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**WELFARE-INCREASING THIRD-DEGREE PRICE
DISCRIMINATION**

Simon Cowan

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Manor Road Building, Manor Road, Oxford OX1 3UQ

Welfare-increasing third-degree price discrimination

Simon Cowan¹

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Abstract

The welfare and output effects of monopoly third-degree price discrimination are analyzed when inverse demand functions are parallel. Welfare is higher with discrimination than with a uniform price when demand functions are derived from the logistic distribution, and from a more general class of distributions. The sufficient condition in Varian (1985) for a welfare increase holds for these demand functions. Total output is higher with discrimination for a large set of demand functions including those derived from strictly log-concave distributions with increasing cost pass-through, such as the normal, logistic and extreme value, and standard log-convex demands.

Keywords: third-degree price discrimination, monopoly, social welfare, output.

JEL Classification: D42, L12, L13.

1 Introduction

Under what conditions does third-degree price discrimination raise social welfare? A monopolist practising this form of price discrimination divides customers into separate markets using exogenous signals such as the customer's age or geographical location. It then sets monopoly prices that vary across the markets. Alternatively the firm may not be able to discriminate, because of regulation or arbitrage, and thus must set a uniform price. The effect on social welfare, defined as consumer surplus plus profits, of discrimination compared to uniform pricing can be positive or negative. Pigou (1920)

¹Department of Economics, University of Oxford, Manor Road Building, Manor Road, Oxford OX1 3UQ, UK, e-mail: simon.cowan@economics.ox.ac.uk.

proved that if demand functions are linear and all markets are served with positive quantities at the uniform price then a move to discriminatory prices keeps total output constant. Output is allocated to maximize profits, with marginal revenues being equalized across markets and set equal to marginal cost. Social welfare is lower with discrimination than with uniform pricing in Pigou's case since the welfare-maximizing way to allocate a given output across markets requires equal marginal utilities, and a uniform price ensures this. Price discrimination always induces a misallocation of total output. If, however, total output rises with discrimination, as may happen when demand functions are non-linear, then social welfare may also be higher. Additional output is socially valuable when prices exceed marginal cost. The challenge is to find families of demand functions for which social welfare is higher with discrimination, so that the social value of the increase in output offsets the social cost of the resulting misallocation, and broader families of demand functions for which total output is higher.

A natural way to model discrimination is to have inverse demands that are parallel. The price, p_i , in market i is related to quantity, q_i , by $p_i = a_i + f(q_i)$, where $f(\cdot)$ is decreasing and common to the markets. In this paper all inverse demands take this form. Markets differ in their values of the shift parameter a_i , with a higher a_i indicating a higher willingness to pay. This implies, in general, that there is a motive for discrimination since the price elasticities of demand differ across markets. In the move to discrimination the firm cuts the price in one market and raises the price in the other. This can be modelled as if the firm faced separate changes in marginal cost in each market. The cost pass-through coefficient - the effect of a cost change on the monopoly price - depends on the shape of $f(\cdot)$ and plays an important role. In particular it determines how much prices change when discrimination becomes feasible.

Two main results are presented. First, welfare is higher with discrimination when demand functions are derived from logistic distributions of customer valuations that differ only in their means, and when demand comes from a generalized class of functions. Second, discrimination raises total output when demands are derived from strictly log-concave distributions with increasing cost pass-through. Many distributions, such as the normal, logistic

and Type I Extreme Value, are in this class. These demand functions have standard features: marginal revenue is downward-sloping and an increase in marginal cost raises the monopoly price by less than the cost shift. The pass-through coefficient is higher in the market where the price falls than in the market where the price rises. When discrimination becomes feasible a consequence is that the price reduction in the former market is sufficiently large compared to the price increase in the other that total output rises. The output result, and the intuition behind it, extends to demand functions which are log-convex, so a market with a higher value of a_i has a lower monopoly price. An example is demand derived from the Pareto distribution. Output is higher with discrimination for a very large set of standard demand functions (though not for all).

In an influential article Varian (1985) derived upper and lower bounds for the effect on social welfare of a change in prices and applied them to the analysis of monopoly price discrimination - see Varian (2012) for a recent presentation. With logistic demand functions (and the more general version) the lower bound is positive. Varian's upper bound is proportional to the change in total output, which confirms that a necessary condition for discrimination to raise welfare is that total output increases. The general formula for the change in total output derived by Cheung and Wang (1994) is used to prove the output results. In all cases the properties of the demand functions and the consequences of profit-maximization fully determine the conclusions.

Other authors have shown that social welfare can be higher with discrimination. Varian (1985) proves, using his lower bound, that discrimination that causes a new market to be served will raise welfare if only one market is served at the uniform price. Kaftal and Pal (2008) analyze market opening when there are multiple markets. Ippolito (1980) considers constant-elasticity demand functions and shows numerically that the effect of discrimination on social welfare can be positive. Aguirre, Cowan and Vickers (2010) - hereafter ACV - show that discrimination can raise welfare for other log-convex demand functions with constant pass-through. All these positive results depend on the parameters, such as how close together the demand functions are and the level of marginal cost. More closely related to this paper is

Proposition 2 of ACV, which says that if the market with the lower price has a sufficiently large pass-through coefficient then social welfare is higher with discrimination. This depends on the somewhat restrictive assumption that profit functions are concave. The formula underlying ACV's Proposition 2, however, turns out to be useful in determining the sign of Varian's lower bound without needing to make any additional assumptions.

A large literature addresses the effect of discrimination on total output - see Aguirre (2009) and ACV (2010) for general analyses. Robinson (1933) showed that total output is higher with discrimination if demand is strictly convex in the market where the price is lower with discrimination and demand is concave in the other market. Shih, Mai and Liu (1988) show that if all demand functions are strictly convex and pass-through is constant then total output is higher with discrimination. This result is generalized by Cheung and Wang (1994), who allow the pass-through coefficient to vary along a demand curve. Aguirre (2006) proves that when all demand functions are in the constant elasticity class total output is higher with discrimination.

Section 2 presents Varian's bounds and the general monopoly pricing problem. The welfare results for logistic demand and its generalization are proved in Section 3. Section 4 shows that for two classes of demand functions total output is higher with discrimination. Conclusions are in Section 5.

2 The welfare bounds and monopoly pricing

2.1 Varian's welfare bounds

The version here is based on that in Varian (1989). There are two markets, 1 and 2. The assumption of two markets is for ease of exposition: all the results apply generally. The utility function of consumers in market i is quasi-linear and strictly concave in the quantity, q_i , supplied to that market. Thus demand, $q_i(p_i)$, is strictly decreasing in the price, p_i , when demand is positive. Demand is independent of the price in the other market.² Equiv-

²Varian (1985) allows for the more general case where demand can depend on the price in another market, and uses convexity of consumer surplus in prices to obtain the bounds.

alently demand can be derived from a distribution of customer valuations, with a customer buying one unit if and only if their valuation exceeds the price. Inverse demand is $p_i(q_i)$. The constant marginal cost of production is $c \geq 0$. Initially a uniform price, \bar{p} , is set for both markets. The discriminatory prices are p_1^* and p_2^* and it is assumed here that $p_1^* \neq p_2^*$, so there is some discrimination. Social welfare in market i , $w_i(q_i)$, is utility minus costs, or consumer surplus plus profits. Welfare is strictly concave in q_i because utility is strictly concave. The effect on welfare of extra output is marginal utility (which equals the price) less marginal cost, i.e. $w_i'(q_i) = p_i(q_i) - c$. The change in total welfare in the move from the uniform price \bar{p} to the discriminatory prices is denoted by ΔW . Strictly concave functions lie below their tangents, so ΔW is bounded above and below:

$$(\bar{p} - c) \sum_{i=1}^2 \Delta q_i > \Delta W > \sum_{i=1}^2 (p_i^* - c) \Delta q_i \quad (1)$$

where $\Delta q_i \equiv q_i(p_i^*) - q_i(\bar{p})$ is the change in the output.³ The upper bound implies that a necessary condition for welfare to be higher with discrimination is that total output rises. The lower bound, which is the focus of this paper, is the sum of the output changes multiplied by the discriminatory profit margins. The bounds in (1) are illustrated in Figure 1, adapted from that in Varian (1985), for a single market where the price falls from \bar{p} to p^* . The change in social welfare is the reduction in the deadweight triangle and is $A + B$. This is bounded by two rectangles: $D + A + B > A + B > B$.

2.2 Monopoly pricing

There is a profit-maximizing monopolist. When discrimination is feasible the firm chooses the discriminatory (or monopoly) output for each market. Marginal revenue, $MR_i(q_i) \equiv p_i + q_i p_i'(q_i)$, is assumed to be strictly decreasing in output. This holds quite generally and a sufficient condition is that the demand function is log-concave. Marginal revenue when output is zero

³The aggregate bounds have strict inequalities because quantity changes in at least one market by the assumption that there is some discrimination.

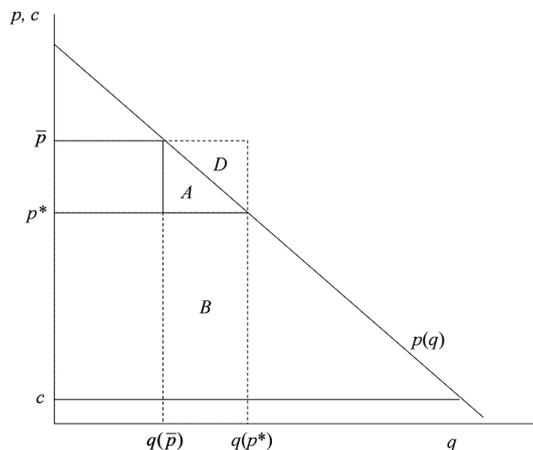


Figure 1: Varian's lower bound

is assumed to be above marginal cost (this holds for all the functions used subsequently). The unique interior solution for the monopoly output is defined by the textbook condition $MR_i(q_i^*) = c$. The resulting discriminatory price is $p_i^* = p_i(q_i^*)$. Equivalently the firm chooses the price in market i to maximize profits $\pi_i(p_i) \equiv (p_i - c)q_i(p_i)$, with the first-order condition being $\pi_i'(p_i^*) = 0$. Because q_i^* is unique and demand is downward-sloping, profit as a function of price is single-peaked.

The cost pass-through coefficient is the rate at which the monopoly price rises as marginal cost increases. It equals the ratio of the slope of inverse demand to the slope of marginal revenue, $\frac{dp_i^*}{dc} = \frac{p_i'(q_i^*)}{MR_i'(q_i^*)}$, and is positive (see Bulow and Pfleiderer, 1983).⁴ Weyl and Fabinger (2013) and Fabinger and Weyl (2012) discuss pass-through comprehensively. There is a one-to-one relationship between demand curvature and pass-through. Defining $\sigma_i(q_i) = -\frac{q_i p_i''(q_i)}{p_i'(q_i)}$ as the curvature of inverse demand (or the elasticity of its slope), pass-through is $\frac{dp_i^*}{dc} = \frac{1}{2 - \sigma_i(q_i^*)}$. Linear demand has zero curvature and thus pass-through equal to 0.5. Demand is log-concave when $\ln(q_i)$ is concave in

⁴From the first-order condition $\frac{dq_i^*}{dc} = \frac{1}{MR_i'(q_i^*)}$. Multiplying by $p_i'(q_i^*)$ yields the pass-through coefficient.

the price, and this implies that $\sigma_i \leq 1$ (with strict inequality when demand is strictly log-concave) and so pass-through is at most 1. Log-convex demand, such as constant-elasticity demand, has pass-through that exceeds 1.

Suppose now that inverse demand is $p_i = a_i + f(q_i)$. Weyl and Fabinger (2013) show that the effect of an increase in a_i on the monopoly price is 1 minus the cost pass-through coefficient. To see why, note that the monopoly quantity depends on $c - a_i$ since higher a_i shifts marginal revenue up and thus raises the quantity. Differentiating $p_i^* = a_i + f(q_i^*(c - a_i))$ yields $\frac{\partial p_i^*}{\partial a_i} = 1 - \frac{\partial p_i^*}{\partial c}$. Thus for strictly log-concave demands $\frac{\partial p_i^*}{\partial a_i}$ is below 1: both cost and demand shifts are passed through partially. A higher a_i is associated with a higher monopoly price and a reduced value of $p_i^* - a_i$. For log-convex demands, however, a rise in a_i causes the monopoly price to fall.

When the firm cannot discriminate it chooses the uniform price, p , that maximizes $\pi_1(p) + \pi_2(p)$. The profit-maximizing uniform price, \bar{p} , is characterized by the first-order condition: $\pi_1'(\bar{p}) + \pi_2'(\bar{p}) = 0$. The sum of the marginal profitabilities is zero. Theorem 1 of Nahata, Ostaszewski and Sahoo (1990) implies that the uniform price is bounded above and below by the two discriminatory prices when each market's profit function is single-peaked. This is intuitive: if the uniform price was, say, above both discriminatory prices then a reduction in the uniform price would increase profits in each market, implying that the initial uniform price was not profit-maximizing. From now on, following the language of Robinson (1933), the market with the lower discriminatory price is called the weak market and the subscript w is used, and that with the higher price is the strong market (using subscript s). Thus $p_s^* \geq \bar{p} \geq p_w^*$. The focus is on the case where both markets are served with positive quantities at the uniform price, with $p_s^* > \bar{p} > p_w^*$. The weak and strong markets can be identified at the uniform price by comparing the local values of the price elasticities of demand: the weak market has the higher price elasticity.

When the firm starts to discriminate it is as if it faces separate shifts in marginal cost in each market (see Cowan, 2012). Define the quantity bought at the uniform price as $\bar{q}_i \equiv q_i(\bar{p})$, with $MR_i(\bar{q}_i)$ being marginal revenue at this output. If marginal cost were equal to $MR_i(\bar{q}_i)$ then \bar{p} would be

the monopoly price in market i . Let $P_i(k)$ be the monopoly price when marginal cost equals k , which can be thought of as virtual marginal cost. The discriminatory price is then $p_i^* = P_i(c)$ and the uniform price is $\bar{p} = P_i(MR_i(\bar{q}_i))$. Pass-through is $P_i'(k)$, with pass-through at the discriminatory price being $P_i'(c)$.

3 Logistic demand and welfare

The logistic demand function is written generally as $N_i(1 + e^{(p_i - a_i)/b})^{-1}$. This can be interpreted as the number of potential customers, N_i , multiplied by the fraction of customers who purchase when the price is p_i . The fraction is 1 minus the cumulative distribution function of the logistic evaluated at the price.⁵ In what follows the number of potential customers in each market is normalized to 1 without loss of generality. Thus $q_i(p_i) = (1 + e^{(p_i - a_i)/b})^{-1}$ is the demand function that will be used. Because $q_i(p_i) > 0$ both markets are always served. Note also that $q_i(p_i) < 1$. The parameter a_i is the mean of the distribution and is the only source of difference across markets. The positive scale parameter b is proportional to the standard deviation of the logistic distribution and is common to the two markets. Inverse demand is $p_i(q_i) = a_i - b \ln\left(\frac{q_i}{1 - q_i}\right)$, which is the mean less the scale parameter times the log-odds ratio. An increase in the mean shifts inverse demand vertically. Inverse demand is S-shaped, with demand being convex for prices above a_i and concave for lower prices. The logistic distribution resembles the normal distribution, but is more peaked and has slightly more weight in its tails. Figure 2 illustrates two inverse demand functions with $a_s = 5$, $a_w = 3$ and $b = 1$. When $q = 0.5$ $p_i = a_i$ in each case.

The slope of inverse demand is $p_i'(q_i) = -b/[q_i(1 - q_i)]$. Marginal revenue is $MR_i = p_i - b/(1 - q_i)$, so the discriminatory price is $p_i^* = c + b/(1 - q_i^*)$. This implies that the market with the higher price is also the one where the monopolist sells a larger quantity (which, from the demand function, is the

⁵ Alternatively this demand function can come from N_i individuals in market i choosing q_i and x_i (the numeraire good) to maximize direct utility $u_i(q_i) = a_i q_i - b[q_i \ln(q_i) + (1 - q_i) \ln(1 - q_i)] + x_i$ subject to a standard budget constraint.

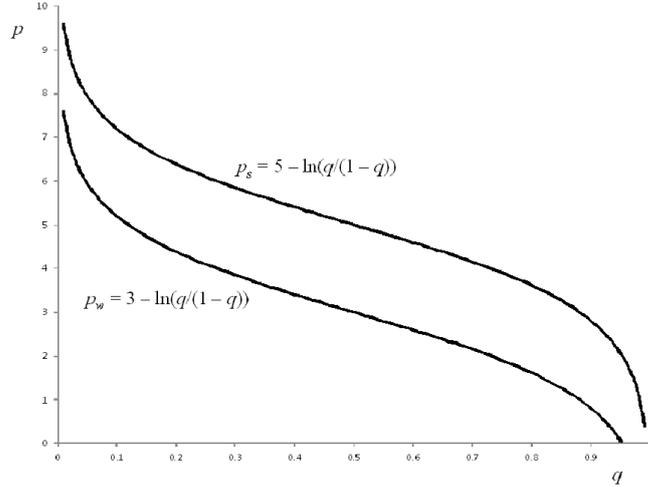


Figure 2: Inverse demand functions for the logistic

market with the lower difference $p_i - a_i$ and thus the higher value of a_i). The slope of marginal revenue is $MR'_i(q_i) = -b/[q_i(1 - q_i)^2]$. Dividing $p'_i(q_i^*)$ by $MR'_i(q_i^*)$ gives the cost pass-through coefficient: $P'_i(c) = 1 - q_i^*$. Pass-through is higher in the market with smaller demand, i.e. the weak market, and is strictly increasing in marginal cost since the monopoly output falls as c increases. The monopoly margin multiplied by the pass-through coefficient, $(p_i^* - c)P'_i(c)$, will be called the pass-through-adjusted margin. For logistic demand this equals b and is thus common to the two markets.

The objective is to show that Varian's lower bound in (1) is always positive. Marginal profitability at the uniform price is $\pi'_i(\bar{p}) = \bar{q}_i - (\bar{p} - c)\bar{q}_i(1 - \bar{q}_i)/b$. Multiply this by the pass-through-adjusted margin, b , and add the resulting expression to Varian's lower bound for market i :

$$(p_i^* - c)\Delta q_i + b\pi'_i(\bar{p}) = \pi_i^* - (p_i^* - c)\bar{q}_i + b\bar{q}_i - \pi_i(\bar{p})(1 - \bar{q}_i) = (1 - \bar{q}_i)\Delta\pi_i. \quad (2)$$

The first equality follows from the definition of Δq_i and the marginal profitability expression. The second equality holds because profit at the discriminatory price can be written as $\pi_i^* = p_i^* - c - b$ and, by definition, the change

in profit is $\Delta\pi_i \equiv \pi_i^* - \pi_i(\bar{p})$. So far profit-maximization with discrimination has been used. To complete the argument equation (2) is rearranged, the resulting expression is added across the markets, and profit-maximization at the uniform price is applied.

Proposition 1. *Price discrimination yields higher social welfare than uniform pricing when demands are derived from logistic distributions that differ only in their means, so inverse demand is $p_i(q_i) = a_i - b \ln\left(\frac{q_i}{1-q_i}\right)$.*

Proof. From (2) $(p_i^* - c)\Delta q_i = (1 - \bar{q}_i)\Delta\pi_i - b\pi'_i(\bar{p})$. Adding this across the markets gives

$$\sum_{i=1}^2 (p_i^* - c)\Delta q_i = \sum_{i=1}^2 [(1 - \bar{q}_i)\Delta\pi_i - b\pi'_i(\bar{p})] = \sum_{i=1}^2 (1 - \bar{q}_i)\Delta\pi_i > 0.$$

The second equality holds because $\sum_i \pi'_i(\bar{p}) = 0$. The strict inequality holds

because both $1 - \bar{q}_i$ and $\Delta\pi_i$ are positive for each market. Thus Varian's lower bound is always positive. It follows from (1) that welfare is higher with discrimination. \square

This seems to be the first demonstration that there is a class of demand functions for which Varian's lower bound is necessarily positive. Since $\bar{q}_s > q_s^* > q_w^* > \bar{q}_w$ the pass-through rate, $1 - q$, in the weak market is always higher than that in the strong market. Intuitively this means that the price reduction in the weak market is large and the price increase in the strong market is small. Discrimination then yields an increase in welfare in the weak market that offsets the fall in welfare in the strong market. The social value of the increase in total output always offsets the misallocation effect. Proposition 1 parallels the total output result of Pigou (1920) for linear demand functions. Linear demand arises naturally from a uniform distribution of customer valuations. Note that the uniform distribution has excess kurtosis of -1.2 , while that of the logistic is 1.2 .

How robust is Proposition 1? First, it does not depend on the absolute

or relative sizes of the markets. Suppose that in one market the number of consumers, N_i , is no longer unity. This will alter the optimal uniform price and the proportions who buy in each market at that price. The size of the lower bound will change, but not its sign.⁶ Second, the shift parameters a_w and a_s can take any distinct values. In contrast Pigou's result for linear demand functions only applies if the demand functions are close to each other so that all markets are served at the uniform price, and similarly the welfare analysis of Aguirre, Cowan and Vickers (2010) depends on demand functions being close. In summary the proposition is robust to multiplicative shifts of direct demand and additive shifts of inverse demand. Third, the number of markets does not matter - Proposition 1 works just as well for more than two markets. Nothing in the proof depended on having two markets. Fourth, it is important that the scale parameter, b , is identical across markets. If the scale parameters differ then the lower bound may have either sign. It is positive if the market with the higher value of b is also the weak one. Otherwise the lower bound will be negative, and the upper bound can also be negative.

In Cowan (2012) I analyze the effect of price discrimination on consumer surplus. With logistic demand functions discrimination raises consumer surplus if demand is always convex and reduces it when demand is always concave. Proposition 1 shows, however, that the welfare effect is always positive, so even when consumer surplus falls the increase in profits is enough to offset that.

Varian's lower bound is positive for a more general class of demand functions. Exponential direct demand is $q_i(p_i) = e^{(a_i - p_i)/b}$, with inverse demand $p_i(q_i) = a_i - b \ln(q_i)$ and a common scale parameter b across the markets. For this the monopoly price, $p_i^* = c + b$, is independent of a_i so when discrimination is feasible the firm chooses not to set different prices. An additive shift in inverse demand is the same as a multiplicative shift of direct demand. Since there is no discrimination the welfare effect is zero. The general class of demand functions for which the lower bound is positive can now be constructed.

⁶With $\Pi_i \equiv (p_i - c)N_i q_i$ denoting profits, the general form of the lower bound is $\sum_i (1 - \bar{q}_i) \Delta \Pi_i > 0$, where \bar{q}_i is the fraction of the potential customers in market i who purchase at the uniform price.

This is when $p_i(q_i)$ is a weighted average of the inverse demands associated with the logistic and exponential demand functions (with the weight on the latter being strictly below 1).

Proposition 2. *Price discrimination yields higher social welfare than uniform pricing when inverse demand is a weighted average of those for the logistic and exponential demands, $p_i(q_i) = a_i - b \ln(q_i) + (1 - w)b \ln(1 - q_i)$, with a_i varying across markets and $w \in [0, 1)$.*

Proof. See Appendix One.

The inverse demands of Proposition 2 have standard features. Demand is decreasing in the price and marginal revenue is decreasing in output. Pass-through is below 1 and increases with marginal cost. Exponential demand is the limiting case where $w = 1$ (which is not included in the proposition), and logistic demand is the special case where $w = 0$. The underlying distributions characterized in Proposition 2 have tails that are fatter than those of a normal but thinner than those of an exponential.⁷ While Proposition 2 encompasses Proposition 1 it is perhaps more natural to emphasize the latter. Nevertheless it is notable that there is a well-defined class of demand functions whose shape alone implies that social welfare is higher with discrimination than without.

4 The effect of discrimination on total output

Total output is higher with discrimination for a large set of demand functions derived from commonly used distributions. These demands satisfy the necessary condition for social welfare to be higher with discrimination and imply that Varian's upper bound is positive. For the first result assume that (i) the slope of demand is strictly log-concave ($\ln(-q')$ is strictly concave in

⁷The logistic and exponential have excess kurtosis of 1.2 and 6 respectively, while, by definition, that of the normal is 0. Gabaix et al. (2013) define another measure of the fatness of tails as (the limit of) $\sigma - 1$. For the demand functions of Proposition 2 σ rises with w at each value of q . Exponential demand has $\sigma = 1$.

p) and (ii) pass-through is (weakly) increasing in marginal cost. Property (i) is equivalent to the underlying probability density function being strictly log-concave. It follows that demand is strictly log-concave, so $\sigma < 1$ and thus $P'(c) < 1$.⁸ Demand that comes from the normal, logistic, Type I Extreme Value, Weibull and Gamma distributions (amongst others) satisfies both properties. Bagnoli and Bergstrom (2005) show that the associated density functions are strictly log-concave, and Fabinger and Weyl (2012) prove that pass-through is increasing for these demands.⁹ Demand that is strictly convex and has constant pass-through below 1 also satisfies the properties, as does the demand function characterized in Proposition 2. Both properties are satisfied by inverse demand that is a strictly convex combination of the inverses of linear and exponential demands.

A feature of demand functions in the class defined by (i) and (ii) is that pass-through is higher in the weak market: $P'_w(c) \geq P'_s(c)$. Section 2.2 showed that for log-concave demand the market with the higher value of a_i has a higher monopoly price, p_i^* , a lower value of $p_i^* - a_i$, and a higher monopoly quantity, q_i^* . This is, therefore, the strong market. As prices move from the uniform price to the discriminatory prices the quantity is always strictly larger in the strong market. By property (ii) cost pass-through is weakly decreasing along a given inverse demand curve as the quantity rises and thus is higher in the weak market. As with the welfare results it is higher pass-through in the weak market that provides the intuition for the fact that total output is higher with discrimination.

Cheung and Wang (1994) use the first-order conditions to derive the exact formula for the change in total output, defined here for two markets as $\Delta Q \equiv \Delta q_s + \Delta q_w$. This can be written as:

$$\Delta Q = \frac{1}{2} \sum_{i \in \{s, w\}} \int_{q_i^*}^{\bar{q}_i} (p_i(q_i) - c) \frac{q_i''}{q_i'} dq_i.$$

⁸Bagnoli and Bergstrom (2005, Theorem 3) show that 1 minus the cumulative distribution function, which here is the demand function, inherits log-concavity from a log-concave density function.

⁹The Weibull and Gamma densities are strictly log-concave when their key parameters exceed 1. In general a sufficient condition for pass-through to be increasing when the demand slope is log-concave is that demand itself is concave.

A derivation is presented in Appendix Two. The connection with property (i) can now be seen. Strict log-concavity implies that $\frac{q''}{q'} = \frac{d}{dp} \ln(-q')$ is decreasing in p and is thus increasing in quantity. With parallel inverse demands $p_i - a_i = f(q_i)$, so q_i and its first and second derivatives are functions of $p_i - a_i$.¹⁰ Define $h(q_i) \equiv \frac{q''}{q'}(f(q_i))$, with $h'(q_i) > 0$. In the logistic case $h(q_i) = (2q_i - 1)/b$. The functional form of $h(\cdot)$ does not depend on the market because $f(\cdot)$ is common to the two markets, as are $q''(\cdot)$ and $q'(\cdot)$. Properties (i) and (ii) together imply that $\Delta Q > 0$.

Proposition 3. *Total output is strictly larger with discrimination if inverse demand is $p_i(q_i) = a_i + f(q_i)$ with a_i varying across the two markets, demand has a strictly log-concave slope, and pass-through is weakly increasing.*

Proof. See Appendix Three.

Holmes (1989), building on Schmalensee (1981), analyzes the output effect using an intuitive, though slightly less general, method than that used for Proposition 3.¹¹ Suppose that the firm chooses its prices to maximize profits subject to $p_s \leq p_w + r$ where r denotes the allowed price difference and thus represents the amount of discrimination. Holmes (1989) shows that as r increases total output rises if $(p_s - c) \frac{q''_s}{q'_s} > (p_w - c) \frac{q''_w}{q'_w}$. These terms are the integrands in the general expression above for the change in output. When inverse demands are parallel and concave it is simple to check that the inequality holds: each term $\frac{q''_i}{q'_i} = h(q_i)$ is positive, and that in the strong market exceeds that in the weak market because the quantity is larger. Since $p_s - c \geq p_w - c$ each marginal increase in the amount of discrimination raises total output. Proposition 3 is a generalization of this.

Proposition 3 does not depend on the size of demand: if demand is Nq

¹⁰A simple example is the normal distribution, for which $q''/q' = (a-p)/s^2$ where $s^2 > 0$ is the variance.

¹¹The Holmes technique, also used by ACV (2010), depends on each profit function being concave in the price. If the demand functions are convex this requires that $a_s - a_w$ is not too large.

then the multiplicative term N cancels when dividing the second derivative by the first. Similarly how far apart the demand functions are, measured by $a_s - a_w$, does not matter. Just outside the conditions of the proposition are the linear and exponential demand functions, for which total output is unchanged. In the linear case the given output is reallocated amongst the markets, while in the exponential case discrimination is not profitable so nothing changes.

The output result extends to demand functions that are log-convex. Cost pass-through exceeds 1, so an upward shift in inverse demand induces a reduction in the monopoly price. Now the weak market is the one with the higher value of a_i and the higher quantities. If pass-through is weakly decreasing in marginal cost (and thus price) it is increasing in quantity, so with this assumption again the weak market will be the one with higher pass-through.

Proposition 4. *Total output is higher with discrimination if inverse demand is $p_i(q_i) = a_i + f(q_i)$ with a_i varying across markets, demands are strictly log-convex, and pass-through is weakly decreasing in marginal cost.*

Proof. Strict log-convexity implies that demand is strictly convex and pass-through is above 1. Thus $\frac{\partial p_i^*}{\partial a_i} = 1 - P_i'(c) < 0$, so the weak market is the one with the higher value of a_i and the higher quantity for all prices $p_w \in [p_w^*, \bar{p}]$ and $p_s \in [\bar{p}, p_s^*]$. Since pass-through is increasing (weakly) in quantity pass-through is (weakly) higher in the weak market. Proposition 2 of Cheung and Wang (1994) states that if demand is strictly convex, and pass-through is at least as large in the weak market, then total output is higher with discrimination. Both conditions apply here. \square

Proposition 4 applies to demand derived from the Pareto distribution (for which pass-through is constant), and for demand derived from the Weibull distribution (when log-convex).¹² Both the output propositions have the

¹²Fabinger and Weyl (2012) caution that when demand comes from the Weibull distribution and is log-convex marginal revenue is increasing at very low prices.

same feature: pass-through is at least as high in the weak market as in the strong one.

Remark. Suppose that the scale parameters differ across markets, so inverse demand is now $p_i = a_i + b_i f(q_i)$ (with a slight change of notation). If demand functions are strictly convex, then both the output propositions continue to apply provided that pass-through is at least as large in the weak market. This is another application of Cheung and Wang's Proposition 2. If pass-through is constant then this holds automatically. If pass-through is not constant then the extension to Proposition 3 holds if $q_s^* \geq q_w^*$, and similarly the generalized version of Proposition 4 applies if $\bar{q}_w \geq \bar{q}_s$.

Propositions 3 and 4 (and their extensions) together imply that the set of demand functions for which total output is higher with discrimination is large. This is not, however, the only possibility when inverse demands are parallel. Output remains constant if demands are linear and are close enough together that both markets are served at the uniform price (Pigou, 1920). Output can also fall. Shih, Mai and Liu (1988) prove that when pass-through is constant and demand functions are strictly concave total output is lower with discrimination.¹³ Again this holds provided both markets are served at the uniform price - if the markets are far enough apart then discrimination opens the weak market and has no effect on the strong market, so total output increases.

5 Conclusion

A positive welfare effect from third-degree price discrimination is possible in standard circumstances when inverse demands are parallel, and is guaranteed for a subset of demand functions. The theme of the paper is that the conditions for welfare to increase are plausible and in particular the necessary condition is commonly met if inverse demands are parallel. Of course

¹³In this case the slope of demand is log-convex.

inverse demands need not be parallel. The scale parameters might differ across markets, in which case the effects of discrimination on output and welfare depend on the parameters. Alternatively direct demands rather than inverse demands might be parallel, in which case a sufficient condition for discrimination to reduce welfare is that the slope of demand is log-concave (Cowan, 2007).

Two recent studies that estimate structural models of discrimination have demand functions related to those used here. Hendel and Nevo (2011) consider intertemporal price discrimination when the product is storable, and apply this to retail pricing of soft drinks by different stores when demand functions take the exponential form. Lazarev (2013) models airlines with monopoly routes setting prices that vary over time and facing logistic demand functions. In both papers discrimination is indirect because customers can choose when to purchase.

The statistical distributions used here in the analysis of monopoly pricing are more commonly used as the bases of demand systems for oligopoly. The natural generalization of the logistic demand function to an oligopoly has a multinomial logit demand structure with an outside option whose utility varies across markets. The proofs of the welfare results depended on the fact that monopoly discrimination raises profits in each market. With oligopoly, however, there is no guarantee that profits will rise in each market, because prices are the result of equilibrium behaviour. Thus alternative approaches will be needed to assess the welfare effect. The output propositions, however, did not depend on profits being higher with discrimination.

Appendix One: Proof of Proposition 2.

The proof is similar to that of Proposition 1. Inverse demand is $p_i(q_i) = a_i - b \ln(q_i) + b(1-w) \ln(1-q_i)$ for $q_i \in (0, 1)$ and $w \in [0, 1)$. The logistic is the special case with $w = 0$. The inverse demand for the exponential is the limiting case where $w = 1$. Subscripts are used only when necessary. The slope of inverse demand is $p'(q) = -b(1-wq)/q(1-q)$, and the slope of marginal revenue is $MR'(q) = -b(1-2wq+wq^2)/q(1-q)^2$. The discriminatory margin is $p^* - c = \frac{b(1-wq^*)}{1-q^*}$ and cost pass-through is $P'(c) = \frac{p'(q^*)}{MR'(q^*)} = \frac{(1-q^*)(1-wq^*)}{1-2wq^*+w(q^*)^2} < 1$. Because $P'(c) < 1$ the monopoly price increases as a rises, but by less than the increase in a . The pass-through-adjusted margin is $(p^* - c)P'(c) = b \frac{(1-wq^*)^2}{1-2wq^*+w(q^*)^2}$. Since the monopoly price increases with a , $\frac{d}{da} [(p^* - c)P'(c)]$ has the same sign as $w(w-1)(1-wq) \leq 0$. This equals 0 when $w = 0$ (or $w = 1$), and otherwise is strictly negative. Thus the pass-through-adjusted margin falls (or stays constant for $w = 0$), as a increases. The weighted sum is therefore non-negative: $\sum_{i=1}^2 (p_i^* - c)P'(c)[- \pi_i'(\bar{p})] \geq 0$. Adding $(p_i^* - c)P'(c)\pi_i'(\bar{p})$ to Varian's lower bound for one market gives:

$$\begin{aligned} & (p_i^* - c)\Delta q_i + (p_i^* - c)P'(c)\pi_i'(\bar{p}) \\ = & \pi_i^* \left(1 - \frac{(1-w)\bar{q}_i}{1-2wq_i^*+w(q_i^*)^2} \right) - \pi_i(\bar{p}) \left(\frac{(1-\bar{q}_i)}{(1-w\bar{q}_i)} \frac{(1-wq_i^*)^2}{(1-2wq_i^*+w(q_i^*)^2)} \right) \end{aligned}$$

which is of the form $A_i\pi_i^* - B_i\pi_i(\bar{p})$ where both A_i and B_i are positive. Since $\pi_i^* > \pi_i(\bar{p})$ this expression is positive if $A_i - B_i \geq 0$. The sign of $A_i - B_i$ is the same as $w(1-w)(q_i^* - \bar{q}_i)^2 \geq 0$. Varian's lower bound is, therefore, positive:

$$\sum_{i=1}^2 (p_i^* - c)\Delta q_i = \sum_{i=1}^2 [A_i\pi_i^* - B_i\pi_i(\bar{p})] + \sum_{i=1}^2 (p_i^* - c)P'(c)[- \pi_i'(\bar{p})] > 0. \quad \square$$

Appendix Two: Derivation of the formula for the change in total output.

Cheung and Wang (1994) use the first-order conditions with quantities as the choice variables. The version here is equivalent but uses the first-order conditions for prices. Profit-maximization with discrimination implies that $q_i(p_i^*) + (p_i^* - c)q_i'(p_i^*) = 0$ in each market, so rearranging and adding across n markets gives total output as $Q^* \equiv \sum_{i=1}^n q_i^* = -\sum_{i=1}^n (p_i^* - c)q_i'(p_i^*)$. Similarly the first-order condition for the uniform price implies that $\bar{Q} \equiv \sum_{i=1}^n \bar{q}_i = -\sum_{i=1}^n (\bar{p} - c)q_i'(\bar{p})$. The change in output is

$$\begin{aligned}
 \Delta Q &= \sum_{i=1}^n [(\bar{p} - c)q_i'(\bar{p}) - (p_i^* - c)q_i'(p_i^*)] \\
 &= \sum_{i=1}^n \int_{p_i^*}^{\bar{p}} [q_i'(p_i) + (p_i - c)q_i''(p_i)] dp_i \\
 &= -\Delta Q + \sum_{i=1}^n \int_{p_i^*}^{\bar{p}} (p_i - c)q_i''(p_i) dp_i \\
 &= \frac{1}{2} \sum_{i=1}^n \int_{p_i^*}^{\bar{p}} (p_i - c)q_i''(p_i) dp_i \\
 &= \frac{1}{2} \sum_{i=1}^n \int_{q_i^*}^{\bar{q}_i} (p_i(q_i) - c) \frac{q_i''(p_i(q_i))}{q_i'(p_i(q_i))} dq_i.
 \end{aligned}$$

The second equation is by the Fundamental Theorem of Calculus. The third and fourth use the fact that $\int_{p_i^*}^{\bar{p}} q_i'(p_i) dp_i = -\Delta q_i$. In the fifth line the variable of integration is changed from p_i to q_i .

Appendix Three: Proof of Proposition 3.

With parallel inverse demands and strict log-concavity $\frac{q''}{q}$ is a decreasing function of $p - a$. Since $p - a = f(q)$, $\frac{q''}{q}$ is increasing in q . Define $h(q) \equiv \frac{q''}{q}(f(q))$, so $h'(q) > 0$. The functional form of $h(\cdot)$ does not depend on the market. Write the change in total output as

$$\Delta Q = \frac{1}{2} \left(\int_{q_s^*}^{\bar{q}_s} (p_s(q_s) - c)h(q_s)dq_s + \int_{q_w^*}^{\bar{q}_w} (p_w(q_w) - c)h(q_w)dq_w \right).$$

Call the lowest value of the first integrand α . It follows that $\int_{q_s^*}^{\bar{q}_s} (p_s - c)h(q_s)dq_s \geq \alpha(\bar{q}_s - q_s^*) = -\alpha\Delta q_s$, because an average value is at least equal to the minimum value. Similarly, let the highest value of the second integrand be β . Since $\bar{q}_w < q_w^*$, $\int_{q_w^*}^{\bar{q}_w} (p_w - c)h(q_w)dq_w \geq -\beta\Delta q_w$. In each case the inequality is strict if the integrand is not constant. Thus $\Delta Q \geq \frac{1}{2}(-\alpha\Delta q_s - \beta\Delta q_w) = \frac{1}{2}(-\alpha\Delta Q + (\alpha - \beta)\Delta q_w)$, so $\Delta Q \geq \frac{(\alpha - \beta)\Delta q_w}{2 + \alpha}$ provided that $2 + \alpha > 0$. There are two cases. First, let $\sigma_s^* \leq 0$ so $h(q_s^*) \geq 0$. As quantity rises from q_s^* to \bar{q}_s demand remains strictly concave (because a strictly log-concave density has at most one peak). The first integrand is always positive so $\alpha \geq 0$ and $2 + \alpha > 0$. For the relevant quantities, $(p_s - c)h(q_s) > (p_w - c)h(q_w)$ as $h(q_s) > h(q_w)$, $h(q_s) \geq 0$ and $p_s \geq p_w$. Thus $\alpha - \beta > 0$ and the lower bound on the change in output is strictly positive. Second, let $\sigma_s^* > 0$. The first-order condition for q_s^* and the fact that $\sigma = qq''/(q')^2$ imply that $(p_s(q_s^*) - c)h(q_s^*) = -\sigma_s^*$. With strictly convex demand $h(q) < 0$ so $\frac{d}{dq_s}(p_s - c)h(q_s) = (p_s - c)h'(q_s) + hp'_s(q_s) > 0$. If demand remains convex as q_s rises then $(p_s - c)h(q_s)$ remains negative and moves closer to zero. If demand becomes concave as the quantity increases then $(p_s - c)h(q_s)$ is a positive number in this region. Either way, the lowest value of the first integrand is $-\sigma_s^*$. Demand in the weak market is always strictly convex over the relevant range of quantities if $\sigma_s^* > 0$, so the maximum value of the second integrand is $-\sigma_w^*$. Thus $\alpha - \beta = \sigma_w^* - \sigma_s^* \geq 0$ (by weakly increasing pass-through). By log-concavity of demand $\sigma_s^* < 1$ so $2 + \alpha = 2 - \sigma_s^* > 0$. Neither integrand is constant, so the change in total output is strictly greater than a non-negative number. \square

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