Econometric Modelling of Climate Systems: The Equivalence of Energy Balance Models and Cointegrated Vector Autoregressions

Felix Pretis$^{1,2,*}$

$^1$Department of Economics & Nuffield College, University of Oxford
$^2$Programme for Economic Modelling, INET at the Oxford Martin School

Abstract

Estimates of both the human impact on climate as well as the economic impacts of climate change are crucial to inform policy decisions. Econometric modelling allows us to quantify these impacts and their uncertainties, but models have to be consistent with the underlying physics and the time series properties of the data. Here, I show that energy-balance models of climate are equivalent to an econometric cointegrated system and can be estimated in discrete time. This equivalence provides a basis for the use of cointegration methods to estimate climate responses and test their feedback. Further, it is possible to use the estimated parameters to quantify uncertainties in integrated assessment models of the economic impacts of climate change. In an application I estimate a system of temperatures, ocean heat content, and radiative forcing including greenhouse gases, and find statistical support for the cointegrated energy balance model. Accounting for structural breaks from volcanic eruptions highlights large parameter uncertainties and shows that previous empirical estimates of the temperature response to increased CO$_2$ concentrations may be misleadingly low due to model-misspecification.

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*Author contact: felix.pretis@nuffield.ox.ac.uk. This version: 9. November 2017
1 Introduction

Climate change is one of the greatest economic challenges facing humanity with potentially large costs from either action or inaction to tackle human-induced (anthropogenic) emissions. Informed policy decisions on climate change rely on both estimates of the magnitudes and uncertainties of the climate response to anthropogenic activity, as well as the climate-driven economic impacts. To quantify the uncertainties in responses and link econometric models with empirical climate models requires modelling approaches that are consistent with the underlying physics and the time series properties of the data. In this paper, I show that a two-component energy-balance model (EBM) of global mean climate is equivalent to an econometric cointegrated system, providing a physical basis for econometric methods that estimate climate responses and test feedback. Formally, I demonstrate the mathematical equivalence of a two-component energy balance model to a cointegrated system with parameter restrictions that can be mapped to a cointegrated vector autoregression (CVAR) and estimated in discrete time.

The motivation for a link between cointegrated econometric systems and energy balance models is two-fold. First, existing projections of economic impacts of climate change have not considered uncertainties around important parameters. The economic impacts of climate change derived from integrated assessment models (IAMs) such as FUND (Waldhoff et al. 2011), PAGE (Hope 2011, Hope et al. 1993), and DICE (Nordhaus 2014) rely on simple energy balance models to model climate internally. However, uncertainties about the values of physical parameters in IAMs beyond the equilibrium climate sensitivity (ECS – the temperature response to a doubling in CO$_2$ concentrations) are conventionally not taken into account.¹ These uncertainties can have large impacts on the projected economic impacts of climate change, because the physical differences in climate models within IAMs already imply a range of economic damages from 5%-30% of future output (Calel & Stainforth, 2016). In particular, parameters are often assumed to be constant over time, and covariances across parameters (such as the ECS and the amount of energy it takes to increase temperatures) are often imposed without a statistical or physical basis. In contrast, a system approach to estimating EBMs based on CVARs can provide empirical parameter estimates and quantify their uncertainties which can subsequently guide economic impact projections.

Second, current climate-econometric research has generally not considered bi-directional feedback between the climate and the economy. Econometric studies beyond IAMs are split into two strands: one side empirically models the impact of climate on the economy, taking climate variation as given (e.g. Burke et al. 2015, Carleton & Hsiang 2016, Dell et al. 2012, Dell et al. 2014), the other side models the impact of anthropogenic (e.g., economic) activity onto the climate by taking radiative forcing – the incoming energy from emitted radiatively active gases such as CO$_2$ – as given (e.g. Estrada et al. 2013, Hendry & Pretis 2013, Kaufmann et al. 2011, Kaufmann et al. 2013, Magnus et al. 2011, Storelvmo et al. 2016). This split in the literature is a concern as each strand considers conditional models, while feedback between the economy and climate likely run in both directions. By definition, both of these approaches cannot be correct at the same time. Most likely, neither is entirely correct; economic and environmental systems are typically determined with interdependencies in both directions. To test the presence and estimate the magnitude of these dependencies requires a framework that allows economic systems – well approximated by VARs (or panel VARs) – to be joined with empirical climate models consistent with the underlying physics.² In this paper, I take a first step in this direction and show that a two-component physics-based EBM can be mapped to a

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¹Nordhaus (2016) considers parameter uncertainties in DICE on the ECS and the carbon cycle, but not on the heat capacity of the atmosphere or ocean.

²Pretis (2017) discusses the validity of such conditioning and the split in climate econometrics, highlighting the importance of modelling the economic and climate sides jointly as a system. This allows us to test whether the magnitudes of feedback and adjustments are small enough to warrant a conditional analysis in either direction.
CVAR allowing adjustments and feedback to be estimated.

Estimation and inference on EBM model parameters in a system is not straightforward because climate observations are predominantly non-stationary time series, often with small numbers of observations (Estrada & Perron 2014, Kaufmann et al. 2013, Kaufmann & Stern 2002, Pretis & Hendry 2013, D. I. Stern & Kaufmann 2000). Econometric cointegration analysis can be used to overcome the inferential difficulties resulting from stochastic trends when that is the only source of non-stationarity, and is applied to test whether there exist combinations of non-stationary variables that are themselves stationary (see Hendry and Juselius 2001 for an overview, Schmith et al. 2012 for an application to sea level observations, and Juselius 2011, and Kaufmann & Juselius 2013 for a paleo-climate analysis). Inference in small samples can be improved through recent developments in bootstrap tests in cointegrated models (Cavaliere et al. 2012, Cavaliere et al. 2015). In the present paper I establish the mathematical equivalence between CVARs and EBMs. Doing so provides a physical science basis for system cointegration, allowing EBMs to be estimated as a system and used to quantify parameter uncertainties within integrated assessment models. Further, the equivalence places the entire CVAR toolkit at the disposal of climate-econometric modelling. This allows researchers to investigate the empirical impacts of shifts from both natural and human sources.

In an application, I estimate a two-component EBM through the use of a CVAR and test it against the observational record. I find that tests against the statistical properties implied by the EBM do not reject the cointegrated model. Specifically, results from both standard cointegration tests as well as tests based on bootstrapping show that time series capturing global mean climate (surface temperatures, ocean heat content, and radiative forcing) form stationary relations and cointegrate in ways consistent with the underlying theory. Using indicator saturation methods (Hendry et al. 2008, Castle et al. 2015) I detect breaks and outliers in the estimated models which highlights the importance of accurately modelling sudden changes in forcing – ranging from volcanic to policy origins. Particularly, my results show that previous empirical estimates of the temperature response to increasing CO$_2$ concentrations (and parameter uncertainties in general) may be misleadingly low due to model mis-specification, partly induced by break-like volcanic aerosol forcing. My model estimates highlight the considerable uncertainty about parameters used in IAMs (such as the ocean heat capacity) which has not been accounted for in existing economic impact projections.

The paper is structured as follows: Section 2 introduces the two-component energy balance model, section 3 shows the equivalence of the EBM and a cointegrated system. Section 4 demonstrates how the continuous-time EBM can be mapped to a discrete-time CVAR and estimated. Section 5 presents the application to a global mean climate system. Section 6 concludes.

2 Energy Balance Models (EBMs)

Energy balance climate models describe the change in temperatures as a function of incoming and outgoing radiation, with the simplest models approximating climate through global mean surface temperatures. They are used within IAMs to model the climate response to different forcings from natural and anthropogenic sources (such as greenhouse gas emissions). Two-component physical energy-balance models (EBMs, see Gregory et al. 2002, Held et al. 2010, Pretis & Allen 2013, Schwartz 2012) extend the simple model to include ocean and atmosphere interactions and have tracked global mean climate well while remaining analytically tractable, but are rarely formally tested as estimation is difficult due to the non-stationarity of observations and multiple-equation nature of the system. Two-component energy balance models of climate are characterised by a mixed upper layer (the shallow ocean/atmosphere) with low heat capacity and a deeper ocean layer with higher heat capacity. The mixed
upper component (denoted by subscript \( m \)) responds more quickly to perturbations (radiative forcing such as an increase in greenhouse gases\(^3\)) and refers to the mixed layers of the atmosphere and shallow ocean. The deeper (denoted by subscript \( d \)) component responds slower and allows for delayed, recalcitrant warming (Held et al. 2010). In other words, if external forcing stopped and the system was in disequilibrium, high temperatures in the deep component would lead to a slow warming of the mixed atmosphere component (and vice versa).

A simple two-component EBM is given here as differential equations in (1)-(2) explicitly taking ocean-atmosphere interactions into account. Let \( T_m \) be the temperature anomaly (deviations from steady state or long-term average common in observed measurements) for the mixed layer, \( C_m \) is the component heat capacity and \( F \) denotes radiative forcing. Heat capacity denotes the amount of energy needed to change the temperature of the component. Net heat flux denoting the net incoming energy is given by \( Q = F - \lambda T_m \) where climate feedback is defined as \( \lambda \), the change in downward net flux with temperatures \( T_m \). The term \( \lambda T_m \) captures the increasing outgoing long-wave radiation with increased temperatures. The upper component of the model describes the changes in heat content relative to the steady state \( C_m \frac{dT_m}{dt} \) as a function of net flux \( F - \lambda T_m \) and the heat exchange \( H \) with the lower component:

\[
C_m \frac{dT_m}{dt} = -\lambda T_m - H + F
\]  
(1)

The change in the heat content of the lower component \( C_d \frac{dT_d}{dt} \) is given by:

\[
C_d \frac{dT_d}{dt} = H
\]  
(2)

where \( C_d \) denotes the effective heat capacity of the deep compartment and \( T_d \) is the associated temperature anomaly. The heat exchange between the bottom and upper component is assumed to be proportional to the difference in temperatures:

\[
H = \gamma (T_m - T_d)
\]  
(3)

Substituting this expression into the above equations yields the two-component model as a system of two differential equations:

\[
C_m \frac{dT_m}{dt} = -\lambda T_m - \gamma (T_m - T_d) + F
\]  
(4)

\[
C_d \frac{dT_d}{dt} = \gamma (T_m - T_d)
\]  
(5)

Parameters \( \lambda, \gamma, C_m, C_d \) are usually calibrated based on theory or general circulation model (GCM) simulations, or estimated using individual equations only. The economic impact models FUND (Waldhoff et al. 2011) and PAGE (Hope 2011, Hope et al. 1993) use a single-equation calibrated energy balance model (without explicit representation of ocean-atmosphere interactions) as in equation (4) setting \( \gamma = 0 \), while DICE (Nordhaus 2014) uses a calibrated two-component energy balance model as outlined here in the system of equations (4) and (5). The parameter \( \lambda \) is of particular interest as it determines the equilibrium response of surface temperatures \( T_m \) to a change in the forcing \( F \) (e.g. the temperature response to increased CO\(_2\) concentration). In the following section I show that this system of differential equations is mathematically

\(^3\)Together with many other factors (e.g. solar irradiance, reflective aerosols from volcanic eruptions, soot, etc.) the effect of GHGs is quantified through the concept of radiative forcing. In simple terms, this captures the change in energy in the atmosphere due to a change in concentration of a GHG or other factor. The radiative forcing effect, measured in watts per square meter (watts m\(^{-2}\)), is calculated for the different forcing agents (e.g. GHGs, aerosols) and links the effect of the concentration of a gas on to temperatures.
equivalent to a cointegrated system which can thus be estimated using econometric methods address the inherent non-stationarity from stochastic trends.

3 The equivalence of EBMs and cointegrated systems

Econometric cointegration analysis can be used to overcome difficulties associated with stochastic trends in time series and applied to test whether there exist combinations of non-stationary series that are themselves stationary. Here I show that the system of differential equations of the two-component EBM is equivalent to a cointegrated system with restrictions on the parameters, providing a physical interpretation of cointegration in this climate system. While there is a debate in the econometric literature on whether climate time series contain stochastic trends (suggesting a cointegration treatment, see Kaufmann et al. 2010), or are better modelled through linear trends with breaks (e.g. Estrada et al. 2010), the analysis here concentrates on the cointegration approach.\footnote{Linking energy balance models to econometric models could equally be done using trend-stationary methods – see e.g. Estrada et al. 2013.}

The energy balance model in (4) and (5) can be expressed as

\[
\frac{dT_m}{dt} = \frac{1}{C_m} \left[ -\lambda T_m + F \right] + \frac{\gamma}{C_m} \left[ T_m - T_d \right]
\]

(6)

\[
\frac{dT_d}{dt} = \frac{\gamma}{C_d} \left[ T_m - T_d \right]
\]

(7)

with no explicit formulation given for the changes in radiative forcing \( \frac{dF}{dt} \) in the EBM. Let \( \alpha \) and \( \beta \) denote \( (p \times r) \) matrices (where \( p = 3, r = 2 \)) corresponding to:

\[
\alpha = \begin{pmatrix}
\frac{1}{C_m} & \frac{\gamma}{C_m} & 0 & \frac{\gamma}{C_d} \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

and

\[
\beta' = (\beta_1, \beta_2)' = \begin{pmatrix}
-\lambda & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}
\]

(8)

Then the two-component EBM in (6) and (7) can be written as the following system:

\[
dY = \alpha \beta' Y dt
\]

(9)

where \( Y = (T_m, T_d, F)' \), a \( (3 \times 1) \) vector, with \( F \) assumed to not adjust to \( T_m \) or \( T_d \), and thus being weakly exogenous for the cointegrating vector \( \beta \). It then holds that \( \beta' Y \) describes the long-run equilibrium relationships between the variables in the system, and the elements of \( \alpha \) describe how the individual variables adjust to this long-run relation. Assuming stochastic processes (e.g. due to imperfect measurement and omitted variables) the two-component EBM can be written with a noise term \( \nu \) as an Ornstein-Uhlenbeck process given by the stochastic differential equation:

\[
dY = \Pi Y dt + d\nu
\]

(10)

where \( \Pi = \alpha \beta' \) is a \( (p \times p) \) matrix of reduced rank \( r = 2 \) such that \( \Pi = \alpha \beta' \). The noise term \( d\nu = D dW \) captures omitted effects from the model, with \( D \) being a \( (p \times p) \) matrix and \( W \) a \( p \)-dimensional Brownian motion. Using the results in Kessler & Rahbek (2004) and Kessler & Rahbek (2001), the formulation of the stochastic two-component energy balance model then implies that, if the continuous Ornstein-Uhlenbeck
process is integrated\(^5\) of order one, I(1), the series in the EBM cointegrate into two stationary relations given by \(\beta_1'Y\) with adjustment coefficients of \(\alpha\). The rank of the matrix \(\Pi\) determines the number of long-run relations, which is equal to two in the EBM. The physical stochastic two-component energy balance model is equivalent to a cointegrated system with parameter restrictions in continuous time in (12) and (13) where the expressions in brackets correspond to the cointegrating vectors, and the coefficients outside the brackets are the \(\alpha\) adjustment coefficients in the cointegration model:

\[
\frac{dT_m}{dt} = \frac{1}{C_m} \left[ -\lambda T_m + F \right] + \frac{\gamma}{C_m} \left[ T_m - T_d \right] + v_1 \tag{12}
\]

\[
\frac{dT_d}{dt} = \frac{\gamma}{C_d} \left[ T_m - T_d \right] + v_2 \tag{13}
\]

The energy balance model implies that the three variables of upper component temperatures, lower component temperatures, and radiative forcing cointegrate into two stationary relations. The first, \((\beta_1'Y = -\lambda T_m + F)\), a relation linking upper component temperatures to radiative forcing describing the net heat flux (Q). The second, \((\beta_2'Y = T_m - T_d)\), a relation describing the heat transfer between the upper and lower component. Upper component temperatures adjust to both cointegrating vectors, lower component temperatures adjust to the second cointegrating vector \((\alpha_{2,1} = 0)\), and radiative forcing is weakly exogenous \((\alpha_{3,1} = \alpha_{3,2} = 0)\). Cointegration between these variables is a testable property, and cointegration analysis can further be used to estimate the EBM as a system and conduct inference on the parameters \(\alpha\) and \(\beta\) for a given set of observations approximated by the model.

4 Mapping of EBMs to CVARs and estimation as a system

The differential equations (12) and (13) describe the two-component energy balance model in continuous time, while sampling happens at discrete intervals. I consider two approaches through which the EBM can be tested and estimated as a CVAR using discrete observations. First, section 4.1 follows the literature on estimating continuous-time differential equations using VARs (see e.g. Phillips 1991 and Kessler & Rahbek 2004). An alternative is to interpret the EBM as a discrete time model (see e.g. Kaufmann et al. 2013 for a single-equation EBM), I consider this approach in section 4.2. The difference between these two approaches in practise is small, and primarily affects the mapping of the adjustment coefficients \(\alpha\) as the following sections show.

4.1 Inference in the cointegrated continuous-time EBM using discrete time

Inference on cointegration and parameter estimates in the continuous-time EBM can be conducted based on discrete observations using the results in Phillips (1991) and Kessler & Rahbek (2004).\(^6\) It is important to note though that the discretization method moving from discrete to continuous time can have effects on the auto-correlation structure of the error term, and thus affect the statistical properties of the estimates (Sargan 1974). Given the specification in equation (11) as an Ornstein-Uhlenbeck process, discrete observations \(Y_t = (T_{m,t}, T_{d,t}, F_t)'\) of \(Y = (T_m, T_d, F)'\) follow a vector-autoregressive (VAR) process in the conditional

\(^5\)In general a series that is I(1), or integrated of order one needs to be differenced once to be stationary, while an I(2) series requires to be differenced twice.

\(^6\)In particular, Theorem 1 in Kessler & Rahbek (2004).
model in equation (14):
\[ Y_t = AY_{t-1} + \epsilon_t \]  
(14)

where \( \epsilon_t \sim N(0, \Sigma) \). The coefficient matrix \( A \) in the VAR in (14) equals the matrix exponential of the coefficient matrix \( \Pi \) in equation (11), \( A = \exp(\Pi) \). This result derives from solving the Ornstein-Uhlenbeck process\(^7\) in equation (11) – see appendix 7.1. Let \( P = A - I \), then (14) can be written in equilibrium correction form as a CVAR:
\[ \Delta Y_t = PY_t + \epsilon_t \]  
(15)

where \( \Delta \) is the first difference operator, such that \( \Delta Y_t = Y_t - Y_{t-1} \). The CVAR describes changes in the endogenous \( Y_t \) variables as a function of past changes in the long run-dynamics as captured through the coefficient matrix \( P \). There is then a direct mapping of the EBM parameters from equation (11) to the parameters in the discrete-time model (15). In particular, the coefficient matrix \( P \) in the CVAR relates to the EBM parameters \( \alpha, \beta \) as:
\[ P = A - I = \exp(\Pi) - I = \exp(\alpha\beta') - I = \alpha\kappa\beta' = \tilde{\alpha}\beta' \]  
(16)

where \( \tilde{\alpha} = \alpha\kappa \) and the \((r \times r)\) matrix \( \kappa = (\beta'\alpha)^{-1} [\exp(\beta'\alpha) - I] \), see appendix 7.1 for a proof. This mapping implies that the two-component continuous-time EBM can be tested and estimated using a CVAR estimated on discrete observations: the rank of the coefficient matrix of the discrete-time VAR model equals that of the EBM, \( \text{rank}(P) = \text{rank}(\Pi) = \text{rank}(\alpha\beta') \), and the matrix \( \hat{P} = \hat{\alpha}\beta' \) yields estimates of the EBM parameters given by \( \beta' \). There is no one-to-one relation between the adjustment coefficients \( \alpha \) in the EBM and the discrete-time CVAR parameters \( \tilde{\alpha} = \alpha\kappa \). Thus, restrictions in \( \alpha \) are not directly preserved in the CVAR formulation. In other words, while in the continuous time EBM the lower component temperatures \( T_d \) do not adjust to the first cointegrating vector, \( \alpha_{2,1} = 0 \), this does not directly imply that the same holds for the discrete-time CVAR, \( \tilde{\alpha}_{2,1} \neq 0 \). An intuitive explanation for this comes from the fact that at the discrete time interval the lower component may adjust to the upper component given the feedback between \( T_m \) and \( T_d \). The extent of this can be tested by assessing whether \( \tilde{\alpha}_{2,1} \) equals zero or not using CVAR estimates.\(^8\) One exception on restrictions linking \( \alpha \) and \( \tilde{\alpha} \) occurs under weak-exogeneity. The forcing series is assumed to be weakly-exogenous (not adjusting to the cointegrating relations) in the EBM, thus \( \alpha_{3,1} = \alpha_{3,2} = 0 \). Given \( \tilde{\alpha} = \alpha\kappa \) this implies that \( \tilde{\alpha}_{3,1} = \tilde{\alpha}_{3,2} = 0 \), forcing should also be weakly exogenous in the CVAR formulation. Writing the model (15) in matrix form I formulate equations (12) and (13) as a discrete CVAR where the CVAR parameters (given by \( \tilde{\alpha} \) and \( \beta \)) correspond to (combinations of) parameters of the EBM:

\[
\begin{pmatrix}
\Delta T_{m,t} \\
\Delta T_{d,t} \\
\Delta F_t
\end{pmatrix} =
\begin{pmatrix}
\tilde{\alpha}_{1,1} \\
\tilde{\alpha}_{2,1} \\
\tilde{\alpha}_{3,1}
\end{pmatrix}
\begin{pmatrix}
\beta_{1,1} & \beta_{1,2} & \beta_{1,3}
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
T_{d,t-1} \\
F_{t-1}
\end{pmatrix}
+ 
\begin{pmatrix}
\tilde{\alpha}_{1,2} \\
\tilde{\alpha}_{2,2} \\
\tilde{\alpha}_{3,2}
\end{pmatrix}
\begin{pmatrix}
\beta_{2,1} & \beta_{2,2} & \beta_{2,3}
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
T_{d,t-1} \\
F_{t-1}
\end{pmatrix} + \epsilon_t
\]  
(17)

The cointegrating vectors and adjustment coefficients are linked to the continuous-time two-component EBM such that: \( \beta_{1,1} = -\lambda, \beta_{1,2} = 0, \beta_{1,3} = 1 \), linking upper component temperatures and radiative

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\(^7\)Where for simplicity the frequency of observations is assumed to be equal to one.

\(^8\)Section 5 investigate this further.
forcing in the first cointegrating relation; $\beta_{2,1} = -\beta_{2,2} = 1$, and $\beta_{2,3} = 0$, linking upper component temperatures and lower component temperatures in the second cointegrating relation; and $\tilde{\alpha}_{3,1} = \tilde{\alpha}_{3,2} = 0$, as forcing is assumed to adjust to neither cointegrating relation. In matrix notation this can be expressed as:

$$
\begin{pmatrix}
\Delta T_{m,t} \\
\Delta T_{d,t} \\
\Delta F_t
\end{pmatrix}
= \begin{pmatrix}
\tilde{\alpha}_{1,1} \\
\tilde{\alpha}_{2,1} \\
0
\end{pmatrix}
\begin{pmatrix}
-\lambda & 0 & 1
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
T_{d,t-1} \\
F_{t-1}
\end{pmatrix}
+ \begin{pmatrix}
\tilde{\alpha}_{1,2} \\
\tilde{\alpha}_{2,2} \\
0
\end{pmatrix}
\begin{pmatrix}
1 & -1 & 0
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
T_{d,t-1} \\
F_{t-1}
\end{pmatrix}
+ \epsilon_t
$$

(19)

This yields the discrete-time CVAR representation of the stochastic energy balance model and can be estimated and tested by estimating (15). The rank of $P$ determines the presence of cointegrating, or long-run, stationary relations, where the theory-EBM suggests there exist two. If the rank of $P = 0$, there are no cointegrating relations, and if $P$ is full rank, the system must be stationary. The matrix $P$ will have full rank if the series tested are stationary ($\sim I(0)$) or if all endogenous stochastic trends can be explained by exogenous variables entering the cointegrating relation in a partial system. The matrix has a rank of zero if the series do not cointegrate, and reduced rank ($p > \text{rank} > 0$) if the series cointegrate. The number of cointegrating relations is chosen using the trace statistic which tests the null hypothesis that there are $r - 1$ cointegrating relations against the null hypothesis that there are $r$ relations (Johansen 1988, Johansen 1995).

To alleviate concerns associated with small samples, Bartlett corrections can be applied (Johansen 2000) or a bootstrapping procedure to test the cointegrating rank (see Cavaliere et al. 2012) can be used. Using a system approach (Johansen 1988) avoids many problems associated with single equation cointegration procedures (e.g. Engle & Granger 1987) and does not require pre-testing of the individual series for stationarity (for a discussion of potential hazards see Pretis & Hendry 2013).

### 4.2 Interpreting the EBM as discrete time model

The previous section 4.1 uses the discrete-time VAR formulation of the continuous Ornstein-Uhlenbeck process. An alternative linking of EBMs to CVARs is to rely on a simple first-order Euler discrete time approximation (see e.g. Calel & Stainforth 2016) interpreting $\frac{dT_m}{dt}$ as $\Delta T_{m,t} = T_{m,t} - T_{m,t-1}$ and $\frac{dT_d}{dt} \approx \Delta T_{d,t} = T_{d,t} - T_{d,t-1}$. The system in (6) and (7) is then approximated as:

$$
\Delta T_{m,t} = \frac{1}{C_m} [ -\lambda T_{m,t-1} + F_{t-1} ] - \frac{\gamma}{C_m} [ T_{m,t-1} - T_{d,t-1} ] + u_{1,t}
$$

(20)

and

$$
\Delta T_{d,t} = \frac{\gamma}{C_d} [ T_{m,t-1} - T_{d,t-1} ] + u_{2,t}
$$

(21)

This system of equations is equivalent to a CVAR with restricted parameters where the expressions in brackets correspond to the cointegrating vectors, and the coefficients outside the brackets are the $\alpha$ adjustment coefficients in the cointegration model. The error terms (here $u_{1,t}$ and $u_{2,t}$) will capture omitted effects from
the model. In matrix notation this approximation is given by:

\[
\begin{pmatrix}
\Delta T_{m,t} \\
\Delta T_{d,t} \\
\Delta F_t
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{C_m} & 0 & 0 \\
0 & -\lambda & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
T_{d,t-1} \\
F_{t-1}
\end{pmatrix}
\tag{22}
\]

\[
+ \begin{pmatrix}
-\gamma C_m & 0 & 0 \\
\gamma C_d & 1 & -\gamma \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
T_{m,t} \\
T_{d,t} \\
F_t
\end{pmatrix}
\]

where \(u_t = (u_{1,t}, u_{2,t}, u_{3,t})\)' and the cointegrating vectors and adjustment coefficients are linked through the approximation to the EBM such that: \(\alpha_{1,1} = 1/C_m, \alpha_{2,1} = \alpha_{3,1} = 0, \beta_{1,1} = -\lambda, \beta_{1,2} = 0, \beta_{1,3} = 1\) further \(\alpha_{2,2} = \gamma/C_d, \alpha_{1,2} = -\gamma/C_m, \alpha_{3,2} = 0, \beta_{2,1} = 1 = -\beta_{2,2}\) and \(\beta_{2,3} = 0\). Note that while the motivation linking EBMs to CVARs in this approximation is slightly different to the one provided in section 4.1 (a simple approximation using Euler-discrete time approach compared to the solution of the Ornstein-Uhlenbeck process), in practice the estimation procedure is near-identical. Cointegration is preserved in this CVAR formulation, testing for the rank and estimating the long-run relations given by \(\beta\) is exactly identical to the results in 4.1. The main difference lies in the interpretation of the adjustment coefficients \(\alpha\). Given the simplifying approximation, the CVAR adjustment coefficients have a structural interpretation and identify the EBM parameters \(\alpha\).

### 4.3 Testable Properties implied by the EBM-CVAR mapping

The translation of a two-component EBM and a CVAR provides direct restrictions and properties which can be tested against observations following an inspection of the residuals to satisfy the underlying assumptions described above. Formally the EBM implies the following properties of the estimated CVAR in equation (15) which can then be tested:

1. Assessment of the model validity in view of the data, and determination of the appropriate lag-length (e.g. using Nielsen et al. 2006).

2. \(\text{rank}(\Pi) = \text{rank}(P) = 2\) (Trace test e.g. using Johansen 1988 or Cavaliere et al. 2012): the time series cointegrate to two stationary relations, linking upper component temperatures with radiative forcing, and lower component temperatures with upper component temperatures.

3. Restrictions in (19) on \(\beta\) are not rejected (likelihood ratio or bootstrap tests), estimates are statistically different from zero and theory-consistent.

4. Restrictions in (19) on \(\tilde{\alpha}\) are not rejected, and estimates are statistically different from zero, (likelihood ratio or bootstrap tests) and theory-consistent.

In particular, for the EBM to hold it is necessary that the three series – upper component temperatures, lower component temperatures, and radiative forcing – cointegrate into two stationary relations: The first, an equation linking upper component temperatures to radiative forcing, describing the net heat flux (Q):

\[
h_{1,t-1} = \beta_{1,1} T_{m,t-1} + \beta_{1,3} F_{t-1}
\]

where \(\beta_{1,1} = -\lambda\), and \(\beta_{1,3} = 1\). The second, a relation describing the heat transfer between the upper and lower component:

\[
h_{2,t-1} = \beta_{2,1} T_{m,t-1} + \beta_{2,2} T_{d,t-1}
\]

(24)
where $β_{2,1} = 1 = −β_{2,2}$. With respect to the physical structure between variables, this implies that any stochastic trends present in the radiative forcing (where the stochastic component is driven by anthropogenic emission of greenhouse gases from stochastic economic activity) will be imparted onto the temperatures of the mixed component, through which it transfers onto the lower component. This generalizes the results in Kaufmann et al. (2013) to the two-component EBM. The flexible nature of the CVAR would also straightforwardly allow extensions to EBMs with more than two-components, and potentially spatial dimensions through cointegrated spatial panel models.

Notably, estimating a two-component EBM through a CVAR avoids potential hazards associated with statistical and econometric climate models. The Johansen cointegration procedure requires no pre-testing of individual series for the order integration (up to I(1)) and takes the system nature of the model into account. The model also relies on the aggregate of radiative forcing, rather than single series which, individually, may suffer from data measurement problems.

The CVAR modelling approach provides a foundations for coupling empirical EBMs with models of the macroeconomy by expanding the system in (15) using an explicit formulation for the forcing series $F_t$ (e.g. endogenously modelling forcing along the lines of Hendry & Pretis 2013). The forcing can be disaggregated into natural and anthropogenic factors, where concentrations of greenhouse gases are modelled as the accumulation of emissions driven by economic activity and social changes. Feedback of climate variables onto economic variables can then be estimated in the CVAR formulation yielding estimates of damages and the social cost of carbon. Further, the CVAR formulation of an EBM yields parameter estimates to assess the impacts of uncertainties in IAMs, and permits large-scale simulated climate models (e.g. general-circulation models – GCMs) to be studied by estimating simple approximations to them using simulated model output and the CVAR approach together with the econometric time-series toolkit.

5 Application

To illustrate the mapping of the two-component EBM to a CVAR with restricted parameters, I estimate a CVAR using global-mean climate time series. The main results presented here do not place additional restrictions on the $α$ adjustment coefficients and so follow the continuous-to-discrete time approach outlined in section 4.1. The results for interpreting the EBM model in discrete time (placing additional restrictions on $α$) are reported in the appendix section 7.3. The empirical illustration here serves to highlight some of the econometric toolkit placed at the disposal of empirical climate models, as well as to discuss challenges in moving from theory to estimation. The section considers model-specification, estimation of EBM parameters, an assessment of stability of the model using recursive estimation, forecasting, and the detection of outliers and structural breaks using indicator saturation. Models are estimated using PcGive (Doornik & Hendry 2013) and CATS in Ox (Doornik 2016).

5.1 Data

The upper/mixed component is specified as the atmosphere at the surface and temperatures are taken as global mean surface temperature anomalies from the GISS dataset (Hansen et al. 2010). The lower component is proxied by the ocean heat content (OHC) anomalies from 0-700m (Levitus et al. 2009). This is a crude approximation as it omits the deep ocean and will therefore provide a high estimate of the adjustment speed of this component. The series are graphed in Figure 1. Effective radiative forcing time series (Figure 2) are taken from Hansen et al. (2011). The aggregate of the forcing used for the model includes well-mixed

\footnote{Data for deeper ocean heat content for 0-2000m is available but only through pentadal averages (Levitus et al. 2012).}
greenhouse gases (WMGHGs)\textsuperscript{10} including CO\textsubscript{2}, stratospheric water vapour (Str. H\textsubscript{2}O), stratospheric ozone (O\textsubscript{3}), black carbon, stratospheric aerosols, snow albedo changes, land use changes, indirect aerosol effects and reflective aerosols. The short time period from 1955-2010 masks the dramatic increase in radiative forcing stemming from WMGHGs since the late 19th century. To estimate the parameters of the EBM here I rely on the aggregate of radiative forcing, an extension in section 5.6.1 considers disaggregates by separating out volcanic forcing in the form of stratospheric aerosols.\textsuperscript{11}

Among the three main IAMs: FUND, DICE, and PAGE, only DICE formally accounts for ocean-atmosphere interactions using a two-componend calibrated energy balance model. Equally few econometric models accounting for non-stationarity in the form of stochastic trends also consider ocean heat content and thus the heat transfer between the atmosphere/top component and the ocean. Two notable exceptions are Stern (2006) using observations for ocean heat content up to 300m depth, and Bruns et al. (2017) who consider a multi-cointegration approach reconstructing ocean heat content up to 2000m for which few observations are available. Relative to these approaches I provide a direct link of the the two-component EBM to a CVAR allowing for joint system estimation while using ocean heat content up to 700m.

Data availability for ocean heat content constrains the sample to a period from 1955 onwards. The estimation period is 1955-2011 at an annual frequency, resulting in T=56 observations. This time period, together with the use of an aggregate of radiative forcing rather than individual series, alleviates some concerns over measurement changes and the data quality of the radiative forcing series (see Pretis and Hendry 2013).

While the time series properties of radiative forcing series are heavily debated within the climate-econometric literature on stochastic trends (Kaufmann & Stern 2002, D. I. Stern & Kaufmann 2000, Beenstock et al. 2012, Pretis and Hendry 2013), the individual order of integration of these series is not directly

\textsuperscript{10}The major gases grouped together as WMGHGs include CO\textsubscript{2}, CH\textsubscript{4}, N\textsubscript{2}O, and SF\textsubscript{6} (Sulfur hexafluoride) and are combined due to their long lifetime and property that they are “well-mixed” – concentrations are at approximately the same level across the globe (Myhre et al., 2013).

\textsuperscript{11}Stern and Kaufmann (2014) test for links between individual forcing series and surface temperatures, further work could focus on more detailed models using individual forcing series found to be significant in Stern and Kaufmann (2014), while series with little impact (e.g. black carbon) could be omitted.
Figure 2: Radiative forcing from 1955-2011 in Wm$^{-2}$. Panel (a) shows global effective total and disaggregated forcing, panel (b) shows the first difference of total forcing, and panel (c) graphs first difference of total forcing excluding stratospheric aerosol forcing from volcanic eruptions.
relevant. Within the EBM formulation it is aggregate forcing that drives upper component temperatures. On
initial inspection it may appear inconsistent that global mean surface temperatures are well approximated by
an I(1) process while well-mixed greenhouse gases (in particular) atmospheric CO\textsubscript{2} concentrations can be
argued to be I(2) – see appendix section 7.5. However, jointly the individual forcing series sum (or cointe-
grate) to an I(1) aggregate of forcing, where the I(1) cointegrating relation is given simply by the sum of the
individual forcing series. Given the same unit of measurement (watts per square metre) and the aggregate
effect, forcings are directly summable – this is a serious concern in the analysis of Beenstock et al. (2012)
who argue that, due to the different time series properties, temperatures and WMGHGs cannot be related.
While individual forcing series may be I(2) or I(0), aggregate forcing appears to be consistent with an I(1)
process, and thus exhibits approximately the same order of integration as global mean surface temperatures.

5.2 Estimation using ocean heat content observations

The EBM characterises changes in temperatures while the data available for applications often are a mixture
of heat content and temperature observations. Given the availability of data, my application uses ocean heat
content (\(O_t\)) for the lower component and temperatures for the upper component. This transformation yields
a re-parametrised, but otherwise identical EBM for estimation where I use \(O_t \approx C_d T_{d,t}\) (see Schwartz 2007
and Stern 2006). The re-parametrised EBM is given as:

\[
\begin{pmatrix}
\Delta T_{m,t} \\
\Delta O_{t} \\
\Delta F_{t}
\end{pmatrix}
= \begin{pmatrix}
\tilde{\alpha}_{1,1} \\
\tilde{\alpha}_{2,1} \\
0
\end{pmatrix} \begin{pmatrix}
-T \lambda & 1 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
O_{t-1} \\
F_{t-1}
\end{pmatrix}
+ \begin{pmatrix}
\tilde{\alpha}_{1,2} \\
\tilde{\alpha}_{2,2} \\
0
\end{pmatrix} \begin{pmatrix}
1 & -1/C_d & 0 \\
0 & 1 & C_d
\end{pmatrix}
\begin{pmatrix}
T_{m,t-1} \\
O_{t-1} \\
F_{t-1}
\end{pmatrix}
+ \epsilon_t
\] (25)

To determine the cointegrating rank of the model, estimation includes an unrestricted constant. If the
rank equals two, then the first corresponding cointegrating vector is restricted and chosen as:

\[
h_{1,t-1} = (\beta_{1,1} T_{m,t-1} + \beta_{1,3} F_{t-1})
\] (26)

The second, a restricted relation of temperature changes in the lower component:

\[
h_{2,t-1} = (\beta_{2,1} T_{m,t-1} + \beta_{2,2} O_{t-1})
\] (27)

For identification both cointegrating vectors are normalized such that \(\beta_{1,3} = 1\) and \(\beta_{2,1} = 1\). The addi-
tional restrictions given by the EBM are: lower component temperatures do not enter the first cointegrating
vector, \(\beta_{1,2} = 0\); radiative forcing does not enter the second cointegrating vector, \(\beta_{2,3} = 0\), and is weakly
exogenous, \(\alpha_{3,1} = \alpha_{3,2} = 0\) which implies that \(\tilde{\alpha}_{3,1} = \tilde{\alpha}_{3,2} = 0\). This is a simplification since some of the
radiative forcing series are determined endogenously, CO\textsubscript{2} concentrations generally vary with temperatures,
as does ice-albedo. However, it is not obvious whether these feedback are measurable and statistically sig-
nificant on a global scale. The weak exogeneity assumption is relaxed in a simple extension of the model
(see model C in Table 2) where an aggregate of radiative forcing is modelled endogenously.

Every parameter in the cointegrating vector of the CVAR has a structural interpretation. To estimate
the model parameters of interest, it then holds that in the second cointegrating vector \((27) \beta_{2,2} = -1/C_d\).
providing an estimate of the heat capacity of the deep component. The climate feedback parameter $\lambda$ can be determined using $\beta_{1.1} = \lambda$. Approximate standard errors and confidence intervals can be derived using the asymptotic mixed normality of the cointegrating parameters. In the case of parameters of interest being non-linear transformations of the cointegrating estimates (such as $-1/C_d = \beta_{2.2}$), I rely on the $\delta$-method to provide approximate standard errors of the estimates. Based on these restrictions all EBM parameters in the approximation can be identified and the over-identifying restrictions can be tested.

5.3 Model Specification

Determining the number of cointegrating relations and conducting inference on the estimated parameters relies on well-specified – congruent – models (Juselius 2006). While the theoretical EBM model does not imply additional lags, the time series properties of the data may support a longer lag length. A more general formulation of the CVAR in equation (25) allowing for longer lags is provided here in equation (28). In equilibrium-correction form CVAR model used for estimation is given by:

$$\Delta Y_t = \Gamma_1 \Delta Y_{t-1} + \ldots + \Gamma_k \Delta Y_{t-k} + PY_{t-1} + \Phi + \epsilon_t$$  \hspace{1cm} (28)

where $Y_t = (Y_{1,t}, Y_{2,t}, Y_{3,t})'$ for the estimation with $Y_{1,t}$ denoting the surface temperature anomaly, $Y_{2,t}$ the 0-700m OHC anomaly, $Y_{3,t}$ radiative forcing as outlined in the data section, and $\Phi$ denotes an unrestricted constant. To formally determine the lag structure I estimate a general unrestricted VAR (as in equation (14)) starting with three lags (equivalent to $k = 2$ in (28)). Removing the third lag is not rejected ($p = 0.08$), while dropping both the second and third lag is rejected ($p = 0.003$). The model with the lowest Schwarz criterion (SC, Schwarz 1978) includes just one lag (SC=3.28, relative to 3.45 for two, and 3.76 for three lags). Assessing the diagnostic tests of the unrestricted models, a VAR(1) model rejects no-residual autocorrelation ($p=0.003$), while a VAR(2) passes the residual vector autocorrelation test ($p=0.59$) and dropping the second lag in the VAR(2) is rejected ($p=0.005$). Many discrete approximations use the average of two periods which further provides a justification for the use of two lags. Additional results on the dynamic stability and unit-root properties of the estimated VAR models are provided in appendix 7.2.

I proceed by estimating three initial variations of the CVAR EBM model. First, model A is estimated with a single lag corresponding to the simple theoretical two-component EBM. This model is included as a comparison despite being mis-specified. Second, model B includes two lags as suggested by the tests determining the lag structure. Once the restrictions implied by the EBM are imposed, both models A and B do not model forcing explicitly but assume it to be weakly exogenous. Nevertheless, some of the forcing series are not exogenous but rather may vary with observed temperatures – particularly on a further disaggregated level, there is evidence of climate variability affecting economic growth, and subsequently GHG emissions from economic activity (Dell et al. 2014, Hsiang 2016, with feedback discussed in Pretis 2017). Third, Model C (estimated with 2 lags) relaxes the strong restrictions from model B by allowing aggregate forcing to adjust to the first cointegrating relation. This allows for a formal test whether forcing measurably adjusts to changes in the temperature-equilibrium on a global scale. This changes the system to three variables, however, no additional variables are included to explain forcing. In other words, radiative forcing is modelled endogenously and is allowed to adjust to the first cointegrating vector ($\tilde{\alpha}_{3,1} \neq 0$). Further research can expand the EBM to disaggregate the forcing series and also incorporate a macroeconomic VAR

\footnote{Given an asymptotically normally distributed random variable $X$ satisfying $\sqrt{n}(X - \mu) \xrightarrow{D} N(0, \sigma^2)$ and a continuous function $g(\mu)$, using a Taylor-expansion we can approximate the distribution of $g(X)$ by $\sqrt{n} (g(X) - g(\mu)) \xrightarrow{D} N(0, \sigma^2 |g'(\mu)|^2)$. However, if likelihood-ratio tests are being used, then linearisation is not required.}
to model the anthropogenic emissions, as well as additional equations modelling natural variability (notably
snow albedo changes). A simple extension is considered in section 5.6.1 for volcanic forcing.

In summary, model A corresponds to the simplest translation of the EBM theory with a single lag, model
B is estimated with two lags to account for additional dynamics and residual autocorrelation, model C is
equivalent to model B with weak-exogeneity of the forcing series relaxed. Section 5.6.1 also investigates an
extension based on modelling volcanic forcing as breaks labelled as model V (for volcanoes).

5.4 Results

The time series of global mean surface temperature anomalies, 0-700m ocean heat content anomalies, and
total radiative forcing cointegrate to two stationary relations consistent with the theory of a two-component
EBM. Likelihood ratio tests cannot reject the presence of two cointegrating vectors (rank = 2) in all the
models estimated here ($p_A=0.58$, $p_{B,C} = 0.76$ for models A, B, C respectively), consistent with the theory
provided by the two-component EBM (1). A single cointegrating relation (rank=1) is rejected in all models
(see Table 1 for full results). To address concerns about small samples, I also apply the bootstrap cointegra-
tion approach proposed by Cavaliere et al. (2012), as well as the Bartlett correction (Johansen 2000) with
results shown in Table 1. Both the bootstrap ($p_A = 0.81$, $p_{B,C} = 0.87$ for rank=2) and Bartlett correction
($p_A = 0.58$, $p_{B,C} = 0.76$ for rank=2) confirm the results of the standard cointegration test.

Table 2 provides the estimated parameters of the CVAR. Fitted values are given in levels in Figure 3 and
first differences in Figure 4. Residual plots are provided in appendix 7.7. The imposed restrictions on the
CVAR are not rejected at the 1% level ($p=0.02$) in model A (see Table 2), though this cannot be relied-on due
to the failure of the residual autocorrelation mis-specification test of the model. The restrictions are, rejected
in model B which controls for residual autocorrelation, both using the standard likelihood ratio tests and the
bootstrap test of restrictions in Cavaliere et al. (2015). To explore the rejection of the restrictions, model C
relaxes weak-exogeneity of the forcing series. Taking into account the error serial correlation through the
extension of the model to two lags, the model restrictions are not rejected in model C once forcing is allowed
to adjust to upper component temperatures ($p=0.12$, bootstrap $p=0.12$). This is a notably strong result – it is
not unusual for restrictions of this complexity to be rejected. For completeness, despite the rejection of the
restrictions in model B and mis-specification in A, the proceeding analysis reports results for all models A,
B, and C.

Normality of the residuals cannot be rejected in diagnostic tests of the models A and B, it is, however,
rejected for model C. The rejection of normality in C likely stems from the forcing series as indicated by
the single-equation diagnostic tests – normality is not rejected in all but the forcing series. This can be
driven by the volcanic impacts resembling structural breaks. Model V in (section 5.6.1 and appendix 7.4),
modelling volcanic forcing as breaks, shows improvements in diagnostic test results (see also the residual
plots in appendix 7.7) and the EBM restrictions (including weak exogeneity of forcing) are then not rejected
($p = 0.08$, bootstrap $p = 0.28$). The fact that there is little evidence of forcing measurably adjusting to
the cointegrating relations on a global scale suggests that temperatures can be modelled conditionally on
forcing without having to consider the marginal process of forcing itself.

The estimated parameters of the EBM in the cointegrating relations are highly significant and estimated
with theory-consistent signs. There is little evidence that the ocean heat content series adjusts to the first
cointegrating vector ($\tilde{\alpha}_{2,1}$ is insignificant in Table 2). This is consistent with the continuous-time EBM
where the lower component only adjusts to the second cointegrating relation in (13). The two cointegrating

\[ \tilde{\alpha}_{2,1} \text{ is insignificant in Table 2.} \]
relations representing the equations of the EBM are reported here as an example for the EBM theory model A:

\[ \hat{h}_{1,t-1} = -2.71 \frac{0.51}{T_{m,t-1} + F_t-1} \] (29)

\[ \hat{h}_{2,t-1} = T_{m,t-1} - 0.044 \frac{0.003}{O_{t-1}} \] (30)

where standard errors are given in parentheses and the coefficients on \( F_t \) in the first equation, and the coefficient on \( T_{m,t} \) in the second equation are normalized to 1 for identification.

Figure 3: EBM fit (estimated as a CVAR). Left panels shows model fit (in colour) and observed (GIS) upper-component (global mean-surface) temperature anomalies (in black). Right panels shows model fit and observed 0-700m ocean heat content (OHC) anomalies (Levitus et al. 2009) which proxy lower component heat content in the EBM.

Table 1: Cointegration Tests of the Two-Component EBM

<table>
<thead>
<tr>
<th>Rank</th>
<th>A: Base, 1-lag</th>
<th>B &amp; C: 2-lag</th>
<th>B &amp; C: 2-lag Bartlett</th>
<th>B &amp; C: 2-lag Bootstrap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank=0</td>
<td>61.52 [p&lt;0.00]**</td>
<td>57.90 [p&lt;0.00]**</td>
<td>54.36 [p&lt;0.00]**</td>
<td>57.90 {cr5% = 41.28} [p&lt;0.00]**</td>
</tr>
<tr>
<td>Rank=1</td>
<td>20.35 [p&lt;0.01]**</td>
<td>22.46 [p&lt;0.01]**</td>
<td>21.69 [p&lt;0.01]**</td>
<td>22.46 {cr5% = 20.56} [p=0.01]**</td>
</tr>
<tr>
<td>Rank=2</td>
<td>0.31 [p=0.58]</td>
<td>0.09 [p=0.76]</td>
<td>0.09 [p=0.76]</td>
<td>0.09 {cr5% = 7.54} [p=0.87]</td>
</tr>
<tr>
<td>Rank=3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Obs.</td>
<td>56</td>
<td>55</td>
<td>55</td>
<td>55 (B=399 Reps.)</td>
</tr>
</tbody>
</table>

Cointegration rank tests using the Johansen (1988) trace test, the Johansen trace test with Bartlett correction (Johansen 2000), and bootstrap cointegration test (Cavaliere et al. 2012) with \( B \) bootstrap replications and reported 5% critical value as ‘{cr5%...}’. p-values are reported in brackets [ ], * indicates rejection at 5%, ** indicates rejection at 1%. Bartlett (and Bootstrap) results for Model A also support rank=2 with p-values for rank=2 of \( p = 0.577 \) (\( p = 0.81 \) for bootstr.) vs. \( p < 0.01 \) (\( p = 0.028 \) for bootstr.) for rank 1.
Table 2: EBM Cointegration Model Parameter Estimates where $\tilde{\alpha} = \alpha \kappa$

<table>
<thead>
<tr>
<th>EBM/CVAR Model</th>
<th>A: Base</th>
<th>B: 2-lag</th>
<th>C: Endo. Forc.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Coint. Relations Vector 1 (Mixed)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{1,1}$</td>
<td>-2.71 (0.51)**</td>
<td>-2.29 (0.41)**</td>
<td>-2.21 (0.32)**</td>
</tr>
<tr>
<td>$\beta_{1,2}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\beta_{1,3}$</td>
<td>-</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td><strong>Adjustment</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}_{1,1}$</td>
<td>0.11 (0.03)**</td>
<td>0.15 (0.03)**</td>
<td>0.11 (0.03)**</td>
</tr>
<tr>
<td>$\tilde{\alpha}_{2,1}$</td>
<td>0.60 (0.33)</td>
<td>0.53 (0.38)</td>
<td>-0.03 (0.41)</td>
</tr>
<tr>
<td>$\tilde{\alpha}_{3,1}$</td>
<td>-</td>
<td>-</td>
<td>-0.55 (0.15)**</td>
</tr>
<tr>
<td><strong>Coint. Relations Vector 2 (Deep)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{2,1}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_{2,2}$</td>
<td>-0.044 (0.003)**</td>
<td>-0.041 (0.004)**</td>
<td>-0.041 (0.004)**</td>
</tr>
<tr>
<td>$\beta_{2,3}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>Adjustment</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\tilde{\alpha}_{1,2}$</td>
<td>-0.41 (0.12)**</td>
<td>-0.46 (0.15)**</td>
<td>-0.47 (0.15)**</td>
</tr>
<tr>
<td>$\tilde{\alpha}_{2,2}$</td>
<td>7.82 (1.48)**</td>
<td>7.43 (1.99)**</td>
<td>7.35 (1.99)**</td>
</tr>
<tr>
<td>$\tilde{\alpha}_{3,2}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>Test of Restrict.</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\chi^2$ (2), C: $\chi^2$ (1)</td>
<td>8.92 [p=0.012]*</td>
<td>16.32 [p=0.003]**</td>
<td>2.47 [p=0.12]</td>
</tr>
<tr>
<td>Bootstrap (B=199)</td>
<td>8.92 cr5%=10.08[p=0.07]</td>
<td>16.32 {cr5% =7.7} [p&lt;0.00]**</td>
<td>2.47 {cr5% = 3.7} [p=0.12]</td>
</tr>
<tr>
<td><strong>Diagnostic Tests</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AR (1-2) F-Test</td>
<td>2.96 [p=0.005]**</td>
<td>1.32 [p=0.24]</td>
<td>1.14 [p=0.32]</td>
</tr>
<tr>
<td>Normality $\chi^2$ (4), C: $\chi^2$ (6)</td>
<td>0.76 [p=0.94]</td>
<td>7.19 [p=0.13]</td>
<td>20.36 [p=0.002]**</td>
</tr>
<tr>
<td>Observations T</td>
<td>56</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>log-likelihood</td>
<td>-72.48</td>
<td>-59.85</td>
<td>-51.84</td>
</tr>
<tr>
<td><strong>EBM Estimates</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$ (W m$^{-2}$ C$^{-1}$)</td>
<td>2.71 (0.51)</td>
<td>2.29 (0.41)</td>
<td>2.21 (0.32)</td>
</tr>
<tr>
<td>ECS (deg. C)</td>
<td>1.37 (0.25)$\dagger$</td>
<td>1.62 (0.29)$\dagger$</td>
<td>1.67 (0.24)$\dagger$</td>
</tr>
<tr>
<td>$C_m$ (W year m$^{-2}$ C$^{-1}$)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$C_d$ (W year m$^{-2}$ C$^{-1}$)</td>
<td>22.72 (1.54)$\dagger$</td>
<td>24.39 (2.38)$\dagger$</td>
<td>24.39 (2.38)$\dagger$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Model estimates based on CVAR estimation. Standard errors are given in parentheses ( ) while p-values are reported in brackets [ ]. Bootstrap test for restrictions (Cavaliere et al. 2015) with B replications, 5% bootstrap critical values reported as ‘{cr5% =...}’. Standard errors are provided where available. If no standard errors are reported, then parameter is restricted or derived from estimated model parameters. * indicates significance at 5%, ** indicates significance at 1%. Standard errors derived using $\delta$-method are marked using $\dagger$. Left column specifies parameters, right columns shows estimation results. Dash - marks imposed restriction and no identification in the case of the structural EBM parameters given the theoretical result of $\tilde{\alpha} = \alpha \kappa$. 

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5.5 Discussion

The CVAR specification places the econometric toolkit at the disposal of estimated EBMs. This makes it possible to test and discover relationships in observational data, provide parameter uncertainties in IAMs, as well as assess simulated climate model output. Here I consider the physical interpretation of the estimated quantities, stability of the estimated model over time, and uncertainties in parameters.

5.5.1 Physical Interpretation of First Cointegrating Vector

Based on the EBM in (1) and the equivalent cointegration formulation (23), the first cointegrating relation used to estimate climate feedback has a direct physical interpretation and describes the net heat flux or net energy coming into the system: $Q = F - \lambda T_m$. As an informal test of the model I compare the estimated cointegrating vector which corresponds to estimated net heat flux $Q$ against the independently observed net heat flux from satellite-based measurements of annually averaged net flux at the top of the atmosphere (TOA) from the CERES Energy Balanced and Filled (EBAF) All Skies data (Loeb et al. 2009). It is important to emphasize that this data series is not used in the estimation of the model but is a different set of observations from satellite measurements of the quantity captured by the first cointegrating relation. Figure 5 plots the first cointegrating relation together with the satellite measurements. While the models under-estimate the volatility and level slightly, the first cointegrating vectors from models B and C remain close to the satellite-observed record. Model A, which does not control for residual autocorrelation (and results in high estimates of climate feedback – see section 5.6.2), consistently lies below satellite-observed TOA net flux.

Figure 4: EBM fit of first differences (estimated as a CVAR). Left panels shows model fit (in colour) and observed first differences of (GIS) upper-component (global mean-surface) temperature anomalies (in black). Right panels shows model fit and first differences of observed 0-700m ocean heat content anomalies (Levitus et al. 2009) which proxy lower component heat content in the EBM.
Figure 5: Physical interpretation of the first cointegrating vector: Net flux based on the first cointegrating vector and observed top of the atmosphere (TOA) net flux. Top panel graphs the net flux as described by the estimated first cointegrating vector for models A-C from 1955-2011 (in colour). Bottom panel graphs the net flux from 2000-2011 together with annually averaged observed TOA net flux from the CERES Energy Balanced and Filled (EBAF) all skies data (black line with points) (Loeb et al. 2009) which have not been used to estimate the model and serve as an informal test of the model against a different set of independently measured observations.
5.6 Forecasting through the hiatus

The cointegrated system model also provides a basis for making forecasts and counter-factual policy analysis (subject to valid conditioning and super-exogeneity – see Engle et al. 1983). As an illustration, to further investigate the performance and the stability of the EBM I re-estimate the models up until 1997 and forecast the remaining period (until 2011) 5-steps (years) ahead using the model estimates. The forecast origin is chosen to coincide with the beginning of the so-called “warming hiatus”, describing a slowdown in surface temperatures despite observing an increase in radiative forcing from greenhouse gases (Meehl et al. 2011 Watanabe et al. 2013). While forecast performance is not a directly useful model evaluation tool (Clements & Hendry 2005), it can provide insights into model stability. The 5-step forecasts used here rely on dynamic forecasting in a closed system – next period’s forecast are used as starting points for succeeding forecasts up to five steps ahead (see e.g. Clements & Hendry 1999), where forcing is assumed to be strongly exogenous in the simple EBM and is thus conditioned on. Figure 6 graphs the 5-step forecasts of the temperature and ocean heat content anomalies. While the forecast confidence intervals are wide, the mean-level forecast tracks observed temperatures and ocean heat content closely. This is consistent with Kaufmann et al. (2011): changes in radiative forcing over this time period together with ocean heat uptake can (in part) account for the slow down in warming. However, it appears that ocean heat uptake is generally slightly under-predicted relative to a model estimated prior to 1997. This may provide some evidence that increased ocean heat uptake accounts for some of the “missing energy” recently debated (Trenberth & Fasullo 2010, Trenberth & Fasullo 2012).

Figure 6: EBM/CVAR Model 5-Year Ahead Forecasts of Surface Temperature (left) and Ocean Heat Content (right) during the Hiatus for models A, B, and C. Shading denotes forecast interval for ±1 standard error (67% int.), ±2 standard errors (≈ 95% int.) and ±2.57 standard errors (≈ 99% int.).
5.6.1 Stability and Detection of Breaks

The stability over time of the EBM can be assessed through formal econometric methods for the detection of outlying observations and structural breaks. Breaks in time series can arise in many shapes and may occur at any point in time – distorting inference in-sample and resulting in forecast failure out-of-sample if not appropriately modelled. For the present application I use impulse and step-indicator-saturation (IIS – see Hendry et al. 2008, and SIS – see Castle, Doornik, Hendry, & Pretis 2015) to identify periods of instability.\textsuperscript{14} Indicator saturation adds a full set of impulse or step-functions at every point in time and removes all but significant ones using general-to-specific selection up to a specified target level of significance $p_\alpha$. This formulates the detection of breaks as a model-selection problem, where selection takes place over a full set of break functions, a subset of which describe any underlying breaks. The false-positive rate of detected breaks and outliers is given by $p_\alpha S_j$ where $S_j$ denotes the number of potential break variables included for $j \in \{iis, sis\}$. The spurious retention of outliers and breaks can therefore be directly controlled by choosing conservative (low) values of $p_\alpha$. IIS includes a zero-one dummy variable for each observations thus $S_{iis} = T$, while SIS includes a full set of $S_{sis} = T - 1$ step-functions additional to the regression intercept. As the saturated model using IIS (or SIS) includes at least as many variables as observations, it cannot be estimated directly. Instead estimation is conducted through block-partitioning of indicator variables. If there are truly no outlying observations or breaks, then despite the inclusion of a large candidate set of dummy variables, the distribution of variables not-selected over (i.e. EBM parameters) remains unaffected beyond small efficiency effects (the asymptotic theory of block-partitioning is derived in Johansen & Nielsen 2009 and Johansen & Nielsen 2016, see also Jiao & Pretis 2017). Indicator saturation does not impose a minimum break length or upper limit on the number of detected breaks, and thus can act as a flexible test for model misspecification without requiring prior knowledge of the form of misspecification or which observations deviate from the estimated model. Individual outliers can be detected through single indicators in IIS, while SIS makes it possible to parsimoniously identify longer breaks. To assess uncertainties around break timings in SIS, the approximate normal distribution of the error terms can be used to compute the probability of a true underlying break falling in any specified interval around a detected break (Hendry & Pretis 2017). This allows for the construction of approximate confidence intervals around detected break dates based on the estimated break magnitude and error variance. Indicator saturation can be implemented using the R-package \textit{gets} (Pretis et al. 2017) or \textit{PcGive} (Doornik & Hendry 2013).

Using IIS and SIS on model B and selecting at $p_\alpha = 1\%$ (which implies a false-positive rate for IIS of 0.01 $\times$ 55 = 0.5 indicators retained spuriously by chance) identifies one outlying observation as an impulse in 1977, and two opposite signed step-shifts in 1982 (99% confidence interval on breakdate: ± 1 year) and 1983 (99% CI: ± 1 year) denoting a negative shock in the early 1980s (Figure 7, blue). This negative structural break in the model of 0-700m OHC in 1982/1983 coincides with the volcanic eruption of “El Chichón”, suggesting that the volcanic impact may not have been modelled appropriately. Indeed, volcanic forcing appears to act more as temporary structural breaks than continuous series (see Pretis et al. 2016 and Schneider et al. 2017), and further may explain some of the auto-correlation in the residuals of the EBM-CVAR models. Volcanic forcing ($F_{t,\text{Volc.}}$) closely resembles an impulse, where the first difference of volcanic forcing ($\Delta F_{t,\text{Volc.}}$) appears similar to a transitory shock dummy e.g. ($\ldots, 0, 0, 1, -1, 0, 0, \ldots$) – see Figure 2. This break-like behaviour likely drives the rejection of normality in Model C. There are multiple different ways in which volcanic forcing can be modelled within the CVAR specification. The aggregate forcing series $F_t$ could be disaggregated and separated into exogenous and endogenous forcing series, alternatively eruptions could be treated as transitory shocks. Here I follow the latter approach in a

\textsuperscript{14}The literature on the detection of structural breaks is vast and many alternatives are available – see Perron 2006.
fourth model specification (reported as Model V – with full results given in appendix 7.4, Table 5) where volcanic aerosol forcing are specified as breaks to avoid concerns of estimating climate sensitivities based on volcanic forcing in general (see e.g Lindzen & Giannitsis 1998). Treating volcanic eruptions as closely resembling transitory breaks, I re-estimate the CVAR EBM with 2-lags, where the forcing series \( \tilde{F}_t = F_t - F_{t,\text{Volc.}} \) enters the first cointegrating relation, and is itself restricted to be weakly exogenous. The first difference of volcanic forcing, \( \Delta F_{t,\text{Volc.}} \) enters the model unrestrictedly, similar to a transitory shock dummy. The model now passes the diagnostic tests for residual autocorrelation (p=0.21) and normality (p=0.22). The EBM restrictions, including weak-exogeneity of the remaining forcing series \( \tilde{F}_t \), are not rejected (p = 0.08, bootstrap \( p = 0.28 \)). Repeating the application of IIS and SIS for model V confirms that the improved model specification – when volcanic forcing is modelled as transitory breaks, no outliers or shifts are detected in indicator saturation (see Figure 7, purple).

Figure 7: Detected breaks using indicator saturation in model B (blue) and V (purple). Top: model fit using IIS and SIS. Bottom: deviations in the coefficient path – the time varying estimate of the intercept as determined using IIS and SIS, together with approximate dating uncertainty around detected step-shifts.
5.6.2 Climate Feedback and Sensitivity

The system’s response to an increase in emissions is crucial for climate policy and assessment of future climate-related economic impacts. The estimates of the climate response to suddenly increasing emissions and thus concentrations of GHGs (i.e. in the form of a step-shift), is characterised in the EBM by climate feedback $\lambda$ and the estimated model dynamics. A simple measure of the response of the system to increased forcing is given by the ECS defined as the equilibrium temperature response to a step-shift doubling of CO$_2$ concentrations. Estimates of climate feedback and equilibrium climate sensitivity (see Table 2) depend on the model specification, where model mis-specification likely leads to an over-estimation of climate feedback (and thus an under-estimation of climate sensitivity). Figure 8 plots the estimated climate feedback ($\lambda$) and sensitivity (ECS) with approximate confidence intervals.

Using model A which does not control for residual-autocorrelation, $\lambda$ is estimated to be 2.71 ($se=0.51$) Wm$^{-2}C^{-1}$, equivalent to an equilibrium climate sensitivity of 1.37 ($se=0.25$) degrees C for a radiative forcing of 3.7Wm$^{-2}$ for a step-shift doubling of CO$_2$. This is lower than IPCC (IPCC 2013) best estimates of around 3 degrees C. A possible reason for why climate feedback may have been over-estimated (and the uncertainty around it as well as ECS under-estimated) when using observation-based approaches (for example Schwartz (2007) finds ECS to be approximately 1.1 degrees C) can be model mis-specification (e.g apparent through residual autocorrelation) and stochastic trends in the data which are not accounted for when relying on simple correlations. Once residual autocorrelation is corrected for through the use of additional lags (model B) the estimates of climate feedback are lower (and those of ECS higher). Model B (two-lags) and model C (endogenous forcing) estimate $\lambda$ at 2.29 ($se=0.41$) and 2.21 ($se=0.32$) Wm$^{-2}C^{-1}$ respectively, with associated equilibrium climate sensitivities of 1.62 ($se=0.29$) degrees and 1.67 ($se=0.24$) degrees C. There are, however, concerns on estimating $\lambda$ and ECS using volcanic forcing data (Lindzen & Giannitsis 1998). Model V makes an attempt in correcting for this by treating volcanic forcing as transitory breaks, entering the CVAR unrestrictedly in first differences. The climate feedback estimate in model V is notably lower than in models A-C, resulting in a higher estimate of ECS of 2.16 ($se=0.56$) degrees C, also associated with higher uncertainty (see Figure 8, model V, purple). To assess the time needed to reach equilibrium, I consider the impulse-response of a step-shift doubling of CO$_2$. Figure 9 plots the impulse response functions of $T_{m,t}$ and ocean heat content to a step-shift doubling of CO$_2$ concentrations using the estimated CVAR EBM models A (green), B (blue), and V (purple). Surface temperatures converge to the estimated equilibrium climate sensitivity, the response of OHC is consistent with the findings of Joos et al. (2013) in orders of magnitude.

For the preferred Model V, half of the temperature response in the upper component is reached 17 years after the pulse (with 90% of the change after 80 years), while half of the change in OHC (the slower component) is reached after 29 years (with 90% reached after around 100 years). This is compared to the mis-specified models A (and B) which do not sufficiently account for dynamics and volcanic impacts and reach half of the temperature change already one year after the pulse (9 years for Model B), and half of the OHC response after 9 years (15 years for Model B).

**Recursive Estimation** To assess the stability of the model over time I recursively estimate the model starting with a base sample of 20 observations from 1955-1976 onwards, expanding the sample up until 2011. This permits an assessment of how the estimates would have changed over time and can provide insight

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15 ECS for a doubling of CO$_2$ is derived from the steady-state equilibrium of the EBM. Using (4) and (5) in equilibrium it holds that $dT_m/dt = dT_d/dt = 0$ and thus $T_m = F/\lambda$. ECS is then defined as the equilibrium temperature response $T_m$ given a radiative forcing of doubling of CO$_2$: ECS = $F_{2xCO_2}/\lambda$, where $F_{2xCO_2} \approx 3.7Wm^{-2}$.

16 Model C is omitted from the impulse-response analysis as it models forcing endogenously.
whether the recent slowdown in increase in surface temperatures – despite an increase in GHG concentrations (the “warming hiatus”) – affects the estimates of the model parameters. Figure 10 graphs the recursively estimated climate feedback and ECS over the sample period for the different model specifications. Consistent with the findings of Otto et al. (2013) (who use a single component EBM), the point-estimates of ECS and climate feedback are relatively stable over the sample prior to 2003. However, recursive estimates in the CV AR indicate a dip in the recursively estimated feedback in 2003, where estimates including the time period thereafter yield a higher ECS with overall higher uncertainty as indicated by the $\pm 2$ standard error interval. This change in 2003 is likely driven by sudden increased ocean heat uptake during this time period (see Figure 1). The inclusion of the hiatus period associated with little increase in surface temperatures, suggests higher uncertainties around the estimates of ECS compared to the 1980s and 1990s.

Figure 8: Estimated climate feedback ($\lambda$, panel A left) and equilibrium climate sensitivity (ECS, panel B right) – the equilibrium temperature response for a doubling of CO$_2$ for models A (theory), B (2-lags), and C (2-lags & endogenous forcing). Model V models volcanic forcing as breaks. MLE marks the maximum likelihood estimate, and shading denotes approximate confidence intervals (CI) from 67% (±1 standard error, darkest) to 99% (lightest). The IPCC “likely” range is shown as gray shading for ECS.

Figure 9: Impulse response of surface temperatures ($T_m$, panel a left) and 0-700m ocean heat content (panel b, middle) to a step-shift doubling of CO$_2$ concentrations at $t = 0$ (panel c, right) using the CV AR estimated EBMs A, B, and V.

17The wide range of the $\pm 2$ standard error interval for ECS in Figure 10 primarily stems from the non-linear transformation of the climate feedback estimate $\lambda$. The estimate for $\lambda$ in model V falls close to zero in 2001-2003 leading to very high values (and uncertainty) of ECS which is a non-linear transformation of $\lambda$. 
Figure 10: Recursive Estimates of Climate Feedback $\lambda$ (top, panel A) and Equilibrium Climate Sensitivity (ECS, bottom, panel B). Top panel shows the recursively estimated climate feedback $\lambda$ starting with a base sample from 1956-1976 extended up to 1956-2011 for models A, B, C, and V. Bottom panel shows the corresponding ECS for a radiative forcing of $3.7 \ W/m^2$ for a doubling of CO$_2$. Purple shading denotes the approximate 95% confidence interval for model V. Standard errors for ECS are derived using the $\delta$-method.

**Transient Climate Response**  ECS describes an equilibrium response to a step-shift doubling of CO$_2$ concentrations outcome, an alternative measure of the climate system’s response to a change in forcing is the transient climate response (TCR) which describes the temperature change at the time of doubling CO$_2$ concentrations when concentrations are increased at the rate of 1% per year (see e.g. Otto et al. 2013). While the ECS can be inferred directly from the first cointegrating vector in the CVAR EBM, the TCR also depends on the short-run dynamic structure of the model. Here I report the estimates$^{18}$ of the TCR for the CVAR EBM when forcing is weakly (does not adjust to the cointegrating vectors) and strongly exogenous (no feedback), the unrestricted constant is dropped (it is insignificant in the equilibrium correction model), and short-run forcing variables only entering the upper-component temperature equation. The TCR estimates$^{19}$ are given by: 1.24°C for model A, 1.38°C for model B, and 1.38°C for the preferred model V. To compare the CVAR estimates of the TCR to those of Otto et al. (2013), I repeat the above analysis for a radiative forcing value for doubling of CO$_2$ of $3.44 W/m^2$. This yields TCR estimates of 1.15 (Model A), 1.28 (Model B), and 1.28 (Model V), close to the preferred estimate of Otto et al. (2013) of around 1.3°C.

### 5.7 Heat Capacity Estimates

The range of economic climate-damage projections due to uncertainties on physical parameters beyond the climate sensitivity in EBMs are not normally considered in IAMs (Calel & Stainforth 2016). Nordhaus (2016) considers uncertainties on parameters in the carbon cycle of the DICE model as well as the ECS.

$^{18}$Computed here for a 1% annual increase in CO$_2$ concentrations which implies a linear increase in forcing up until the time of doubling at a value of $3.71 W/m^2$.

$^{19}$Model C endogenizes forcing and so cannot be directly used to condition on a prescribed forcing path to compute TCR due to the presence of feedback.
However, does not examine the impact of uncertainties on heat capacity in the underlying EBM. My approach provides a way to jointly estimate climate sensitivity as well as the heat capacities entering the models used to project economic damages. The lower component heat capacity $C_d$ is given by the CVAR mapping as $\beta_{2,2} = -1/C_d$. Estimates of the heat capacity of the lower component using observational data and a two-component EBM estimated as a CVAR from model A suggests an effective ocean 0-700m heat capacity of 22.72 (se=2.70) W year $m^{-2}C^{-1}$. This increases to 24.39 (se=2.38) when two lags are considered (models B) and when we allow forcing to be endogenous (Model C), and 23.8 (se=2.26) when volcanic forcing is modelled unrestrictedly (model V) – see Figure 11.

Thus, when estimating the EBM as a system and taking the time series properties into account, the raw estimate is higher than found by Schwartz (2012), who finds the heat capacity of the lower component up to 700m to be 14.1 W year $m^{-2}C^{-1}$ using simple OLS regression and suggests that his finding is a strong under-estimation. We may still under-estimate the total effective heat capacity of the system, mostly due to omission of the deeper ocean (>700m). Equally, other heat sinks are omitted in this simple model. If we follow Schwartz’s approach of correcting the 0-700m estimate upwards (by 30% for deeper ocean, and 19% for other heat sinks), this yields an estimate of 36.34 W year $m^{-2}C^{-1}$ (using models B, C). To assess the dependence between estimates of heat capacities and climate feedback $\lambda$ as well as the ECS, I bootstrap the CVAR model and generate a sample of the cointegrating vectors. I find a negative relationship between parameter estimates of the heat capacity of the lower component $C_d$ and climate feedback $\lambda$ (see appendix 7.6). The quantified relationship between parameter estimates can help alleviate arbitrarily imposed covariances in EBMs in CVARs (Calel & Stainforth 2016).

There is no one-to-one mapping between upper-component heat capacity $C_m$ (the amount of energy needed to change the temperature) and parameters in the discrete CVAR representation of the continuous-time EBM. However, when imposing additional restrictions using the Euler-discretisation we can obtain an estimate of $C_m$ (reported in supplementary Table 4). Crucially, the estimates show high uncertainties around heat capacities (Figure 11), a 95% confidence interval suggests that the true value of $C_d$ falls within a range of ±20%, and ±50% for $C_m$ for model V. These ranges can be used in the calibration of IAMs.

Figure 11: Estimated heat capacities for Models A,B,C, and V. Left panel (a) shows the estimated heat capacity of the upper (mixed) component $C_m$ which is identified when additional restrictions from section 4.2 are imposed. Right panel (b) shows the estimates of the heat capacity of the deep component $C_d$ without additional restrictions on adjustment coefficients. MLE marks the maximum likelihood estimate, and shading denotes approximate confidence intervals (CI) from 67% (±1 standard error, darkest) to 99% (lightest).
6 Conclusion

I showed how econometric system models of climate can be estimated in ways consistent with the underlying physics. Doing so, I provide a basis for empirical models of climate-economic systems, which are vital to correctly assess the economic costs of climate change. Formally, I demonstrated the mathematical equivalence of a two-component energy balance model (EBM) to a cointegrated system that can be mapped to a cointegrated vector autoregression (CVAR) in discrete time. The equivalence provides a physical basis for the use of econometric CVARs in climate-economic research, placing the entire tool-kit of CVARs at the disposal of energy balance models. The resulting system-based estimation can be used directly to quantify parameter uncertainties in projections of the economic damages from climate change in integrated assessment models.

In an application, I estimated a two-component EBM using global mean surface temperature anomalies, 0-700m ocean heat content anomalies, and aggregates of radiative forcing showed the model not to be rejected. Standard and bootstrap cointegration tests confirmed that the time series form two stationary relations and cointegrate in ways consistent with theory. Individual parameters were statistically significant with theory-consistent signs and independent satellite observations closely tracked the estimates of the first cointegrating vector. CVAR estimation of a two-component EBM showed that point estimates of the temperature responses to increased forcing (e.g., from CO$_2$ concentrations) were relatively stable over the sample of observations, although, the recent slowdown in warming is associated with higher parameter uncertainties. I found that model mis-specification (apparent through residual autocorrelation) results in high values of the observationally-determined climate feedback, and in turn, may lead to an under-estimation of the equilibrium climate sensitivity. By improving model specification through the detection and modelling of structural breaks, I showed that uncertainties on key parameters beyond climate sensitivity (such as heat capacities) are high and need to be accounted for in economic impact projections.

This paper provides a foundation for the use of CVARs – which are commonly used to model the macroeconomy – in estimating climate econometric systems consistent with physical laws. This allows socio-economic models to be linked to climate models by estimating a system where the forcing series are endogenized and driven by economic activity. More generally, this enables the coupling of empirical climate models with empirical macroeconomic systems to address concerns with existing climate-economic models (Burke et al. 2016, Pindyck 2013, N. Stern 2013, N. Stern 2016). Furthermore, using econometric methods to identify shifts in models (Castle et al. 2015, Estrada et al. 2013, Pretis & Allen 2013) allows an assessment of the impacts of policy effectiveness and natural shocks. A system model that is consistent with physical laws also permits the feedback of climate variables into the economy to be estimated, opening the door to data-driven system models of interactions between the economy and climate.
Acknowledgements

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7 Appendix

7.1 Further Results on the Continuous to Discrete Mapping

The energy balance model in (11) is given as:

\[ dY = \Pi Y dt + DdW \]  

(31)

where \( \Pi = \alpha \beta' \). The solution to the Ornstein-Uhlenbeck process can be found using a change-of-variable approach. Define \( Z \) such that:

\[ Z = \exp (-\Pi t)Y \]  

(32)

where \( \exp (-\Pi t) \) denotes the matrix exponential of \( -\Pi t \), where for a \( n \times n \) matrix \( A \) \( \exp (A) \) is defined as:

\[ \exp (A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \ldots \]  

(33)

with the inverse of \( \exp (A) \) given by \( \exp (A)^{-1} = \exp (-A) \). Pre-multiplying (32) by \( \exp (\Pi t)^{-1} \) yields:

\[ Y = \exp (\Pi t)Z \]  

(34)

Using (32) we can write \( dZ \) as:

\[ dZ = -\Pi \exp (-\Pi t)Y dt + \exp (-\Pi t)dY \]  

(35)

\[ = -\Pi \exp (-\Pi t)Y dt + \exp (-\Pi t) (\Pi Y dt + DdW) \]  

(36)

\[ = \exp (-\Pi t)DdW \]  

(37)

Integrating from 0 to \( t \) yields:

\[ Z = Z_0 + \int_0^t \exp (-\Pi u)DdW_u \]  

(38)

Substituting for \( Z \) in (34) provides the solution to the Ornstein-Uhlenbeck process as:

\[ Y = \exp (\Pi t)Y_0 + \exp (\Pi t) \int_0^t \exp (-\Pi u)DdW_u \]  

(39)

\[ = \exp (\Pi t) \left( Y_0 + \int_0^t \exp (-\Pi u)DdW_u \right) \]  

(40)

Discrete observations \( Y_t \) of \( Y \) at a frequency of one then follow a VAR process as:

\[ Y_t = AY_{t-1} + \epsilon_t \]  

(41)

where \( A = \exp (\Pi), \) and \( \epsilon_t \sim N (0, \Sigma) \). In equilibrium correction form the VAR is written as:

\[ \Delta Y_t = PY_t + \epsilon_t \]  

(42)

with

\[ P = A - I = \exp (\Pi) - I = \exp (\alpha \beta') - I = \alpha \kappa \beta' \]  

(43)
where $\kappa$ is a $(r \times r)$ matrix $\kappa = (\beta' \alpha)^{-1} \left[ \exp (\beta' \alpha) - I \right]$. Proof:

\[
\begin{align*}
P &= \exp (\alpha \beta') - I \\
    &= I + \alpha \beta' + \alpha \beta' \alpha \beta' \frac{1}{2!} + \cdots - I \\
    &= \alpha \left( I + \beta' \alpha \frac{1}{2!} + \cdots \right) \beta' \\
    &= \alpha \left( (\beta' \alpha)^{-1} \left[ \beta' \alpha + \beta' \alpha \beta' \alpha \frac{1}{2!} + \cdots \right] \right) \beta' \\
    &= \alpha \left( (\beta' \alpha)^{-1} \left[ I + \beta' \alpha + \beta' \alpha \beta' \alpha \frac{1}{2!} + \cdots - I \right] \right) \beta' \\
    &= \alpha \left( (\beta' \alpha)^{-1} \left[ \exp (\beta' \alpha) - I \right] \right) \beta' \\
\end{align*}
\]

(44)

For additional results see Kessler & Rahbek (2004).

### 7.2 Eigenvalues of the Companion Matrix

To further assess dynamic stability, Table 3 reports the moduli of the eigenvalues of the companion matrix of models A and B & C for the unrestricted VAR and reduced rank estimates ($r=2$) without the EBM theory restrictions imposed. The indicative results of the eigenvalue analysis suggest a single unit root: one eigenvalue lies on the unit circle, all other eigenvalues are within the unit circle. There is no evidence of an explosive process and the system does not appear to be I(2) – once a reduced rank of two is imposed, all other eigenvalues are well within the unit circle.

<table>
<thead>
<tr>
<th>Table 3: Eigenvalues of the companion matrix of models A and B &amp; C for the unrestricted VAR and reduced rank ($r=2$) estimates (without EBM theory restrictions).</th>
</tr>
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<tbody>
<tr>
<td><strong>Model A (1 × 3 roots)</strong></td>
</tr>
<tr>
<td>VAR(1)</td>
</tr>
<tr>
<td>r=2 1.00</td>
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</table>

<table>
<thead>
<tr>
<th>Models B &amp; C (2 × 3 roots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAR(2)</td>
</tr>
<tr>
<td>r=2 1.00</td>
</tr>
</tbody>
</table>
7.3 Estimation Results when the model is interpreted in discrete time

If the discrete time approximation is considered, then the EBM implies additional restrictions on the adjustment coefficients and provides further estimates of the structural parameters. These restrictions are: lower component temperatures only adjust to the second cointegrating vector, \( \tilde{\alpha}_{2,1} = 0 \); the adjustment of the lower component to the second cointegrating vector \( \tilde{\alpha}_{2,2} \) equals the coefficient \( \gamma \) determining the rate of heat transfer. This provides a cross-equation over-identifying restriction since \( \gamma \) is also identified in the first cointegrating relation, therefore \( \tilde{\alpha}_{2,1} = -\tilde{\alpha}_{2,2}\tilde{\alpha}_{1,1} = \gamma/C_m \). Upper component heat capacity is given through \( \tilde{\alpha}_{1,1} = 1/C_m \).

I re-estimate models A-C, and V, using the restrictions in (22), the models with additional restrictions are denoted as A*-C* and V*. Estimation results when the additional restrictions on \( \alpha \) are imposed are provided here in Table 4 for models A* (1-lag, EBM theory), B* (2-lags), C* (2-lags, endogenous forcing), and model V* (2-lags unrestricted volcanic forcing). The additional restrictions on \( \alpha \) here can also be motivated by the fact that the adjustment coefficients \( \tilde{\alpha}_{2,1} \) of ocean heat to the first cointegrating vector in Table 2 in section 5 are insignificant.

Comparing Table 2 to 4 there is little difference in estimation results of the models with additional \( \alpha \) restrictions relative to those used in section 5. Estimates of ECS are generally higher with additional \( \alpha \) restrictions imposed, with Models C* and V* estimates for ECS of 1.76 (0.29)K and 3.10 (1.72)K respectively.

Given the structural interpretation of the adjustment coefficients here, estimates for \( C_m \) can be obtained using \( \tilde{\alpha}_{1,1} = 1/C_m \). Theory consistent, the effective heat capacity of the mixed component (here proxied through the use of global mean surface temperatures) is lower than that of the deeper component. The estimates of 12.19 (2.68) W year \( m^{-2}C^{-1} \) for model A*, (8.33 for models B* and 11.11 for model C*) are relatively close in orders of magnitude to the lower component heat capacity. This result likely stems from the fact that the deep component in the present application does not directly represent the deep ocean, but rather only the first 700m. Temperatures for this range are therefore likely to be close to the mean surface temperature.
### Table 4: Appendix Model – EBM Cointegration Model Parameter Estimates with additional α restrictions

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Vector 1 (Mixed)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \beta_{1,1} )</td>
<td>-2.57 (0.94)**</td>
<td>-1.61 (0.77)*</td>
<td>-2.10 (0.35)**</td>
<td>-1.20 (0.67)</td>
</tr>
<tr>
<td>( \beta_{1,2} )</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>( \beta_{1,3} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Adjustment**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{1,1} )</td>
<td>0.082 (0.018)**</td>
<td>0.12 (0.03)*</td>
<td>0.09 (0.02)**</td>
</tr>
<tr>
<td>( \alpha_{2,1} )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \alpha_{3,1} )</td>
<td>-</td>
<td>-</td>
<td>-0.55 (0.13)**</td>
</tr>
</tbody>
</table>

| Vector 2 (Deep) |          |           |                |                 |
| \( \beta_{2,1} \) | 1        | 1         | 1              | 1               |
| \( \beta_{2,2} \) | -0.043 (0.005)** | -0.037 (0.005)** | -0.041 (0.004)** | -0.038 (0.005)** |
| \( \beta_{2,3} \) | -        | -        | -              | -               |

**Adjustment**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_{1,2} )</td>
<td>-0.52</td>
<td>-0.65</td>
<td>-0.57</td>
</tr>
<tr>
<td>( \alpha_{2,2} )</td>
<td>6.28 (1.24)**</td>
<td>5.31 (1.14)**</td>
<td>6.20 (1.53)**</td>
</tr>
<tr>
<td>( \alpha_{3,2} )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

**LR Test of Restrict.**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi^2(4), C: \chi^2(3) )</td>
<td>12.87 [p=0.012]*</td>
<td>19.45 [p=0.001]**</td>
</tr>
</tbody>
</table>

**Diagnostic Tests**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>AR (1-2) F-Test</td>
<td>2.82 [p=0.01]**</td>
<td>1.03 [p=0.42]</td>
<td>0.89 [p=0.59]</td>
</tr>
<tr>
<td>Normality ( \chi^2(4), C: \chi^2(6) )</td>
<td>1.25 [p=0.87]</td>
<td>5.57 [p=0.23]</td>
<td>19.34 [p=0.004]**</td>
</tr>
<tr>
<td>Observations T</td>
<td>56</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>log-likelihood</td>
<td>-74.46</td>
<td>-61.42</td>
<td>-53.36</td>
</tr>
</tbody>
</table>

**EBM Estimates**

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda (Wm^{-2}C^{-1}) )</td>
<td>2.57 (0.94)</td>
<td>1.61 (0.77)</td>
<td>2.10 (0.35)</td>
</tr>
<tr>
<td>( \text{ECS (deg. C)} )</td>
<td>1.44 (0.53)†</td>
<td>2.30 (1.10)†</td>
<td>1.76 (0.29)†</td>
</tr>
<tr>
<td>( C_m (W \text{ year } m^{-2}C^{-1}) )</td>
<td>12.19 (2.68)†</td>
<td>8.33 (2.08)†</td>
<td>11.11 (2.47)†</td>
</tr>
<tr>
<td>( C_d (W \text{ year } m^{-2}C^{-1}) )</td>
<td>23.25 (2.70)†</td>
<td>27.02 (3.65)†</td>
<td>24.39 (2.38)†</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>6.28 (1.24)</td>
<td>5.31 (1.14)</td>
<td>6.20 (1.53)</td>
</tr>
</tbody>
</table>

Model estimates based on CVAR estimation. Standard errors are given in parentheses () while p-values are reported in brackets [ ]. Standard errors are provided where available. If no standard errors are reported, then parameter is restricted or derived from estimated model parameters. * indicates significance at 5%, ** indicates significance at 1%. Standard errors derived using \( \delta \)-method are marked using †. Left column specifies parameters, right columns shows estimation results. Dash - marks imposed restriction, specifically the restrictions based on the EBM are: \( \beta_{1,2} = 0, \beta_{1,3} = 1, \alpha_{2,1} = 0, \alpha_{3,1} = 0, \beta_{2,1} = 1, \alpha_{1,2} = -\alpha_{2,2}\alpha_{1,1}, \alpha_{3,2} = 0. \)
7.4 Additional Results on Modelling Volcanic Forcing as Breaks

Table 5 provides the estimation results where model V refers to the model including the first difference of volcanic forcing unrestrictedly.

Table 5: EBM Cointegration Model Parameter Estimates - Unrestricted Volcanic Forcing

<table>
<thead>
<tr>
<th>EBM/CVAR Model</th>
<th>V: Volc. Breaks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coint. Relations V: Volc. Breaks</td>
<td></td>
</tr>
<tr>
<td>Vector 1 (Mixed)</td>
<td></td>
</tr>
<tr>
<td>$\beta_{1,1}$</td>
<td>-1.71 (0.44)**</td>
</tr>
<tr>
<td>$\beta_{1,2}$</td>
<td>-</td>
</tr>
<tr>
<td>$\beta_{1,3}$</td>
<td>1</td>
</tr>
<tr>
<td>Adjustment</td>
<td></td>
</tr>
<tr>
<td>$\hat{\alpha}_{1,1}$</td>
<td>0.21 (0.06)**</td>
</tr>
<tr>
<td>$\hat{\alpha}_{2,1}$</td>
<td>0.09 (0.81)</td>
</tr>
<tr>
<td>$\hat{\alpha}_{3,1}$</td>
<td>-</td>
</tr>
<tr>
<td>Vector 2 (Deep)</td>
<td></td>
</tr>
<tr>
<td>$\beta_{2,1}$</td>
<td>1</td>
</tr>
<tr>
<td>$\beta_{2,2}$</td>
<td>-0.042 (0.004)**</td>
</tr>
<tr>
<td>$\beta_{2,3}$</td>
<td>-</td>
</tr>
<tr>
<td>Adjustment</td>
<td></td>
</tr>
<tr>
<td>$\hat{\alpha}_{1,2}$</td>
<td>-0.65 (0.17)**</td>
</tr>
<tr>
<td>$\hat{\alpha}_{2,2}$</td>
<td>6.43 (2.04)**</td>
</tr>
<tr>
<td>$\hat{\alpha}_{3,2}$</td>
<td>-</td>
</tr>
<tr>
<td>LR Test of Restrict.</td>
<td>5.09 [p=0.08]</td>
</tr>
<tr>
<td>$\chi^2(2)$, C: $\chi^2(1)$</td>
<td>Bootstrap (B=199) 5.09 {cr5% =10.2} [p=0.28]</td>
</tr>
<tr>
<td>Diagnostic Tests</td>
<td></td>
</tr>
<tr>
<td>AR (1-2) F-Test</td>
<td>1.37 [p=0.21]</td>
</tr>
<tr>
<td>Normality $\chi^2(4)$, C: $\chi^2(6)$</td>
<td>5.75 [p=0.22]</td>
</tr>
<tr>
<td>Observations T</td>
<td>55</td>
</tr>
<tr>
<td>log-likelihood</td>
<td>84.11</td>
</tr>
<tr>
<td>EBM Estimates</td>
<td></td>
</tr>
<tr>
<td>$\lambda$ (W m$^{-2}$ C$^{-1}$)</td>
<td>1.71 (0.44)</td>
</tr>
<tr>
<td>ECS (deg. C)</td>
<td>2.16 (0.56)$^\dagger$</td>
</tr>
<tr>
<td>$C_m$ (W year m$^{-2}$ C$^{-1}$)</td>
<td>-</td>
</tr>
<tr>
<td>$C_d$ (W year m$^{-2}$ C$^{-1}$)</td>
<td>23.8 (2.26)$^\dagger$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-</td>
</tr>
</tbody>
</table>

Model estimates based on CVAR estimation. Bootstrap test for restrictions (Cavaliere et al. 2015) with B replications, 5% bootstrap critical values reported as ‘{cr5% =...}’. Standard errors are given in parentheses () while p-values are reported in brackets [ ]. Standard errors are provided where available. If no standard errors are reported, then parameter is restricted or derived from estimated model parameters. * indicates significance at 5%, ** indicates significance at 1%. Standard errors derived using $\delta$-method are marked using $^\dagger$. Left column specifies parameters, right columns shows estimation results. Dash - marks imposed restriction and no identification in the case of the structural EBM parameters given the theoretical result of $\hat{\alpha} = \alpha \kappa$. 
7.5 Univariate Unit Root Tests

While the Johansen cointegration procedure does not require pre-testing for unit-roots, for completeness I provide the results of uni-variate augmented Dickey-Fuller (ADF) (Dickey & Fuller 1981) unit root tests on both levels and first differences for global mean surface temperatures, 0-700m ocean heat content, total radiative forcing, total radiative forcing excluding stratospheric aerosols (volcanic forcing) and well-mixed greenhouse gases (WMGHGs). The null hypothesis $H_0$ is that the series has a unit root, rejecting $H_0$ suggests no unit-root non-stationarity. D-lag specifies the number of lags included in the ADF test where choice of lag length is based on the lowest AIC. A constant is included in the ADF test specification. Test outcomes: ** indicates rejection of $H_0$ at 1% and * at 5%.

Table 6: GIS Temperature ADF Unit Root Tests

<table>
<thead>
<tr>
<th>D-Lag</th>
<th>$t$-adf</th>
<th>AIC</th>
<th>D-Lag</th>
<th>$t$-adf</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-0.1707</td>
<td>-4.252</td>
<td>3</td>
<td>-4.329**</td>
<td>-4.263</td>
</tr>
<tr>
<td>2</td>
<td>-0.5416</td>
<td>-4.229</td>
<td>2</td>
<td>-6.850**</td>
<td>-4.291</td>
</tr>
<tr>
<td>1</td>
<td>-1.168</td>
<td>-4.161</td>
<td>1</td>
<td>-8.252**</td>
<td>-4.262</td>
</tr>
<tr>
<td>0</td>
<td>-1.949</td>
<td>-4.1</td>
<td>0</td>
<td>-10.26**</td>
<td>-4.172</td>
</tr>
</tbody>
</table>

Table 7: 0-700m OHC ADF Unit Root Tests

<table>
<thead>
<tr>
<th>D-Lag</th>
<th>$t$-adf</th>
<th>AIC</th>
<th>D-Lag</th>
<th>$t$-adf</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.02</td>
<td>0.6989</td>
<td>3</td>
<td>-3.065*</td>
<td>0.6295</td>
</tr>
<tr>
<td>2</td>
<td>0.8619</td>
<td>0.6704</td>
<td>2</td>
<td>-5.301**</td>
<td>0.6821</td>
</tr>
<tr>
<td>1</td>
<td>0.3142</td>
<td>0.7122</td>
<td>1</td>
<td>-7.107**</td>
<td>0.6468</td>
</tr>
<tr>
<td>0</td>
<td>-0.1528</td>
<td>0.7377</td>
<td>0</td>
<td>-9.019**</td>
<td>0.675</td>
</tr>
</tbody>
</table>

Table 8: Total Radiative Forcing ADF Unit Root Tests

<table>
<thead>
<tr>
<th>D-Lag</th>
<th>$t$-adf</th>
<th>AIC</th>
<th>D-Lag</th>
<th>$t$-adf</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-1.655</td>
<td>-1.476</td>
<td>3</td>
<td>-4.969**</td>
<td>-1.446</td>
</tr>
<tr>
<td>2</td>
<td>-1.789</td>
<td>-1.515</td>
<td>2</td>
<td>-5.400**</td>
<td>-1.457</td>
</tr>
<tr>
<td>1</td>
<td>-2.867</td>
<td>-1.454</td>
<td>1</td>
<td>-7.166**</td>
<td>-1.488</td>
</tr>
<tr>
<td>0</td>
<td>-2.299</td>
<td>-1.422</td>
<td>0</td>
<td>-6.182**</td>
<td>-1.335</td>
</tr>
</tbody>
</table>
Table 9: Total Forcing (excluding Stratospheric Aerosols) ADF Unit Root Tests

<table>
<thead>
<tr>
<th>D-Lag</th>
<th>t-adf</th>
<th>AIC</th>
<th>D-Lag</th>
<th>t-adf</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-0.5852</td>
<td>-6.789</td>
<td>3</td>
<td>-4.764**</td>
<td>-6.917</td>
</tr>
<tr>
<td>2</td>
<td>-0.7223</td>
<td>-6.763</td>
<td>2</td>
<td>-3.794**</td>
<td>-6.821</td>
</tr>
<tr>
<td>1</td>
<td>-0.7475</td>
<td>-6.801</td>
<td>1</td>
<td>-3.257*</td>
<td>-6.791</td>
</tr>
<tr>
<td>0</td>
<td>-0.7114</td>
<td>-6.43</td>
<td>0</td>
<td>-3.543*</td>
<td>-6.829</td>
</tr>
</tbody>
</table>

Table 10: Well-Mixed Greenhouse Gases (WMGHGs) ADF Unit Root Tests

<table>
<thead>
<tr>
<th>WMGHGs</th>
<th>D-Lag</th>
<th>t-adf</th>
<th>AIC</th>
<th>D-Lag</th>
<th>t-adf</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-0.9117</td>
<td>-8.967</td>
<td>3</td>
<td>-1.793</td>
<td>-8.997</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-0.6269</td>
<td>-8.902</td>
<td>2</td>
<td>-2.114</td>
<td>-8.989</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-0.5831</td>
<td>-8.903</td>
<td>1</td>
<td>-2.898</td>
<td>-8.933</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-0.6148</td>
<td>-8.619</td>
<td>0</td>
<td>-3.980**</td>
<td>-8.935</td>
<td></td>
</tr>
</tbody>
</table>
7.6 Relation between Estimates of Heat Capacities and Climate Feedback

To investigate the covariance between estimates of heat capacities and climate feedback I bootstrap the CVAR model to obtain a sample of EBM cointegration parameter estimates. I focus on $\lambda$, the ECS, and the heat capacity of the lower component, $C_d$, which can be estimated using the cointegrating vectors. I create 5000 bootstrap samples and for each bootstrap sample estimate the cointegrating vectors under the imposed reduced rank of two. Figure 12 shows (using Model B) the relationship between estimated values of $C_d$ and $\lambda$, as well as between $C_d$ and the ECS. To quantify the degree of dependence between parameters, I estimate a simple linear (OLS) model between the parameters across the bootstrap sample, as well as a robust MM estimator (Yohai 1987, Rousseeuw et al. 2009) to account for outlying bootstrap draws.\(^{20}\) The results show that there is a significant negative relationship between $C_d$ and $\lambda$, and a subsequent positive relationship between $C_d$ and ECS.

Figure 12: Relation between estimated heat capacity ($C_d$) and climate feedback $\lambda$ (left, panel $a$), and equilibrium climate sensitivity (right, panel $b$). Scatter plots show the parameter estimates obtained over 5000 bootstrap samples for Model B. Regression lines shown using OLS (dashed) as well as a robust estimator to remove outlying bootstrap observations. The p-values shown in square brackets correspond to tests on the slope coefficients of $C_d$ on $\lambda$ and $ECS$ respectively.

\(^{20}\)An alternative to bootstrapping is to use the estimated covariance matrix of parameters in the CVAR.
7.7 Residual Plots

Figure 13 plots the scaled model residuals from models A-C (see section 5) and model V (see section 7.4). The outlying observations in 1981/1982 are associated with strong volcanic forcing from the “El Chichón” eruption. Model V where volcanic forcing is modelled as transitory breaks alleviates some of these problems.

![Figure 13: Scaled model residuals from models A-C (section 5) and model V (section 7.4)](image)