. Part 2 gives a necessary condition for the nonemptiness of the core and part 3 gives necessary conditions for coalitions which are core stable.

Theorem 3.5. *Let* $a_{\mathcal{I}J}^{q,r}$ *be a general apex game and let* φ *be a strongly monotonic power index which satisfies anonymity, the null player property, and efficiency. Let* $J' \subseteq J$ *with* $|J'| = r$*.*

1. *If $\varphi_j(I \cup J') \leq \frac{1}{q}$ for all $j \in J'$ then $\mathcal{C}(N, a_{\mathcal{I}J}, \varphi) \neq \emptyset$. Suppose, additionally, that $r < \frac{1}{2}|J|$ and let $S \subseteq N$ be null player free. Then S is core stable if and only if $S \subseteq J$ and $|S| = q$.*

2. *If $\frac{1}{q} < \varphi_j(I \cup J') \leq \frac{1}{2r}$ for all $j \in J'$ then $\mathcal{C}(N, a_{\mathcal{I}J}, \varphi) = \emptyset$.*

3. *If $\max\left\{\frac{1}{q}, \frac{1}{2r}\right\} < \varphi_j(I \cup J')$ for all $j \in J'$ then*

$$\frac{1}{2} \leq \frac{|S \cap J|}{|J|} \leq \frac{|S \cap J|}{q} \leq r\varphi_j(I \cup J')$$

for each $S \in \mathcal{C}(N, a_{\mathcal{I}J}, \varphi)$.

Note that the cases of Theorem 3.5 are exhaustive due to symmetry between minor players. However, Theorem 3.5 is not a complete if and only if statement: In the third case we give a necessary condition for the existence of core stable coalitions, namely that they cannot be blocked by a minimal winning coalition of minor players. But this does not guarantee that a coalition which satisfies this condition is not blocked at all.

The crucial point in Theorem 3.5 is the relation between $\varphi_j(I \cup J')$ and the parameters q and r of the game. The power of minor players in $I \cup J'$ depends on the collection $\mathcal{I}$. Although it is impossible to compare arbitrary collections of apex sets, strong monotony of a power index φ gives us an idea.[3] Consider two collections $\mathcal{I}_1$ and $\mathcal{I}_2$ on $I = N \setminus J$ with $\mathcal{I}_1 \subsetneq \mathcal{I}_2$. Let q and r be fixed and denote the corresponding general apex games by v^1 and v^2. By strong monotony of φ we find that $\varphi_j\left(v^1_{I \cup J'}\right) < \varphi_j\left(v^2_{I \cup J'}\right)$. An implication of Theorem 3.5 is now the following: If $J'' \in \mathcal{C}(N, v_2, \varphi)$ then $J'' \in \mathcal{C}(N, v_1, \varphi)$ for each $J'' \subseteq J$ with $|J''| = q$. We will further elaborate this ideas in the next section (see Corollary 4.2 and Example 4.4), where we consider the Shapley-Shubik index and the Banzhaf index.

4 Applications

4.1 The Shapley-Shubik Index

Among the many different power indices the probably best known is the Shapley-Shubik index (Shapley, 1953), which is defined as

$$\sigma_i(v) = \sum_{S \in \mathcal{P}_i} \frac{(|N| - |S| - 1)!\,(|S|)!}{|N|!}\left(v(S \cup \{i\}) - v(S)\right)$$

for any proper simple game v and any $i \in N$. It is well known that σ satisfies the conditions of all of the previous theorems. But we can strengthen these results even further for a special subclass of general apex games, namely those

[3] Thanks to the referees for bringing up this very interesting point.

with $r = 1$. As $r + q \geq |J| + 1$ we must have that $q = |J|$. We therefore simply write $a_{\mathcal{I}J}$ for such games and see

$$a_{\mathcal{I}J}(S) = \begin{cases} 1, & \text{if } J \subseteq S \text{ or } (S \cap I \in \mathcal{I} \text{ and } S \cap J \neq \emptyset), \\ 0, & \text{else.} \end{cases}$$

The next theorem gives a characterization of those general apex games $a_{\mathcal{I}J}$ for which core stable coalitions exist.

Theorem 4.1. *Let $a_{\mathcal{I}J}$ be a general apex game. Then $\mathcal{C}(N, a_{\mathcal{I}J}, \sigma) \neq \emptyset$ if and only if $\sigma_j(I \cup \{j\}) \leq \frac{1}{|J|}$ for all $j \in J$. In this case a null player free coalition $S \subseteq N$ is core stable if and only if $S = J$.*

This result is very useful as it is easy to check whether or not core stable coalitions exist. But it also enables us to compare different games as we did at the end of the previous section.

Corollary 4.2. *Let $a_{\mathcal{I}_1 J}$, $a_{\mathcal{I}_2 J}$ be two general apex games with $\mathcal{I}_1 \subseteq \mathcal{I}_2$. Then $\mathcal{C}(N, a_{\mathcal{I}_2 J}, \sigma) \subseteq \mathcal{C}(N, a_{\mathcal{I}_1 J}, \sigma)$.*

The game $a_{\mathcal{I}J}$ may model the following situations: All minor players have the same information and the coalition J is winning if this information is not leaked. Further, any other coalition is winning only if it has this information. Then only one minor player is needed to pass the information to a coalition. The next example illustrates this idea.

Example 4.1. In a city there is a drug market on which several dealers are active. These dealers can form coalitions in order to control this market and make profit. A group of dealers which is able to drive out the remaining players and control the market on its own is called a *syndicate*. In particular, there cannot be two disjoint syndicates. The city's police department wants to dry out the drug market and combats the dealers by raids. As long as no policeman leaks out time and place of a raid, the syndicates will not be able to make any profit. However, if a syndicate can bribe at least one of these policemen then the police department is powerless and this syndicate can control the drug market. The situation can be modelled as an apex game $a_{\mathcal{I}J}$, where $\mathcal{I}$ is the collection of syndicates, J is the police force, and $r = 1$. If a syndicate S bribes a policeman $j \in J$ then $S \cup \{j\}$ is winning; if J stays together then J wins.

4.2 The Banzhaf Power Index

Another example of a strongly monotonic power index is the Banzhaf index η (see Banzhaf, 1965; Coleman, 1971). We follow Owen (1978), where the Banzhaf index of a player $i \in N$ in a game $v \in \mathcal{V}$ is defined as

$$\eta_i(v) = \frac{1}{2^{|N|-1}} \sum_{S \in \mathcal{P}_i} (v(S) - v(S \setminus \{i\})).$$

It can easily be verified that η satisfies the null player property and anonymity on $\mathcal{V}$. In particular, Theorems 3.3 and 3.1 apply; however, as η is not efficient, Theorem 3.5 does not apply.

We will, again, consider a general apex game $a_{\mathcal{I}J}$ with $r = 1$ and $q = |J|$. Before we do so, we make the following observation. Given any proper simple game v on a player set $I \subseteq N$ with $|I| \leq |N| - 3$ we can construct a general apex game by defining

$$\mathcal{I} = \{I' \subseteq I;\ v(I) = 1\}$$

and $J = N \setminus I$. Vice versa, each collection of apex sets can be interpreted as a collection of winning coalitions of a proper simple game on $I = \bigcup_{I' \in \mathcal{I}} I$. The next theorem connects this game with the apex game.

Theorem 4.3. *Let $a_{\mathcal{I}J}$ be a generalized apex game and let v be the simple game on I defined by the collection of winning coalitions $\mathcal{I}$. If $I \in \mathcal{C}(I, v, \eta)$ then $N \in \mathcal{C}(N, a_{\mathcal{I}J}, \eta)$.*

The condition in Theorem 4.3 might seem a bit awkward on a first sight. However, the following example gives a nice application.[4]

Example 4.2. Consider a club with two kinds of membership, the two (disjoint) sets of members are denoted by I and J. The voting rule for accepting or rejecting a proposal is defined by three integers p, q, r where $p \geq \frac{1}{2}|I|$, $q > \frac{1}{2}|J|$ and $q + r \geq |J| + 1$. A proposal is accepted if at least p members of I and at least r members of J agree; or if q members of J agree, independent of the behavior of members of I. In this case we end up with a general apex game with

$$\mathcal{I} = \mathcal{I}_p = \{I' \subseteq I;\ |I'| \geq p\}.$$

Hence, not only minor players, but also players in the apex sets are symmetric. The game v which is defined on I by the collection of apex sets as winning coalitions is a symmetric game, and it can easily be shown that I is core stable in the hedonic game induced by v and η. Hence, by Theorem 4.3 we have that N is core stable in the hedonic game defined by the general apex game $a^{q,r}_{\mathcal{I}_p J}$ and η.

4.3 The Normalized Banzhaf Index

Although η is not efficient, there is a natural way to construct an efficient version (see for instance Owen, 1978; Dubey and Shapley, 1979), namely

$$\beta_i(v) = \frac{\eta_i(v)}{\sum_{j \in N} \eta_j(v)} = \frac{\mu_i(v)}{\sum_{j \in N} \mu_j(v)},$$

where $\mu_i(v)$ is the number of coalitions $S \subseteq N$ in which player i is pivotal with respect to v. As before we will write $\mu(S)$ for $\mu(v_S)$ if the game v is known.

In Sagonti (1991) the question is asked - but not answered - whether or not β is strongly monotonic. The following example shows that in general it is not.

[4] Thanks to the editor in charge for this idea

Example 4.3. Let $A = \{1, 2\}, B = \{3, 4, 5\}$, and $N = A \cup B$. Define v on N as

$$v(S) = \begin{cases} 1, & \text{if } A \subseteq S \text{ or } (|S \cap A| \geq 1 \text{ and } |S \cap B| \geq 2), \\ 0, & \text{else.} \end{cases}$$

Let $S = \{1, 3, 4\}$ and $T = \{1, 3, 4, 5\}$. If β were strongly monotonic then $\beta_1(T) > \beta_1(S)$. But $\mu_1(S) = \mu_3(S) = \mu_4(S) = 1$, and on the other hand $\mu_1(T) = 3$ and $\mu_3(T) = \mu_4(T) = \mu_5(T) = 2$. Hence, $\beta_1(S) = \frac{1}{3} = \frac{3}{9} = \beta_1(T)$. This implies that β is not strongly monotonic.

This example shows that β does not satisfy the conditions of Lemmas 3.2 and 3.4. However, we can show that β satisfies conclusions which are similar to those of Lemma 3.2 (see Lemma 1 in the appendix). In particular, it remains true that for each general apex game $a_{\mathcal{I}J}^{q,r}$ each core stable coalition must contain at least half of the minor players. The following theorem considers the case $r = 1$ as in the previous subsections. It is very similar to Theorem 4.1, however, as the Banzhaf index is not strongly monotonic it is in general not true that $\beta_j(S) \leq \beta_j(I \cup \{j\})$ for all S with $|S \cap J| < |J|$.

Theorem 4.4. *Let $a_{\mathcal{I}J}$ be a general apex game. Then $\mathcal{C}(N, a_{\mathcal{I}J}, \beta) \neq \emptyset$ if and only if $\beta_j(S) \leq \frac{1}{|J|}$ for all $S \subseteq N$ and all $j \in S \cap J$. In this case a null player free coalition $S \subseteq N$ is core stable if and only if $S = J$.*

We see again, that the key for core stability lies in the set of minor players. Either each minimal winning coalition of minor players is core stable or there is no core stable coalition. The following Corollary summarizes our results of this section for apex games in the classical sense. It can easily be be derived from Theorems 4.1, 4.3, and 4.4.

Corollary 4.5. *Let a_{iJ} be the apex game with apex player i and minor players J. Then $\mathcal{C}(N, a_{iJ}, \sigma) = \mathcal{C}(N, a_{iJ}, \beta) = \emptyset$. However, $N \in \mathcal{C}(N, a_{iJ}, \eta)$.*

We close this section by coming back to the symmetric apex games from Example 4.2. We give conditions for the existence of core stable coalitions and we compare different collections of apex sets with respect to the existence of core stable coalitions. The relevant calculations can be found in the appendix.

Example 4.4. Consider the general apex game defined in Example 4.2. It is clear that no coalition containing q minor players which is not minimal winning can be core stable. Also, each core stable coalition must contain at least half of the minor players. Let $J' \subseteq J$ with $|J'| = r$. It can easily be seen that $\beta_j(I \cup J') \geq \beta_j(S)$ for all S with $|S \cap J| < q$. We define

$$\xi = \frac{\sum_{j \in J'} \mu_j(I \cup J')}{\sum_{i \in I} \mu_i(I \cup J')} = r \sum_{k=p}^{|I|} \frac{(|I| - p)!\,(p-1)!}{k!\,(|I| - k)!}.$$

It turns out that a coalition $J'' \subseteq J$ with $|J''| = q$ is core stable if and only if

$$\xi \leq \frac{r}{q - r}. \tag{2}$$

Moreover, there cannot be any (null player free) core stable coalition besides those of type J'' if

$$\xi < \frac{|J|}{2q - |J|} \tag{3}$$

as in this case J'' blocks $I \cup J'''$ for each J''' with $|J'''| \geq \frac{1}{2}|J|$.

If now $r < \frac{1}{2}|J|$, it can be shown that if J'' is core stable, there are no other (null player free) core stable coalitions. The same holds true if $r = \frac{1}{2}|J|$ and $p < |I|$. If, on the other hand, $r > \frac{1}{2}|J|$ the core is always nonempty and contains at least one of the coalitions J'' and $I \cup J'$.

For $r < \frac{1}{2}|J|$ we can now find the analogue to Corollary 4.2: For the family of apex games $a^{q,r}_{\mathcal{I}_p J}$ we find $\mathcal{C}\left(N, a^{q,r}_{\mathcal{I}_p J}, \beta\right) \subseteq \mathcal{C}\left(N, a^{q,r}_{\mathcal{I}_{p'} J}, \beta\right)$ whenever $p \leq p'$. In particular, there is p^* such that $\mathcal{C}\left(N, a^{q,r}_{\mathcal{I}_p J}, \beta\right) \neq \emptyset$ if and only if $p > p^*$ (note that $p^* = |I|$ is possible so that $p \leq |I|$).

For $r > \frac{1}{2}|J|$ we find something similar: There is p^* (which does not need to be an integer) such that

$$\begin{aligned} J'' \in \mathcal{C}\left(N, a^{q,r}_{\mathcal{I}_p J}, \beta\right) \quad &\Leftrightarrow \quad p \leq p^* \\ I \cup J' \in \mathcal{C}\left(N, a^{q,r}_{\mathcal{I}_p J}, \beta\right) \quad &\Leftrightarrow \quad p \geq p^*. \end{aligned}$$

5 Conclusion

Although the structure of a apex game is very general - we have seen that each proper simple game can be mapped one to one to a general apex game on a larger player set - many results about the existence of core stable coalitions can be derived without deep analysis of the collection of apex sets. This is good news: Normally the analysis of coalition formation games, even if derived from a proper simple game and a power index, is very complicated as previous works have shown (see for instance Hart and Kurz, 1984; Aumann and Myerson, 1988; Dimitrov and Haake, 2008b). We have shown that in case of the Shapley-Shubik index and the normalized Banzhaf index, for a minor quota $r > \frac{1}{2}|J|$ a stable outcome always exists, and for $r = 1$ the only possible (null player free) core stable outcome is the minimal winning coalition of minor players.

In previous works (Shenoy, 1979; Dimitrov and Haake, 2008a) the following property of power indices has been used to guarantee the existence of core stable coalition: A power index φ *does not exhibit the paradox of smaller coalitions on a game* v, if for each non minimal winning coalition S there is a coalition $T \subsetneq S$ such that $\varphi_i(v_T) \geq \varphi_i(v_S)$. It can be shown that this property together with (a weaker form of) anonymity is sufficient for the existence of core stable coalitions; in particular, the only (null player free) core stable coalitions are minimal winning coalitions of minimal cardinality. However, recalling the conclusions of Lemma 3.2, we see that a strongly monotonic power index must exhibit the paradox on any general apex game. In particular, the Shapley-Shubik index and the Banzhaf index (both normalized and normalized) do so.

We therefore introduced strong monotony as the key property of power indices as it is a very natural property and satisfied for instance by the Shapley-Shubik index and the Banzhaf index. It becomes clear from the proofs though, that we only need a weaker monotony property. Namely, monotony as stated in the conclusions of Lemma 3.2, and satisfied by the normalized Banzhaf index as well (which is not strongly monotonic). *The more players can substitute each other in a game, the more powerful are those players who cannot be substituted.* In case of the Shapley-Shubik index this property enables us to boil down the question of existence of core stable coalitions to a single inequality which can easily be checked.

Although many results rely on the structure of apex sets only as much as certain inequalities must be checked, we can also compare the cores of apex games which differ only in the collection of apex sets. Strong monotony implies that the power of minor players in a general apex game (and its subgames) increases if further apex sets are added. If we consider the Shapley-Shubik index on a general apex game with apex quota $r = 1$, the consequence is that if a collection of apex sets results in a nonempty core, each of its subcollections does as well. Nevertheless, the core might be empty if the minor quota is small and the collection of apex sets is too large.

A Proofs

Proof of Lemma 2.2. We prove the only if part. Let $q \leq \frac{1}{2}|J|$. The there is $J' \subseteq J$ with $|J'| = q$ and $|J \setminus J'| \geq q$. Hence, $a^{q,r}_{\mathcal{I}J}(J') = a^{q,r}_{\mathcal{I}J}(J \setminus J') = 1$. Consequently, $a^{q,r}_{\mathcal{I}J}$ is not proper.

Let $r + q \leq |J|$. Then there is $J' \subseteq J$ with $|J'| = q$ and $|J \setminus J'| \geq r$. Hence, $a^{q,r}_{\mathcal{I}J}(J') = a^{q,r}_{\mathcal{I}J}(N \setminus J') = 1$. Therefore, a is not proper. ■

Proof of Theorem 3.1. Let $S \subseteq N$ such that $\varphi_j(S) \geq \varphi_j(S')$ for all $S \in \mathcal{P}_j$. Because of anonymity S does not depend on the choice of j, that is $\varphi_j(S) \geq \varphi_j(S')$ for all $S \in \mathcal{P}_j$ and all $j \in S \cap J$. As $r > \frac{1}{2}|J|$ we have that there is a minor player $j' \in S \cap S'$ for each winning coalition S'. Hence, S' cannot block S as $\varphi_{j'}(S') \leq \varphi_{j'}(S)$. ■

Proof of Lemma 3.2. Since all $j \in J$ are symmetric with respect to $a^{q,r}_{\mathcal{I}J}$ and since φ satisfies anonymity, it can be assumed without loss of generality that $J_2 \subsetneq J_1$. Let v be the restriction of $a^{q,r}_{\mathcal{I}J}$ on $I' \cup J_1$ and u be the restriction of $a^{q,r}_{\mathcal{I}J}$ on $I' \cup J_2$.

1. Let $i \in I'$ not be a null player with respect to u. Let $S \subseteq N$ be such that that i is pivotal in S with respect to u. Then i is also pivotal in S with respect to v. Hence, $v(S) - v(S \setminus \{i\}) \geq u(S) - u(S \setminus \{i\})$ for all $S \subseteq N$. Let $J' \subseteq J$ such that $|J' \cap J_2| < r \leq |J' \cap J_1|$ and let $S' = (S \setminus J) \cup J'$. Then i is pivotal in S' with respect to v but $u(S') = 0$; hence, by strong monotony of φ we have $\varphi_i(I' \cup J_1) = \varphi_i(v) > \varphi_i(u) = \varphi_i(I' \cup J_2)$.

2. By part 1 of the proof $\varphi_i(I' \cup J_1) > \varphi_i(I' \cup J_2)$ for each $i \in I'$ who is not a null player in $I' \cup J_2$. As there is at least one such i, we have

$$\begin{aligned}\sum_{j \in J_1} \varphi_j(I' \cup J_1) &= 1 - \sum_{i \in I'} \varphi_i(I' \cup J_1) \\ &< 1 - \sum_{i \in I'} \varphi_i(I' \cup J_2) = \sum_{j \in J_2} \varphi_j(I' \cup J_2).\end{aligned}$$

■

Proof of Theorem 3.3. If $r > \frac{1}{2}|J|$ there is nothing to show, so let $r \leq \frac{1}{2}|J|$. Let $S \in \mathcal{C}(N, a^{q,r}_{\mathcal{I}J}, \varphi)$ and assume that $|S \cap J| < \frac{1}{2}|J|$. As each coalition $T \in \pi$ except S is winning we have that the restriction of $a^{q,r}_{\mathcal{I}J}$ on T is the zero game ω. Let $z = \varphi_k(\omega)$ for $k \in N$ (this does not depend on k by anonymity). As $J \setminus S$ is not winning, we have $|J \setminus S| < q$. Let $I' = S \setminus J$. We show that $I' \cup (J \setminus S)$ is a deviation of S. Indeed, since I' is not winning and since all $j \in J \setminus S$ are symmetric with respect to $a^{q,r}_{\mathcal{I}J}$, none of them is a null player $I' \cup (J \setminus S)$. Therefore, $\varphi_j(I' \cup (J \setminus S)) > z$ for all $j \in J \setminus S$. Since φ satisfies anonymity, and since $|S \cap J| < |J \setminus S| < q$, part 1 of Lemma 3.2 applies. Hence, $\varphi_i(I' \cup (J \setminus S)) > \varphi_i(S)$ for all $i \in S \setminus J$. Thus, S is blocked. ■

Proof of Lemma 3.4. Let $j \in J$, let v be the apex game restricted to $I \cup J'$ and let u be $a^{q,r}_{\mathcal{I}J}$ restricted to $I' \cup J'$. If all $i \in I \setminus I'$ are null players with respect to $a^{q,r}_{\mathcal{I}J}$ then u and v coincide, and hence, $\varphi(v) = \varphi(u)$. Let $I' \in \mathcal{I}$ such that there is $i \in I \setminus I'$ who is not a null player with respect to v. If $T \subseteq N$ is such that $j \in T$ is pivotal in T with respect to u then $T \cap I' \in \mathcal{I}$ and $|T \cap J'| = r$. In this case T is also winning in v as $T \cap I' \subseteq T \cap I$; and j is also pivotal in T with respect to v. As i is not a null player with respect to v, there is $T \subseteq N$ such that both i and j are pivotal in T with respect to v. However, $u(T) \leq v(T \setminus \{i\}) = 0$ and hence, j is not pivotal in T with respect to u. Therefore, $\varphi_j(I \cup J') = \varphi_j(v) > \varphi_j(u) = \varphi_j(I' \cup J')$ due to strong monotony of φ. ■

Proof of Theorem 3.5. Let $J'' \subseteq J$ with $|J''| = q$.

1. Let $\varphi_j(I \cup J') \leq \frac{1}{q}$. Then J'' cannot be blocked by any coalition T with $|T \cap J| \geq q$ by efficiency and anonymity of φ. By the initial condition J'' cannot be blocked by $I \cup J'$. As $\varphi_j(S) \leq \varphi_j(I \cup J')$ for each S with $|S \cap J| < q$, J'' cannot be blocked at all.

   Let now $r < \frac{|J|}{2}$ and let S be core stable. Then $|S \cap J| \geq \frac{1}{2}|J| > r$ by Theorem 3.3. If $|S \cap J| > q$ then S is blocked by J'' due to anonymity and efficiency. Assume that $|S \cap J| < q$. From Lemmas 3.2 and 3.4 follows that

$$|S \cap J| \cdot \varphi_j(S) < r \cdot \varphi_j(S \setminus J \cup J') \leq r \cdot \varphi_j(I \cup J') \leq \frac{r}{q} < \frac{|J|}{2q}.$$

Hence, $\varphi_j(S) < \frac{|J|}{2q}\frac{1}{|S\cap J|} \leq \frac{1}{q}$ for all $j \in S \cap J$. So, S is blocked by J'' and therefore S is not core stable.

Hence, $|S \cap J| = q$. If S contains $i \in S \setminus J$ which is not a null player then $\varphi_i(S) > 0$ and consequently, $\varphi_j(S) < \frac{1}{q}$ for all $j \in S \cap J$ by anonymity. Thus, S is blocked by J''. Hence, the only null player free core stable coalitions are those of the form of J''.

2. Let $\frac{1}{q} < \varphi_j(I \cup J') \leq \frac{1}{2r}$ and let $S \subseteq N$ be winning. First, let $|S \cap J| \geq q$. If there is $i \in S \setminus J$ which is not a null player in S then S is blocked by $S \cap J$ by anonymity and efficiency. If all $i \in S \setminus J$ are null players in S then S is blocked by $I \cup J'$.

   Now, let $|S \cap J| < q$. If $|S \cap J| < \frac{1}{2}|J|$ then S cannot be core stable by Theorem 3.3. Hence, let $\frac{1}{2}|J| \leq |S \cap J| < q$. By the initial condition we have $2r < q \leq |J|$ and therefore $r < |S \cap J|$. With Lemma 3.2 we see $|S \cap J| \cdot \varphi_j(S) < r \cdot \varphi_j(I \cup J')$ and therefore

$$\varphi_j(S) < \frac{r}{|S \cap J|} \cdot \varphi_j(I \cup J') \leq \frac{r}{\frac{1}{2}|J|} \cdot \frac{1}{2r} \leq \frac{1}{|J|} \leq \frac{1}{q}.$$

   Thus, S is blocked by J''. Therefore, there are no core stable coalitions.

3. Let $\varphi_j(I \cup J') > \max\left\{\frac{1}{2r}, \frac{1}{q}\right\}$. Let further $S \subseteq N$ be core stable. As before, Theorem 3.3 implies the first inequality. The second inequality is clear. Since $\varphi_j(J'') = \frac{1}{q}$, J'' is blocked by $I \cup J'$. Therefore, each coalition S with $|S \cap J| \geq q$ is either blocked by J'' or by $I \cup J'$. Using Lemma 3.2 as before implies $\varphi_j(S) \leq \frac{r}{|S\cap J|}\varphi_j(I \cup J')$. If now $r\varphi_j(I \cap J') < \frac{|S\cap J|}{q}$ then $\varphi_j(S) < \frac{1}{q}$. In this case S is blocked by J''. Hence, $\frac{|S\cap J|}{q} \leq r\varphi_j(I \cup J')$.

■

Proof of Theorem 4.1. As σ satisfies the conditions of Theorem 3.5, it is sufficient to show that $\sigma_j(I \cup \{j\}) \leq \frac{1}{2r} = \frac{1}{2}$ for all $j \in J$. In this case the conditions of part 3 of Theorem 3.5 are never satisfied. We see that

$$\begin{aligned}
\sigma_j(I \cup \{j\}) &= \sum_{T \subseteq I} \frac{(|I| - |T|)!\,|T|!}{(|I|+1)!} \left(a_{\mathcal{I}J}(T \cup \{j\}) - a_{\mathcal{I}J}(T)\right) \\
&= \frac{1}{|I|+1} \cdot \frac{1}{2} \left[\sum_{T \subseteq I} \binom{|I|}{|T|}^{-1} \left(a_{\mathcal{I}J}(T \cup \{j\}) - a_{\mathcal{I}J}(T)\right) \right. \\
&\quad \left. + \sum_{T \subseteq I} \binom{|I|}{|I \setminus T|}^{-1} \left(a_{\mathcal{I}J}((I \setminus T) \cup \{j\}) - a_{\mathcal{I}J}(I \setminus T)\right) \right].
\end{aligned}$$

The intersection of two apex sets must be nonempty by definition, that is if $T' \cup \{j\}$ is winning then $I \setminus T'$ is not an apex set. Consequently, $a_{\mathcal{I}J}((I \setminus T') \cup \{j\}) +$

$a_{\mathcal{I}J}(T' \cup \{j\}) \le 1$. Hence, we find

$$\begin{aligned}\sigma_j(I \cup \{j\}) &\le \frac{1}{|I|+1} \cdot \frac{1}{2} \sum_{T \subseteq I} \binom{|I|}{|T|}^{-1} \cdot 1 \\ &= \frac{1}{|I|+1} \cdot \frac{1}{2} \sum_{k=0}^{|I|} \binom{|I|}{k} \binom{|I|}{k}^{-1} \\ &= \frac{1}{2}.\end{aligned}$$

This completes the proof. ■

Proof of Theorem 4.3. Let $I \in \mathcal{C}(I, v, \eta)$. Then for all $I' \in \mathcal{I}$ there is $i \in I'$ such that

$$\sum_{S \subseteq I'} (v(S) - v(S \setminus \{i\})) \le 2^{|I'|-|I|} \sum_{S \subseteq I} (v(S) - v(S \setminus \{i\})). \tag{4}$$

Assume now that N is not core stable. Then there must be a coalition T which blocks N. Let $I' = T \cap I$ and let $t = |T \cap J|$. Then $\eta_i(I') > \eta_i(N)$ for all $i \in I'$, that is

$$\frac{2^t - 1}{2^{|I'|+t-1}} \sum_{S \subseteq I'} (v(S) - v(S \setminus \{i\})) > \frac{2^{|J|} - 2}{2^{|I|+|J|-1}} \sum_{S \subseteq I} (v(S) - v(S \setminus \{i\}))$$

or equivalently,

$$\sum_{S \subseteq I'} (v(S) - v(S \setminus \{i\})) > \frac{2^t - 1}{2^t} \frac{2^{|J|}}{2^{|J|} - 2} \cdot 2^{|I'|-|I|} \sum_{S \subseteq I} (v(S) - v(S \setminus \{i\})).$$

As $t \le |J| - 1$ we have that $\frac{2^t-1}{2^t} \frac{2^{|J|}}{2^{|J|}-2} \ge 1$ and thus

$$\sum_{S \subseteq I'} (v(S) - v(S \setminus \{i\})) > 2^{|I'|-|I|} \sum_{S \subseteq I} (v(S) - v(S \setminus \{i\}))$$

for all $i \in I'$. But this contradicts Equation (4). ■

Lemma 1. *Let $a^{q,r}_{\mathcal{I}J}$ be a general apex game, let $J_1, J_2 \subsetneq J$ with $r \le |J_2| < |J_1| < q$, and let $I' \in \mathcal{I}$. Then*

1. *If $i \in I'$ is not a null player in $I' \cup J_2$ with respect to $a^{q,r}_{\mathcal{I}J}$ then $\beta_i(I' \cup J_1) > \beta_i(I' \cup J_2)$.*
2. *It holds that $\sum_{j \in J_1} \beta_j(I' \cup J_1) < \sum_{j \in J_2} \beta_j(I' \cup J_2)$.*

Proof. Let $S = I' \cup J_1$ and let $\mu(S) = \sum_{i \in S \setminus J} \mu_i(S)$. Note that $j \in J_1$ is pivotal in $T \subseteq S$ if and only if T is winning and $|T \cap J_1| = r$. There are exactly $|\mathcal{I}'| \cdot \binom{|J|-1}{r-1}$ such sets containing j where $\mathcal{I}'$ denotes the collection of apex sets contained in I'. For each such set T there are $2^{|J|-|J_1|}$ subsets $U \subseteq J \setminus J_1$ such that j is pivotal in $T \cup U$ with respect to the apex game restricted to S. Hence, there are $|\mathcal{I}'| \cdot \binom{|J|-1}{r-1} \cdot 2^{|J|-|J_1|}$ different coalitions $T \subseteq S$ in which j is pivotal with respect to the restriction of $a^{q,r}_{\mathcal{I}J}$ to S.

On the other hand let $i \in I'$. Then i is pivotal in $T \subseteq S$ if and only if T is winning and i is pivotal in $(T \setminus J_1) \cup J'$ with $J' \subseteq J_1$ and $|J'| \geq r$. Hence, there are $\mu_i (I' \cup J') \cdot \sum_{k=r}^{|J_1|} \binom{|J_1|}{k}$ different coalitions $T \subseteq S$ such that i is pivotal in T with respect to $a^{q,r}_{\mathcal{I}J}$. For each of theses sets there are $2^{|J|-|J_1|}$ subsets U of $J \setminus J_1$ such that i is pivotal in $T \cup U$. Altogether,

$$\begin{aligned}
\beta_j (S) &= \frac{|\mathcal{I}'| \cdot \binom{|J_1|-1}{r-1} \cdot 2^{|J|-|J_1|}}{|J_1| \cdot |\mathcal{I}'| \cdot \binom{|J_1|-1}{r-1} \cdot 2^{|J|-|J_1|} + \mu (I' \cup J') \cdot \sum_{k=r}^{|J_1|} \binom{|J_1|}{k} \cdot 2^{|J|-|J_1|}} \\
&= \frac{|\mathcal{I}'| \cdot \binom{|J_1|-1}{r-1}}{|J_1| \cdot |\mathcal{I}'| \cdot \binom{|J_1|-1}{r-1} + \mu (I' \cup J') \cdot \sum_{k=r}^{|J_1|} \binom{|J_1|}{k}} \\
\beta_i (S) &= \frac{\mu_i (I' \cup J') \cdot \sum_{k=r}^{|J_1|} \binom{|J_1|}{k} \cdot 2^{|J|-|J_1|}}{|J_1| \cdot |\mathcal{I}'| \cdot \binom{|J_1|-1}{r-1} \cdot 2^{|J|-|J_1|} + \mu (I' \cup J') \cdot \sum_{k=r}^{|J_1|} \binom{|J_1|}{k} \cdot 2^{|J|-|J_1|}} \\
&= \frac{\mu_i (I' \cup J')}{|J_1| \cdot |\mathcal{I}'| \cdot x (|J_1|) + \mu (I' \cup J')}
\end{aligned}$$

where

$$x (|J_1|) = \frac{\binom{|J_1|-1}{r-1}}{\sum_{k=r}^{|J_1|} \binom{|J_1|}{k}}.$$

To prove the first part of the Lemma we have to show that $x (|J_1|) < x (|J_2|)$. Note that for an integer $m \geq r$ we have

$$\begin{aligned}
\frac{1}{x (m+1)} &= \binom{m}{r-1}^{-1} \sum_{k=r}^{m+1} \binom{m+1}{k} \\
&= \binom{m}{r-1}^{-1} + \binom{m-1}{r-1}^{-1} \frac{m-r+1}{m} \sum_{k=r}^{m} \binom{m}{k} \frac{m+1}{m-k+1} \\
&= \binom{m}{r-1}^{-1} + \binom{m-1}{r-1}^{-1} \sum_{k=r}^{m} \binom{m}{k} \frac{m+1}{m-k+1} \frac{m-r+1}{m} \\
&> \binom{m-1}{r-1}^{-1} \sum_{k=r}^{m} \binom{m}{k} = \frac{1}{x (m)}
\end{aligned}$$

and therefore $x (m) > x (n)$ if $n > m$. Hence, $x (|J_1|) < x (|J_2|)$. The proof of the second part follows from efficiency and the null player property of β and is identical to the proof of part 3 of Lemma 3.2. ■

Lemma 2. *Let $a^{q,r}_{\mathcal{I}J}$ be a general apex game. Then $\beta_j (S) \leq \frac{1}{r+1}$ for all $j \in S \cap J$.*

Proof. If $|S \cap J| \geq q > r$, the claim follows from efficiency and anonymity of β. So let $|S \cap J| < q$. We use Corollary 1 from Dubey and Shapley (1979):

Let v be a (not necessarily proper) simple game on a player set N. Let ω be the number of winning coalitions with respect to v and let ν be the number of losing coalitions with respect to v. Then

$$\sum_{i \in N} \mu_i(v) \geq \lambda \cdot \lfloor |N| - \log_2(\lambda) \rfloor, \tag{5}$$

where $\lambda = \min(\omega, \nu)$ and $\lfloor x \rfloor$ is the greatest integer k with $k \leq x$.

By part 2 of Lemma 1 it is sufficient to show that the inequality holds for each winning coalition $S \subseteq N$ with $|S \cap J| = r$. Hence, let $J' = S \cap J$ for such a coalition S and let $I' = S \setminus J$. Then each $j \in J'$ is a veto player in the restriction v of $a^{q,r}_{\mathcal{I}J}$ to S. Hence, $\mu_j(v) = \omega(v) = |\mathcal{I}'|$ where $\mathcal{I}'$ denotes the collection of apex sets contained in I'. As apex sets are pairwise non disjoint, we have $|\mathcal{I}'| \leq 2^{|I'|-1} = 2^{|S|-r-1}$. As there are $2^{|N|-|S|}$ different subsets of $N \setminus S$, we find

$$\mu_j(S) = \omega(v) = |\mathcal{I}'| \cdot 2^{|N|-|S|} \leq 2^{|S|-r-1} \cdot 2^{|N|-|S|} = 2^{|N|-r-1} < \nu(v).$$

With equation (5) we see

$$\begin{aligned} \sum_{k \in S} \mu_k(S) &\geq \omega(v) \cdot \lfloor |N| - \log_2(\omega(v)) \rfloor \\ &\geq \mu_j(S) \cdot \left\lfloor |N| - \log_2\left(2^{|N|-r-1}\right) \right\rfloor \\ &= \mu_j(S) \cdot \lfloor |N| - |N| + r + 1 \rfloor \\ &= (r+1)\, \mu_j(S). \end{aligned}$$

Thus, $\beta_j(S) = \frac{\mu_j(S)}{\sum_{k \in S} \mu_k(S)} \leq \frac{1}{r+1}$. ■

Proof of Theorem 4.4. Let $S \subseteq N$ be winning. If $J \subseteq S$ and S contains $i \in N \setminus J$ who is not a null player in S then S is blocked by J.

Let $J \nsubseteq S$. With help of part 1 of Lemma 1 we can show that if $|S \cap J| < \frac{1}{2}|J|$, S cannot be core stable (the proof in analogous to the proof of Theorem 3.3). Hence, let $|S \cap J| \geq \frac{1}{2}|J|$ and let $j' \in J \setminus S$. By part 2 of Lemma 1 we have that

$$\sum_{j \in S \cap J} \beta_j(S) < \beta_{j'}((S \setminus J) \cup \{j'\}).$$

Together with Lemma 2 and recalling that $r = 1$ we see

$$\beta_j(S) < \frac{1}{|S \cap J|} \beta_j((S \setminus J) \cup \{j'\}) \leq \frac{1}{\frac{1}{2}|J|} \frac{1}{2} \leq \frac{1}{|J|}$$

for each $j \in S \cap J$. So, S is blocked by J. We see that if $S \subseteq N$ is core stable then $J \subseteq S$ and all $i \in S \setminus J$ must be null players. Hence, J is the only candidate for a null player free core stable coalition. Indeed, J is core stable if and only if $\beta_j(S) \leq \frac{1}{|J|}$ for all $S \subseteq N$. ■

Proof of Example 4.4. We start with the proof of the formula of ξ. We have

$$\begin{aligned}
\xi &= \frac{\sum_{j\in J'} \mu_j (I \cup J')}{\sum_{i\in I} \mu_i (I \cup J')} \\
&= \left(r \sum_{k=p}^{|I|} \binom{|I|}{k} \right) \left(|I| \binom{|I|-1}{p-1} \right)^{-1} \\
&= r \sum_{k=p}^{|I|} \frac{|I|!}{(|I|-k)!k!} \frac{1}{|I|} \frac{(|I|-p)!\,(p-1)!}{(|I|-1)!} \\
&= r \sum_{k=p}^{|I|} \frac{(|I|-p)!\,(p-1)!}{k!\,(|I|-k)!}.
\end{aligned}$$

Coalition J'' is core stable if and only if it is not blocked by $I \cup J'$, that is

$$\beta_j (I \cup J') \leq \frac{1}{q}$$

for all $j \in J'$. This is equivalent to

$$r + |I| \binom{|I|-1}{p-1} \left(\sum_{k=p}^{|I|} \binom{k}{|I|} \right)^{-1} = \frac{1}{\beta_j (I \cup J')} \geq q$$

or after subtracting r and inverting the inequality (note that $q - r > 0$)

$$\frac{\xi}{r} = |I|^{-1} \binom{|I|-1}{p-1}^{-1} \left(\sum_{k=p}^{|I|} \binom{k}{|I|} \right) \leq \frac{1}{q-r}.$$

Multiplication with r delivers Equation (2).

Next we have to show that $\beta_j (I \cup J') \geq \beta_j (S)$ for all S with $|S| < q$. For this purpose note that within coalition $I \cup J'$ it does not play any role which players are minor players or apex players since players of both groups are symmetric to each other and $|J'| < q$. Hence, part 1 of Lemma 1 can be used to show the required result.

For coalitions of the form $I' \cup J'''$ we find with help of Lemma 1

$$\begin{aligned}
\beta_j (I' \cup J''') &\leq \frac{r}{|J'''|} \beta_j (I \cup J') = \frac{r}{|J'''|} \frac{\mu_j (I \cup J')}{r\mu_j (I \cup J') + |I| \mu_i (I \cup J')} \\
&= \frac{r}{|J'''|} \frac{1}{r + r\frac{|I|\mu_i(I\cup J')}{r\mu_j(I\cup J')}} = \frac{1}{|J'''|} \frac{1}{1+\xi^{-1}} \leq \frac{2}{|J|} \frac{1}{1+\xi^{-1}}.
\end{aligned}$$

Hence, $I' \cup J'''$ is blocked if $\frac{2}{|J|} \frac{1}{1+\xi^{-1}} < \frac{1}{q}$ or equivalently $\xi < \frac{|J|}{2q-|J|}$.

If now $r < \frac{1}{2} |J|$ then $\frac{r}{q-r} = \frac{2r}{2q-2r} < \frac{|J|}{2q-|J|}$, that is, Inequality (2) implies Inequality (3). Let now $r = \frac{1}{2} |J|$ and assume that J'' is core stable, so that inequality (2) holds. As in this case $\frac{|J|}{2q-|J|} = \frac{r}{q-r}$ we have that $I \cup J'$ is the only possible candidate for further core stable coalitions. However, if $p < |I|$ and $I' \subsetneq I$ with $|I'| \geq p$, this coalition is blocked by $I' \cup (J \setminus J')$ as $|J \setminus J'| = r$. Hence again, if J'' is core stable (and $p < |I|$), there is no other (null player free) core stable coalition.

Finally, let $r > \frac{1}{2}|J|$ and suppose there is no core stable coalition containing less than q players. Then coalitions of the form $I \cup J'$ must be blocked. However, each blocking coalition must contain at least one of the minor players in $I \cup J'$ as $r > \frac{1}{2}|J|$. Hence, $I \cup J'$ can only be blocked by J''. That means that either J'' or $I \cup J'$ must be core stable. ■

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