Eliciting the just-noticeable difference*

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October, 2017

JOB MARKET PAPER

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Abstract

This paper provides a method of eliciting consumer preference from observable choices when the agent can not perfectly discern between bundles that are similar. The evidence from psychophysics suggest that people are unable to discriminate between alternatives unless the options are sufficiently different. Since such a behaviour requires for indifferences to be non-transitive, it can not be reconciled with utility maximisation. We approach the issue of noticeable differences by modelling consumer choice using *semiorders*. Following the tradition of Afriat (1967), we introduce a necessary and sufficient condition under which a finite dataset of consumption bundles and corresponding budget sets can be rationalised with such a relation. The result can be thought of as an extension of the well-known Afriat's theorem to semiorder, rather than utility maximisation. Our approach is constructive and allows us to infer both the "true" preferences of the consumer (i.e., as if perfect discrimination were possible), as well as the value of the just-noticeable difference that is sufficient for the agent to discern between alternatives. Furthermore, we argue that the latter constitutes a natural, behaviourally founded measure of departures from utility maximisation. We conclude by applying our method to household-level scanner panel data of food expenditures.

Keywords: revealed preference, testable restrictions, semiorder, just-noticeable difference, GARP, Afriat's efficiency index, money-pump index

JEL Classification: C14, C60, C61, D11, D12

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*I am grateful to John Quah for his invaluable help during writing this paper. Moreover, I would like to thank Abi Adams, Miguel Ballester, Laurens Cherchye, Ian Crawford, Federico Echenique, John Hey, Ole Jann, Josh Lanier, Paulo Natenzon, Ludovic Renou, Bram de Rock, David Ronayne, Bart Smeulders, Tomasz Strzalecki, and André Veiga, as well as the seminar participants at Time, Uncertainties and Strategies in Paris (December 2016), Johns Hopkins University (May 2017), European Workshop in General Equilibrium Theory in Salamanca (June 2017), Bounded Rationality in Choice in London (June 2017), Summer Workshop in Economic Theory in Paris (June 2017), European Meeting of the Econometric Society in Lisbon (August 2017), Transatlantic Theory Workshop in Paris (September 2017), and University of St Andrews (October 2017) for helpful comments and suggestions. Finally, I am indebted to Chris Chambers for the encouragement and early discussions on this project.

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1 Introduction

The evidence from psychophysics suggest that people are unable to discriminate between two physical stimuli unless their intensities are sufficiently different.[1] A similar idea was applied by Luce (1956) to the problem of consumer choice. Referring to his classical thought experiment, suppose an agent is presented with two cups of coffee containing n and $(n+1)$ grains of sugar, respectively. Given that the two commodities are indistinguishable, for any n, we would expect the consumer to behave as if she were indifferent between the options, regardless of the actual taste for sugar. Since utility maximisation requires for indifferences to be transitive, it could explain such observations only if we assumed that the consumer was indifferent between any two amounts of sugar in her coffee. However, such an approach would be neither realistic nor useful. Specifically, the revealed indifference between the cups need not exhibit the actual preference; rather, it might be a result of the consumer's inability to discern between the goods.

The aim of this paper is to provide a method of inferring consumer preference from observable choices when the agent is unable to discriminate perfectly between alternatives. Moreover, we are interested in eliciting the just-noticeable difference that is sufficient for the decision maker to distinguish between options. In particular, we claim that the latter serves as an informative measure of departures from utility maximisation.

We consider our research question to be important for three main reasons. First of all, even though the theory of utility maximisation is being applied throughout economics, the empirical evidence suggest that observable choices of consumers are generally inconsistent with this hypothesis. Introducing the notion of noticeable differences provides an intuitive, behavioural explanation for such violations and allows us to measure their significance by referring to the agent's cognitive inability to perceive small differences among alternatives. Additionally, inferring the "true" preferences enables us to perform welfare analysis and extrapolation as if perfect discrimination between options were possible. Second of all, as shown in Argenziano and Gilboa (2015), noticeable differences may provide a foundation for interpersonal welfare comparisons and utilitarianism. In particular, evaluating noticeable differences for individual consumers may allow us to determine welfare improvements that would exploit heterogeneity in the agent's insensitivity to small changes in allocations. Finally, this question plays a significant role in

[1] See, e.g., Laming (1997) or a short survey in Algom (2001) .

marketing. Manufacturers and marketers endeavour to evaluate the relevant noticeable difference in order to determine changes in bundle sizes or qualities of their products that would be perceived or neglected by consumers. Our result provides a general method of evaluating noticeable differences from observable choices over multi-product bundles. In particular, this enables us to account for complementarities or substitutabilities among commodities when estimating the agent's inability to discern between options.

The main difficulty in our analysis follows from our departure from the model of utility maximisation. This makes the classical tools from the revealed preference analysis no longer applicable. In particular, in order to evaluate noticeable differences and the underlying "true" consumer preference, we need to exploit relations between particular sequences of observable choices, rather than transitive closures of revealed preference relations, as in the standard approach of Afriat (1967) or Varian (1982). Therefore, new arguments have to be constructed in order to properly address our question.

Following Luce (1956) or Beja and Gilboa (1992), we address the problem of noticeable differences by modelling consumer choice using *semiorders*. The formal definition of this notion is postponed until Section 2.[2] Nevertheless, a semiorder can be thought of as an asymmetric binary relation P such that xPy if and only if $u(x) > u(y)+1$, for some utility function u. Thus, option x is strictly preferred to y if and only if it yields a sufficiently higher utility than y, where the strictly positive threshold is normalised to 1. Since its symmetric counterpart I is given by: xIy if and only if $\left|u(x) - u(y)\right| \leq 1$, the latter relation is *not* transitive. In addition, note that any semiorder P is inherently related to a weak order induced by the utility function u.

Our main results are presented in Section 3. We begin by determining the testable implications for semiorder maximisation. Suppose we observe an agent making a choice from ℓ goods. Following the tradition of Afriat (1967), an observation consists of a budget set $B \subset \mathbb{R}^\ell_+$ and a bundle $x \in B$ selected from it. We say that a semiorder P rationalises a finite set $\mathcal{O}$ of such observations, whenever

$$(B, x) \in \mathcal{O} \text{ and } y \in B \ \text{ implies } \ \textit{not } yPx.$$

Without further restrictions, the above notion has no empirical content, as an arbitrary dataset can be rationalised with the semiorder $P = \emptyset$.[3] To obtain any observable

[2]See also Luce (1956), Scott and Suppes (1958), Fishburn (1975), Manders (1981), Beja and Gilboa (1992), Gilboa and Lapson (1995), or more recently Argenziano and Gilboa (2015, 2017).

[3]This is to say that xIy, for all x, $y \in \mathbb{R}^\ell_+$. An analogous issues arises in the case of utility maximi-

implications of the model and capture a notion of noticeable differences, we restrict our attention to a class of semiorders P for which there exists a number $\lambda > 1$ such that

$$\lambda' \geq \lambda \ \text{ implies } \ (\lambda' y) P y, \tag{1}$$

for all non-zero $y \in \mathbb{R}^{\ell}_{+}$. This condition is inspired by the so-called *Weber-Fechner law* which postulates that people can not discriminate between two physical stimuli unless the ratio between their magnitudes exceeds a particular value — the Weber's constant.[4] In our specification, the parameter λ is interpreted as the ratio between sizes of two bundles that is sufficient for the agent to correctly discern between the options. One example of such preferences is discussed in Argenziano and Gilboa (2017), where the authors characterise semiorders P that satisfy: $(x_i, x_{-i})P(y_i, x_{-i})$ if and only if $x_i/y_i > \delta_i$, for some predetermined $\delta_i > 1$, for all $i = 1, \ldots, \ell$. That is, the agent prefers one bundle over another if the ratio between volumes of at least one commodity i exceeds the constant δ_i. In particular, Argenziano and Gilboa show that such relations can be represented with a Cobb-Douglas utility function u, in the sense specified previously. Clearly, such preferences satisfy condition (1) for any $\lambda \geq \min\left\{\delta_i : i = 1, \ldots, \ell\right\}$.

We find condition (1) to be convenient for our analysis. Apart from allowing us to formalise a notion of noticeable differences, the above class of semiorders is appealing for empirical applications. Specifically, by restricting our attention to such relations, we find it possible to construct computationally efficient algorithms that determine whether a set of observations is rationalisable by a semiorder P satisfying the restriction for some $\lambda > 1$. Nevertheless, in Section B.4 of the Online appendix we extend our analysis to other classes of semiorders that admit alternative properties.

Axiom 1 constitutes a necessary and sufficient condition under which a dataset $\mathcal{O}$ is rationalisable by a semiorder P satisfying property (1) for some $\lambda > 1$. We summarise this observation in the Main Theorem. The restriction has a structure similar to the generalised axiom of revealed preference, or GARP for short, introduced in Afriat (1967) and Varian (1982). GARP requires that the dataset admits no revealed preference cycles, where bundle x is revealed preferred to y if $(B, x) \in \mathcal{O}$ and $y \in B$. It is well-established that this is both a necessary and sufficient condition for the set of observations to be

sation. Clearly, any set of observations can be rationalised with a constant utility function u. Unless we impose some additional restrictions on u, e.g., local non-satiation, this model has no testable implications.

[4]Although *Stevens' power law* seems to better explain sensory discrimination than Weber-Fechner's law, the latter is considered to be the best first approximation. See, e.g., Algom (2001).

rationalisable by a locally non-satiated utility function. Along similar lines, the purpose of our condition is to exclude a particular type of cycles from the dataset. However, the class of sequences admissible by our restriction is broader than in the case of GARP. In fact, we argue that $\mathcal{O}$ satisfies the latter if and only it is rationalisable by a semiorder satisfying property (1) for *all* $\lambda > 1$. Thus, GARP is a special case of Axiom 1.

The proof of our main result is constructive and provides a method of formulating a semiorder P, along with the corresponding utility function u, that rationalises the dataset in the aforementioned sense. Since the number of observations is finite, such an ordering need not be unique, nor it has to coincide with the semiorder that originally generated the data. We address this issue in Section 3.4, where we show how to recover (partially) such orderings from observed choices. Specifically, we introduce two revealed preference relations P^* and $\succeq^*$ that are consistent with any semiorder P and the corresponding utility u that generated the observations, respectively.

Representing consumer choice with a semiorder can be interpreted as a utility maximisation with thick indifference curves. In Section 4 we show that, whenever a set of observations $\mathcal{O}$ is rationalisable by a semiorder P satisfying condition (1) for some $\lambda > 1$, there is a utility function u such that, for any observation (B, x) in $\mathcal{O}$:

$$u(z) \geq u(x) \;\text{ implies }\; u(\lambda z) > u(y), \;\text{ for all }\; y \in B \;\text{ and non-zero }\; z \in \mathbb{R}^{\ell}_{+}.$$

Even though the consumer determines the choice using her utility u, she is making a consistent error in evaluating which bundles are superior to x. Despite an available option y such that $u(y) > u(x)$, the choice x remains optimal unless there is some z such that $u(y) \geq u(\lambda z)$ and $u(z) \geq u(x)$. That is, the indifference curves intersecting bundles y and x must be sufficiently far apart for y to be chosen over x. Hence, the agent is maximising her utility as if the indifference curves were thick, where the thickness is determined by the parameter λ. Consider the example in Figure 1.

In fact, the opposite implication is also true. That is, if there is a utility function u rationalising the set of observations in the above sense for some $\lambda > 1$, the data can be supported by a semiorder P satisfying property (1) for λ. Therefore, the two models of consumer choice are equivalent in terms of testable implications.

The above observation highlights three properties of our approach. First of all, by eliciting the utility u, we are able to infer the "true" preferences of the consumer, i.e., as if perfect discrimination were possible. Second of all, we can evaluate the parameter λ

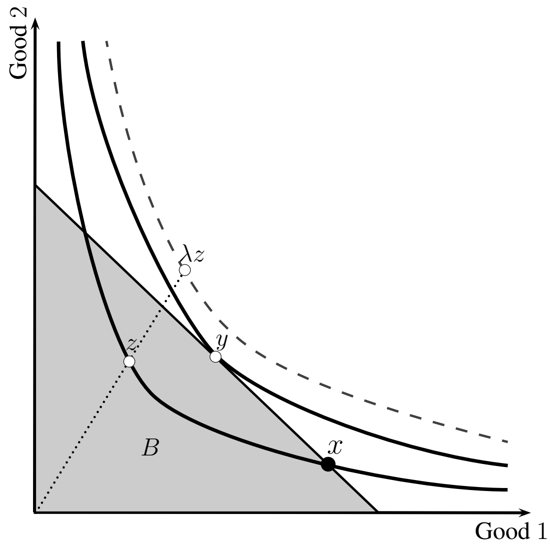

Figure 1: The dashed line depicts set $\{\lambda z : u(z) = u(x)\}$. Although $y \in B$ and $u(y) > u(x)$, we have $u(\lambda z) > u(y)$, for all z with $u(z) \geq u(x)$. Thus, bundle x is optimal.

that determines the thickness of indifference curves that the agent is using when comparing alternatives. Finally, since grater values of λ imply thicker indifference curves, this underlines the sense in which noticeable differences measure departures from rationality.

We explore the latter in Section 5. Let λ_* denote the infimum over all parameters $\lambda > 1$ for which a dataset $\mathcal{O}$ satisfies Axiom 1 — henceforth, the *just-noticeable difference*. We argue that this value, or rather its inverse $(1/\lambda_*)$, constitutes an informative measure of how severely the set of observations violates GARP. We claim that this approach combines the desirable properties of the well-known Afriat's efficiency index and the money-pump index, while responding to the main criticism addressed at those two measures. Additionally, since our index can be evaluated using computationally efficient methods, it is convenient for empirical implementations.

We are not the first to analyse consumer choice under semiorder maximisation. Fishburn (1975) characterises such relations in terms of choice correspondences. The author imposes conditions on correspondence $C(B) := \{x \in B : \text{ if } y \in B \text{ then not } yPx\}$, defined over all subsets B of the domain, that are both necessary and sufficient for the relation P to be a semiorder. The axiomatisation crucially depends on the fact that the researcher observes the value of the correspondence C for *any* subset B of the domain and that $C(B)$ contains *all* choices from the set B. In contrast, we follow Afriat (1967) by assuming that the observer gathers only partial information about C, by monitoring a single choice from each of a finite collection of budget sets.

We conclude the paper with an empirical illustration of our methods. In Section 6 we apply our test to household-level scanner panel data containing time series of grocery purchases. We elicit the value of the just-noticeable difference for 396 households whose choices violated GARP, out of 494 available in the dataset, and find that the median value of λ_* is approximately 1.07. Hence, a 7% increase in the size of a bundle is sufficient for the median consumer to perceive the change.

Proofs of the results not displayed in the main body of the article are postponed till the Appendix. The Online appendix provides additional discussion and extensions.

2 Semiorders

A *semiorder* over a set of alternatives X is an *irreflexive* binary relation P that satisfies (S1) the *interval order condition*, i.e., if xPy and $x'Py'$ then xPy' or $x'Py$; and (S2) is *semitransitive*, i.e., if xPy and yPz then xPy' or $y'Pz$, for any y' in X. In particular, condition (S1) guarantees that any semiorder is both transitive and asymmetric.[5] Throughout this paper, we interpret P as the *strict preference*. If neither xPy nor yPx, we say that x is *indifferent* to y and denote it by xIy. Clearly, the later relation is reflexive and symmetric, but not transitive. We denote xRy whenever xPy or xIy.

The notion of semiorders is inherently related to weak orders. Recall that a *weak order* over X is a complete, reflexive, and transitive binary relation $\succeq$, with its symmetric and asymmetric parts denoted by $\sim$ and $\succ$, respectively. Theorem 1 in Luce (1956) guarantees that any semiorder P induces the following weak order: Let $x \succ y$ whenever there is some z such that either (W1) xPz and zIy; or (W2) xIz and zPy. Specifically, if xPy then $x \succ y$. Moreover, we set $x \sim y$ if neither $x \succ y$ nor $y \succ x$. In the remainder of our paper, we say that such a weak order is *induced* by the semiorder P.

Even though the indifference relation I is not transitive, it does convey some information about the semiorder P and the induced weak order $\succeq$. The following property plays a significant role in our main argument presented in Section 3. As the proof of this result is rather extensive, we postpone it till the Appendix.

Proposition 1. *For any finite sequence* $\big\{(x_i, x_{i+1})\big\}_{i=1}^{m}$ *with* $x_i R x_{i+1}$, *for all* $i = 1, \ldots, m$,

[5] Property (S1) implies that if xPy and yPz then xPz or yPy. Since P is irreflexive, it must be that xPz. Thus, relation P is transitive. Clearly, it has to be asymmetric as well.

let $\mathcal{P}$ and $\mathcal{I}$ denote sets of pairs (x_i, x_{i+1}) such that $x_i P x_{i+1}$ and $x_i I x_{i+1}$, respectively. Whenever $|\mathcal{P}| \geq |\mathcal{I}|$ then $x_1 \succ x_{m+1}$, while $|\mathcal{P}| > |\mathcal{I}|$ implies $x_1 P x_{m+1}$.

The above proposition implies that, for any sequence x_1, x_2, …, x_m, x_{m+1} of elements in X such that $x_i R x_{i+1}$, for all $i = 1, \ldots, m$, whenever the number of strict comparisons P is strictly greater than the number of indifferences I, then $x_1 P x_{m+1}$. Moreover, if the number of strict comparisons is equal to the number of indifferences, then $x_1 \succ x_{m+1}$, where $\succeq$ is the weak order induced by P. This property proves to be crucial in determining the testable implications for the model under consideration.

An important characteristic of semiorders concerns their utility representation. Following the result by Scott and Suppes (1958), for any semiorder P over a finite domain X there exists a function $u : X \to \mathbb{R}$ such

$$xPy \text{ if and only if } u(x) > u(y) + 1. \tag{2}$$

This implies that xRy is equivalent to $u(x) \geq u(y) - 1$, while the weak order $\succeq$ induced by P satisfies $x \succeq y$ if and only if $u(x) \geq u(y)$, without loss of generality.[6] Under some regularity conditions, the representation result can be extended to semiorders defined over infinite spaces. See Manders (1981) or Beja and Gilboa (1992) for details.

3 Revealed noticeable difference

A *set of observations* $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ consists of a finite number of pairs (B, x) of a budget set $B \subset \mathbb{R}^\ell_+$ and the corresponding choice $x \in B$.[7] In this section we provide the testable restrictions for the model of consumer choice with a *noticeable difference* λ. That is, we determine a tight condition on set $\mathcal{O}$ under which there is a semiorder P satisfying $(\lambda' y) P y$, for all $\lambda' \geq \lambda$ and non-zero $y \in \mathbb{R}^\ell_+$, such that

$$(B, x) \in \mathcal{O} \text{ and } y \in B \text{ implies } \textit{not } yPx.$$

Before we proceed, we introduce the framework in which we perform our analysis.

[6] This is to say that, in general, function u satisfying condition (2) does not represent the induced weak order $\succeq$. However, there always exists a function u for which this is true.

[7] We endow space $\mathbb{R}^\ell_+$ with the natural product order $\geq$. This is to say that $x \geq y$ if and only if $x^i \geq y^i$, for all $i = 1, 2, \ldots, \ell$. Moreover, we denote $x \gg y$ whenever $x^i > y^i$, for all $i = 1, 2, \ldots, \ell$.

3.1 Preliminaries

Throughout this paper we consider consumer choices over *generalised budget sets*, defined as in Forges and Minelli (2009). Specifically, for any observation (B, x) in $\mathcal{O}$, we require that the set B is compact and comprehensive.[8] Additionally, we assume that there is some $y \in B$ such that $y \gg 0$. With a slight abuse of the notation,

$$\partial B := \big\{ y \in B : \text{ if } z \gg y \text{ then } z \notin B \big\}$$

denotes the *upper bound* of the set B. Suppose that, for any vector $y \in \partial B$ and scalar $\theta \in [0,1)$, we have $\theta y \in B \setminus \partial B$. That is, for any element $y \in \mathbb{R}^{\ell}_{+}$, ray $\big\{\theta y : \theta \geq 0\big\}$ intersects the boundary ∂B exactly once. Finally, we assume that $x \neq 0$.[9]

Even though originally Afriat (1967) and Varian (1982) formulated the *generalised axiom of revealed preference* (or GARP) for linear budget sets, it is possible to extend this restriction to our general framework.[10] Following Forges and Minelli (2009), a set of observations $\mathcal{O}$ obeys GARP if for any sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, we have $x_s \in \partial B_t$, for all $(t,s) \in \mathcal{C}$. That is, bundle x_s belongs to the upper bound of set B_t, for all pairs (t,s) in the cycle. Specifically, this includes all one-element sequences. That is, we have $x_t \in \partial B_t$, for all $t \in T$.

Proposition 3 in Forges and Minelli (2009) states that a dataset $\mathcal{O}$ is rationalisable by a locally non-satiated utility function u if and only if it satisfies GARP. That is, the condition is necessary and sufficient for the utility u to satisfy:

$$\text{if } \ (B,x) \in \mathcal{O} \text{ and } y \in B \ \text{ then } \ u(x) \geq u(y).$$

In addition, without loss of generality, we may assume that the utility function u is both continuous and (weakly) increasing, while $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero $y \in \mathbb{R}^{\ell}_{+}$. This extends the original result by Afriat (1967) and Varian (1982).

3.2 The axiom

In this subsection we introduce and thoroughly discuss the condition imposed on a set of observations that is essential to our analysis. Consider the following axiom.

[8] Set B is *comprehensive* if $y \in B$ and $z \leq y$ implies $z \in B$, for all $z \in \mathbb{R}^{\ell}_{+}$.

[9] This assumption is not without loss of generality. However, it substantially simplifies our analysis and seems to be insignificant from the empirical point of view.

[10] For any observation $(B,x) \in \mathcal{O}$, the budget set B is *linear* if $B = \big\{y \in \mathbb{R}^{\ell}_{+} : p \cdot y \leq p \cdot x\big\}$, for some price vector $p \in \mathbb{R}^{\ell}_{++}$. Clearly, this class of budget sets is included in our framework.

Axiom 1. *Dataset $\mathcal{O}$ satisfies the axiom for some $\lambda > 1$ if for an arbitrary sequence $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, we have*

$$-\sum_{(t,s)\in\mathcal{C}} k_{ts} \; < \; |\mathcal{C}|,$$

where we denote $k_{ts} := \inf\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\}$, for all $t,\, s \in T$.[11]

To better understand the above restriction, consider a sequence $\mathcal{C}$ specified as in the thesis of the axiom. Note that, by assumptions imposed on set B_t, integer k_{ts} must be negative, for any index pair (t,s) that belongs to the cycle. In particular, $(-k_{ts})$ determines the maximal number of times one could scale set B_t by $(1/\lambda)$ without excluding x_s. Equivalently, it is the greatest integer k for which bundle $(\lambda^k x_s)$ belongs to B_t. Axiom 1 requires that, for any cycle $\mathcal{C}$, the sum of such integers is strictly less than the total number $|\mathcal{C}|$ of its components. Equivalently, this is to say that the average value of $(-k_{ts})$ along each cycle must be strictly less than 1.

Alternatively, number $\lambda^{k_{ts}}$ is the least power of λ for which bundle x_s belongs to $\lambda^{k_{ts}} B_t$. This value may serve as an approximation of the relative distance between x_s and the boundary of the budget set B_t. Notice that, for any cycle $\mathcal{C}$ specified as in Axiom 1, the previously stated restriction may be reformulated as

$$\sqrt[|\mathcal{C}|]{\textstyle\prod_{(t,s)\in\mathcal{C}} \lambda^{k_{ts}}} \; > \; \lambda^{-1}. \tag{3}$$

Thus, the geometric mean of such powers $\lambda^{k_{ts}}$, over any cycle $\mathcal{C}$, has to be strictly greater than the inverse of λ. In other words, our condition imposes a bound on the average approximate distance between bundle x_s and the boundary of set B_t, along any cycle. We discuss this condition further in Section 5.

The property stated in Axiom 1 has to be satisfied by all one-element sequences. This requires for bundle x_t to be outside of the set $\lambda^{-1}B_t$, for all $t \in T$. Therefore, the least integer k for which the vector x_t is in $\lambda^k B_t$ must be equal to 0. Nevertheless, unlike GARP, the condition admits observations (B_t, x_t) in which x_t does not belong to the upper boundary of B_t. For instance, see Figure 2.

A closer look at Axiom 1 reveals three important properties of this restriction. First of all, for any set of observations $\mathcal{O}$ there is a sufficiently large number $\lambda > 1$ for which the set obeys the axiom. For example, take any number λ such that $1/\lambda$ is strictly less than

[11] For any scalar θ, we denote $\theta B := \{\theta y : y \in B\}$.

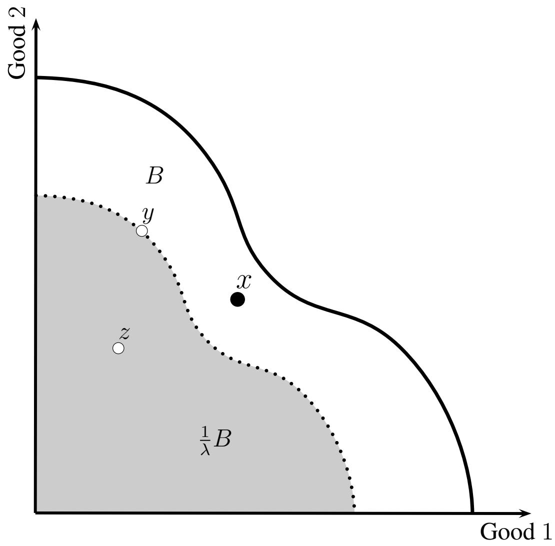

Figure 2: A singleton set $\mathcal{O} = \{(B, x)\}$ obeys Axiom 1 for $\lambda > 1$. However, since x does not belong to ∂B, it can not be supported by a locally non-satiated utility function. At the same time, sets $\{(B, y)\}$ and $\{(B, z)\}$ violate Axiom 1 for this value of λ.

$\inf\{\theta > 0 : x_s \in \theta B_t,\ \text{for all}\ t, s \in T\}$. By construction, it must be that $x_s \notin \lambda^{-1}B_t$, for all t, $s \in T$, which implies that number k_{ts}, specified as in the thesis of Axiom 1, must be weakly greater than 0, for all t, $s \in T$. This suffices for $\mathcal{O}$ to satisfy the condition.

Second of all, Axiom 1 is monotone in λ. That is, whenever a dataset satisfies the restriction for some $\lambda > 1$, it obeys the condition for all $\lambda' \geq \lambda$. Take some $\lambda > 1$ and a cycle $\mathcal{C}$, and evaluate the corresponding integers k_{ts}, for all $(t, s) \in \mathcal{C}$. For any $\lambda' \geq \lambda$, define $k'_{ts} := \inf\{k \in \mathbb{Z} : x_s \in (\lambda')^k B_t\}$, for all $(t, s) \in \mathcal{C}$. Since bundle x_s belongs to set B_t, for any pair (t, s) in the cycle, it must be that $k_{ts} \leq k'_{ts}$. Thus,

$$-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}| \quad \text{implies} \quad -\sum_{(t,s)\in\mathcal{C}} k'_{ts} < |\mathcal{C}|.$$

In particular, the set of observations satisfies Axiom 1 for λ only if it obeys the condition for λ'. This means that the focus of the researcher may be restricted solely to the least value of the parameter λ for which the property stated in the axiom holds.

Finally, our condition is weaker that the generalised axiom of revealed preference stated previously. We formalise this claim below.

Proposition 2. *Set $\mathcal{O}$ obeys GARP if and only if it satisfies Axiom 1 for all $\lambda > 1$.*

Proof. Consider a cycle $\mathcal{C} = \{(a, b), (b, c), \dots, (z, a)\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t, s) \in \mathcal{C}$. Whenever the set of observations obeys GARP, it must be that $x_s \in \partial B_t$, for

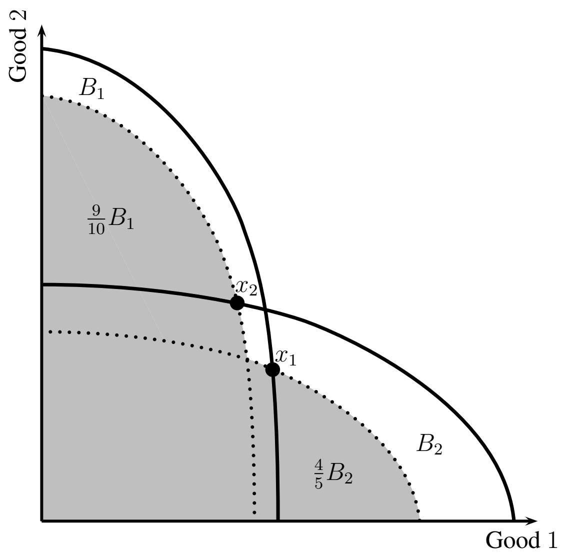

Figure 3: Set $\mathcal{O} = \{(B_1, x_1), (B_2, x_2)\}$ violates GARP. In fact, it fails to satisfy the weak axiom of revealed preference. However, it obeys Axiom 1 for any $\lambda > \sqrt{5/4}$.

all $(t, s) \in \mathcal{C}$, where ∂B_t denotes the upper boundary of set B_t, defined as in Section 3.1. This implies that the integer k_{ts} is equal to 0, for any pair (t, s) and $\lambda > 1$. Hence, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} = 0 < |\mathcal{C}|$, which suffices for $\mathcal{O}$ to obey Axiom 1 for any $\lambda > 1$.

We prove the converse by contradiction. Suppose that set $\mathcal{O}$ obeys Axiom 1 for all $\lambda > 1$, but fails to satisfy GARP. In such a case, there is a cycle $\mathcal{C}$ and a pair $(t, s) \in \mathcal{C}$ such that $x_s \notin \partial B_t$. Thus, we have $\inf\{\theta > 0 : x_s \in \theta B_t\} < 1$. Take any $\lambda > 1$ such that $\lambda^{-|\mathcal{C}|}$ is greater than the above infimum. By construction, we have $x_s \in \lambda^{-|\mathcal{C}|} B_t$, which implies that $k_{ts} \leq -|\mathcal{C}|$. Therefore, it must be that $-\sum_{(i,j)\in\mathcal{C}} k_{ij} \geq |\mathcal{C}|$, which contradicts that the set of observations $\mathcal{O}$ satisfies Axiom 1 for all $\lambda > 1$. □

We find the above result intuitive. The generalised axiom requires that, for any cycle $\mathcal{C}$, bundle x_s belongs to the upper bound of set B_t, for all $(t, s) \in \mathcal{C}$. This implies that the corresponding integers k_{ts} are equal to 0, regardless of the value of λ. Thus, such a set of observations satisfies Axiom 1 for any value of the parameter. Conversely, the sum of numbers k_{ts} is lower than the cardinality of $\mathcal{C}$, for any value of λ, only if each bundle x_s lays at the boundary of the corresponding set B_t. This is equivalent to GARP.

Consider the example depicted in Figure 3. Clearly, it violates GARP, as $x_1 \in B_2$ and $x_2 \in B_1$, but none of the bundles lay on the upper bound of the respective sets. Nevertheless, those observations obey Axiom 1 for all $\lambda > \sqrt{5/4}$. Take any such λ. Since $x_1 \in \partial B_1$ and $x_2 \in \partial B_2$, it suffices to verify the condition specified in the axiom for cycle

$\mathcal{C} = \{(1,2),(2,1)\}$. Given that $x_2 \in (9/10)\partial B_1$ and $(9/10) > \lambda^{-1}$, the integer k_{12} is equal to 0. Similarly, as $x_1 \in (4/5)\partial B_2$ and $(4/5) > \lambda^{-2}$, we obtain $k_{21} > -2$. Therefore, we have $-k_{12} - k_{21} < 2$, which is sufficient for the dataset to satisfy Axiom 1. At the same time, $\sqrt{5/4}$ constitutes the lower bound for all such parameters λ.

Verifying whether a set of observations satisfies Axiom 1 is computationally efficient. Specifically, the test can be reduced to the minimum cost-to-time ratio cycle problem.[12] Take any $\lambda > 1$. Notice that set $\mathcal{O}$ obeys the condition if and only if the minimum of $\sum_{(t,s)\in\mathcal{C}} k_{ts}/|\mathcal{C}|$ over all cycles $\mathcal{C}$ specified in the axiom is strictly greater than -1, where integers k_{ts} are defined as previously. Construct a graph with each vertex $t \in T$ corresponding to a single observation in $\mathcal{O}$. Additionally, let the graph contain a directed arc (t,s) with the corresponding cost $c_{ts} = k_{ts}$ and time $\tau_{ts} = 1$ if and only if $x_s \in B_t$, i.e., if $k_{ts} \leq 0$. Clearly, solving the minimum cost-to-time ratio cycle problem for the above graph is equivalent to determining the least value of $\sum_{(t,s)\in\mathcal{C}} k_{ts}/|\mathcal{C}|$ over all cycles $\mathcal{C}$ induced by $\mathcal{O}$. Since the above class of problems is solvable in a polynomial time, determining whether a set of observations obeys Axiom 1 is efficient.[13]

3.3 The main result

Consider the main theorem of this paper.

Main Theorem. *Set of observations $\mathcal{O}$ is rationalisable by a semiorder P with a noticeable difference $\lambda > 1$ if and only if it obeys Axiom 1 for λ.*

We devote this subsection to the proof of this result. Given that the argument highlights both the properties of Axiom 1 and its applicability, we find it convenient to present it in several steps. First, we show that the restriction is necessary for a set of observations to be rationalisable with some noticeable difference $\lambda > 1$.

Consider a cycle $\mathcal{C} = \{(a,b),(b,c),\dots,(z,a)\}$ in $T \times T$ with $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$. Take any pair (t,s) in $\mathcal{C}$ and evaluate the corresponding integer k_{ts}, which has to be negative. Let $y_k := \lambda^k x_s$, for all $k = 1,\dots,(-k_{ts})$, and construct sequence

$$\mathcal{S}_{ts} = \{(x_t, y_{-k_{ts}}),\ (y_{-k_{ts}}, y_{-(k_{ts}+1)}),\ \dots\ ,\ (y_2, y_1),\ (y_1, x_s)\}$$

[12] See Chapter 5.7 in Ahuja, Magnanti, and Orlin (1993) for a handbook introduction to this method.

[13] Note that, in order to reject the axiom, it suffices to find at least one cycle that violates the condition. Hence, it is not necessary to determine the minimal price-to-time ratio to conclude that the set of observations does not meet our restriction. Thus, the algorithm can be rendered even more efficient.

consisting of $(1 - k_{ts})$ elements. Since semiorder P rationalises set $\mathcal{O}$ and $y_{-k_{ts}} \in B_t$, we have $x_t R y_{-k_{ts}}$, while yPy' for any other pair (y, y') in $\mathcal{S}_{ts}$. Therefore, the number of elements (y, y') in $\mathcal{S}_{ts}$ for which yPy' is greater than $(-k_{ts})$.

Since the above observation holds for any $(t, s) \in \mathcal{C}$, combining all such sequences $\mathcal{S}_{ts}$ allows us to construct another sequence $\big\{(z_i, z_{i+1})\big\}_{i=1}^{m}$, where $m = |\mathcal{C}| - \sum_{(t,s)\in\mathcal{C}} k_{ts}$, with $z_i R z_{i+1}$, for all $i = 1, \ldots, m$, and $z_1 = z_{m+1} = x_a$. Moreover, the total number of pairs (z_i, z_{i+1}) satisfying $z_i P z_{i+1}$ is greater than $-\sum_{(t,s)\in\mathcal{C}} k_{ts}$. By Proposition 1, this requires that $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$. Otherwise, it would have to be that $x_a = z_1 \succ z_{m+1} = x_a$, contradicting that the induced strict relation $\succ$ is irreflexive.

A careful read of the above argument provides a deeper understanding of the revealed preference relation that is induced by the set of observations. Note that, for any $\lambda > 1$ and sequence $\mathcal{S} = \big\{(a, b), (b, c), \ldots, (y, z)\big\}$ such that $x_s \in B_t$, for all $(t, s) \in \mathcal{S}$:

$$-\sum_{(t,s)\in\mathcal{S}} k_{ts} \geq |\mathcal{S}| \;\text{ implies }\; x_a \succ x_z \quad\text{and}\quad -\sum_{(t,s)\in\mathcal{S}} k_{ts} > |\mathcal{S}| \;\text{ implies }\; x_a P x_z,$$

where integers k_{ts} are defined as previously. Therefore, any sequence satisfying one of the above conditions reveals that the bundle x_a is strictly superior to x_z — either with respect to the induced weak order $\succ$ or the original semiorder P. Clearly, as both relations are irreflexive, this excludes the possibility that $x_a = x_z$. In fact, the exact role of Axiom 1 is to guarantee that the set of observations $\mathcal{O}$ admits no such cycles. We exploit this observation in Section 3.4, where we present a method of recovering preferences.

Establishing the sufficiency part of the Main Theorem is more demanding. For this reason, we conduct the remainder of the proof in three stages. In the lemma that follows, we argue that whenever a set of observations satisfies Axiom 1 for some $\lambda > 1$, there exists a solution to a particular system of linear inequalities.

Lemma 1. *If set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$, there are numbers $\{\phi_t\}_{t\in T}$ and $\mu > 1$ such that $k_{ts} \leq 0$ implies $\phi_s - 1 \leq \phi_t + \mu k_{ts}$, where $k_{ts} := \inf\big\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\big\}$, for $t, s \in T$.*

Proof. Notice that, a set of observations $\mathcal{O}$ satisfies Axiom 1 if and only if, for any cycle $\mathcal{C} = \big\{(a, b), (b, c), \ldots, (z, a)\big\}$ in $T \times T$ such that $k_{ts} \leq 0$, for all $(t, s) \in \mathcal{C}$, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$. The rest follows from Proposition A.1 in the Appendix. □

As discussed in Section 3.2, there is an efficient algorithm that allows us to verify whether a set of observations satisfies Axiom 1. Nevertheless, the above lemma provides an alternative method that pertains to linear programming.[14] According to the result, determining existence of a solution to the above system of inequalities is necessary for the dataset $\mathcal{O}$ to obey Axiom 1. In fact, as we argue in the remainder of this section, it is also a sufficient condition. Given that linear programs are solvable in a polynomial time, the alternative method is computationally efficient.

Apart from establishing a way of verifying Axiom 1, Lemma 1 provides us with a foundation for constructing a semiorder that rationalises the set of observations. In the next result, we argue that determining a solution to the linear system specified above is sufficient for existence of a particular utility function.

Lemma 2. *Whenever there is a solution to the system of inequalities specified in Lemma 1, there is a utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfies $u(\lambda' y) > u(y) + 1$, for all $\lambda' \geq \lambda$ and non-zero $y \in \mathbb{R}^\ell_+$, while $(B, x) \in \mathcal{O}$ and $y \in B$ implies $u(x) + 1 \geq u(y)$.*

Proof. Define function $g_t : \mathbb{R}^\ell_+ \to \mathbb{Z}$ by $g_t(y) := \inf\big\{k \in \mathbb{Z} : y \in \lambda^k B_t\big\}$, for all $t \in T$. Note that, we have $g_t(y) \leq 0$ if and only if $y \in B_t$. Take any non-zero $y \in \mathbb{R}^\ell_+$ and $\lambda' \geq \lambda$. For any $k \in \mathbb{Z}$, if $(\lambda' y) \in \lambda^{k+1} B_t$ then $(\lambda'/\lambda) y \in \lambda^k B_t$. Since B_t is comprehensive, this implies that $y \in \lambda^k B_t$. Thus, we obtain $g_t(\lambda' y) \geq g_t(y) + 1$. Finally, observe that $g_t(x_s) = k_{ts}$, where k_{ts} is defined as previously, for all t, $s \in T$.

Take any numbers $\{\phi_t\}_{t\in T}$ and μ that satisfy the inequalities in Lemma 1. Moreover, let $\nu > 1$ be such that $\phi_s - 1 \leq \phi_t + \nu k_{ts}$, for all t, s in T satisfying $k_{ts} > 0$. Clearly, it exists. Construct function $v_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ as follows:

$$v_t(y) := \begin{cases} \phi_t + \mu g_t(y) & \text{if } \; g_t(y) \leq 0, \\ \phi_t + \nu g_t(y) & \text{otherwise.} \end{cases}$$

First, we claim that $v_t(\lambda' y) > v_t(y) + 1$, for any $\lambda' \geq \lambda$ and non-zero $y \in \mathbb{R}^\ell_+$. In fact, whenever we have $g_t(y) \leq 0$ and $g_t(\lambda' y) \leq 0$, then it must be

$$v_t(\lambda' y) - v_t(y) \;=\; \mu\big[g_t(\lambda' y) - g_t(y)\big] \;\geq\; \mu \;>\; 1.$$

As $\nu > 1$, the claim holds analogously if $g_t(y) > 0$ and $g_t(\lambda' y) > 0$. Therefore, it suffices to prove the property for $g_t(y) \leq 0$ and $g_t(\lambda' y) > 0$. In such a case, we have

$$v_t(\lambda' y) - v_t(y) \;=\; \nu g_t(\lambda' y) - \mu g_t(y) \;>\; g_t(\lambda' y) - g_t(y) \;\geq\; 1,$$

[14] Note that, this linear system is similar to *Afriat's inequalities*, as in Forges and Minelli (2009).

since μ, $\nu > 1$. Thus, our initial claim. Since $g_s(x_t) = k_{st}$, by construction of numbers $\{\phi_t\}_{t\in T}$, μ, and ν, we obtain $\phi_t - 1 \leq \min\{v_s(x_t) : s \in T\}$, for all $t \in T$.

Define function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ by $u(y) := \min\{v_s(y) : s \in T\}$. Given the properties of functions v_s, for $s \in T$, it is clear that $\lambda' \geq \lambda$ implies $u(\lambda' y) > u(y) + 1$, for any non-zero y.[15] To complete the proof, take any $t \in T$ and $y \in B_t$. Since $g_t(y) \leq 0$,

$$u(y) - 1 \;\leq\; \phi_t + \mu g_t(y) - 1 \;\leq\; \phi_t - 1 \;\leq\; \min\{v_s(x_t) : s \in T\} \;=\; u(x_t),$$

which concludes our argument. □

In order to complete the proof of the Main Theorem, take any function u specified as in Lemma 2 and define a binary relation P by:

$$xPy \text{ if and only if } u(x) > u(y) + 1.$$

It is straightforward to show that P is a semiorder. Moreover, if $\lambda' \geq \lambda$ then $(\lambda' y)Py$, for all non-zero y. Finally, since $(B, x) \in \mathcal{O}$ and $y \in B$ implies $u(x) + 1 \geq u(y)$, it can never be that yPx. Thus, the semiorder P rationalises the set $\mathcal{O}$.

The argument presented above establishes a fourfold equivalence. First of all, it implies that Axiom 1 is necessary and sufficient for a set of observations $\mathcal{O}$ to be rationalisable by a semiorder P for a particular noticeable difference $\lambda > 1$. Moreover, our restriction is equivalent to existence of a solution to the system of linear inequalities specified in Lemma 1. Third, solving the linear problem is both necessary and sufficient for the set of observations to admit a utility function u that satisfies the two properties listed in Lemma 2. This establishes the fourth equivalence, according to which the semiorder P induced by any such function u rationalises $\mathcal{O}$ with the noticeable difference λ.

The proof supporting Lemma 2 proposes only one possible utility function u that satisfies the conditions listed in its thesis. In Section B.2 of the Online appendix we provide an alternative proof of the Main Theorem. Specifically, we show that, without loss of generality, the utility function specified in the lemma is continuous, increasing, and satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y.[16] More importantly, we argue that u represents the weak order $\succeq$ induced by the corresponding semiorder P. That is, we have $x \succeq y$ if and only if $u(x) \geq u(y)$. This will be significant in Section 4.

[15] By definition, there is some t, $s \in T$ such that $u(\lambda' y) = v_t(\lambda' y)$ and $u(y) = v_s(y) \leq v_t(y)$. This guarantees that $u(\lambda' y) - u(y) = v_t(\lambda' y) - v_s(y) \geq v_t(\lambda' y) - v_t(y) > 1$.

[16] In addition, if the complement of set B_t is convex, for all $t \in T$, then the function is quasiconcave.

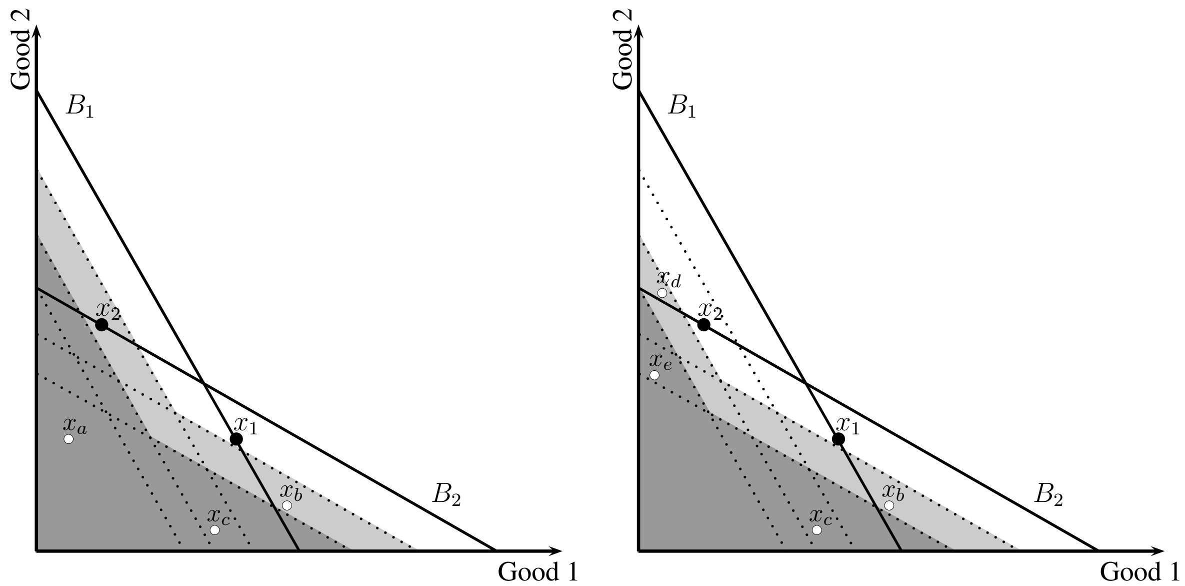

Figure 4: Recovering preferences from observations (B_1, x_1) and (B_2, x_2). The dotted lines represent successive scalings of sets B_1 and B_2 by λ. The shaded areas depict bundles that are revealed strictly inferior to x_1 (left) and x_2 (right) with respect to $\succ^*_\lambda$ and P^*_λ.

3.4 Recoverability and extrapolation

In this subsection we apply the tools developed in the preceding part of the paper to the problem of recovering preferences from the observable data and forecasting choices from hypothetical budget sets. As it was pointed out previously, our framework crucially depends on two types of preferences revealed in the data: the semiorder P that the agent is using when making a choice, and the induced weak order $\succeq$. Below we show how to recover both relations from a set of observations.

For any $\lambda > 1$, bundle x is *revealed strictly preferred* to y with respect to the revealed relation $\succ^*_\lambda$, denoted by $x \succ^*_\lambda y$, if there is a sequence $\mathcal{S} = \big\{(a,b),(b,c),\dots,(z,0)\big\}$ of pairs of indices in $T \cup \{0\}$ such that $x_a = x$, $x_0 = y$, $x_s \in B_t$, for all $(t,s) \in \mathcal{S}$, and

$$-\sum_{(t,s)\in\mathcal{S}} k_{ts} \;\geq\; |\mathcal{S}|,$$

where $k_{ts} := \inf\big\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\big\}$, for all t, $s \in T \cup \{0\}$. In particular, the element x_0 in the above sequence need not belong to the set of observable choices. Whenever the above inequality is strict, then x is *revealed strictly preferred* to y with respect to the revealed relation P^*_λ, denoted by $xP^*_\lambda y$. It is straightforward to verify that both relations are transitive, while $xP^*_\lambda y$ implies $x \succ^*_\lambda y$.

To better understand the two revealed relations, consider the example depicted in Figure 4. As shown in the graph on the left, since bundle x_2 belongs to $\lambda^{-1}B_1$, we have

$k_{12} \leq -1$, which implies that $x_1 \succ^*_\lambda x_2$. Analogously, given that x_a is in $\lambda^{-2}B_1$, it must be that $x_1 P^*_\lambda x_a$. By applying the same argument, we can show that $x_2 \succ^*_\lambda x_b$ and $x_2 P^*_\lambda x_c$, as it is represented in the graph on the right.

Regarding the remaining revealed comparisons, transitivity of $\succ^*_\lambda$ guarantees that $x_1 \succ^*_\lambda x_2$ and $x_2 \succ^*_\lambda x_b$ implies $x_1 \succ^*_\lambda x_b$, as shown on the left. Moreover, since $k_{12} = -1$ and $k_{2c} \leq -2$, there is a sequence $\mathcal{S} = \big\{(1,2),(2,c)\big\}$ such that $-k_{12} - k_{2c} \geq 3 > |\mathcal{S}|$. Thus, we obtain $x_1 P^*_\lambda x_c$. Similarly, observe that $k_{21} = 0$ and $k_{1d} = -2$, which implies $x_2 \succ^*_\lambda x_d$. Using an analogous argument, we can show that $x_2 P^*_\lambda x_e$.

Proposition 3. *Suppose that set $\mathcal{O}$ is rationalisable by a semiorder P with a noticeable difference $\lambda > 1$. The revealed relation P^*_λ is consistent with P and the revealed relation $\succ^*_\lambda$ is consistent with the strict part $\succ$ of the weak order induced by P.*

The above result is a direct application of Proposition 1 and the argument presented in Section 3.3. Whenever $x \succ^*_\lambda y$, it is possible to construct a sequence $\big\{(z_i, z_{i+1})\big\}_{i=1}^m$ such that $z_i R z_{i+1}$, for all $i = 1, \ldots, m$, while $z_1 = x$ and $z_{m+1} = y$. In any such sequence, the number of pairs (z_i, z_{i+1}) for which $z_i P z_{i+1}$ is greater than the number of couples with $z_i I z_{i+1}$. By Proposition 1, this guarantees that $x \succ^*_\lambda y$ implies $x \succ y$. Similarly, if $x P^*_\lambda y$ then xPy. The corollary below proposes an alternative characterisation of Axiom 1.

Corollary 1. *Set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$ if and only if $\succ^*_\lambda$ is irreflexive.*

We turn to the problem of extrapolation. Suppose that set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$, or equivalently, it is rationalisable by a semiorder P with the noticeable difference λ. The revealed relations P^*_λ and $\succ^*_\lambda$ summarize all the information on consumer preference that is contained in the data. In particular, by Proposition 3, any semiorder P and the induced weak order $\succeq$ supporting the observations must contain the respective revealed relations. In the remainder of this subsection we are interested in out-of-sample predictions of consumer choices. That is, given a budget set B_0 that has not been previously observed, which bundles $x_0 \in B_0$ are consistent with the past behaviour? In other words, we would like to determine which vectors belong to set

$$S_\lambda(B_0) := \Big\{x_0 \in B_0 : \text{set } \mathcal{O} \cup \big\{(B_0, x_0)\big\} \text{ satisfies Axiom 1 for } \lambda\Big\}.$$

Given that set $\mathcal{O}$ satisfies Axiom 1 for $\lambda > 1$, it is straightforward to show that bundle x_0 belongs to $S_\lambda(B_0)$ if and only if for any cycle $\mathcal{C} = \big\{(0,a),(a,b),\ldots,(z,0)\big\}$ of pairs of indices in $T \cup \{0\}$ such that $k_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$, where integers k_{ts} are defined as previously. Verifying the above condition directly might be cumbersome. Therefore, we find it convenient to provide some necessary conditions that have to be satisfied by each element in $S_\lambda(B_0)$. In particular, those properties make an extensive use of the revealed preference relations defined above.

Suppose that x_0 is an element of $S_\lambda(B_0)$. Clearly, we require that $x_0 \notin \lambda^{-1}B_0$. Moreover, it must be that: (i) $x_t P^*_\lambda x_0$ implies $x_t \notin B_0$; (ii) $x_t \succ^*_\lambda x_0$ implies $x_t \notin \lambda^{-1}B_0$; and (iii) $x_0 \in B_t$ implies $x_t \notin \lambda^{-2}B_0$. To show that (i) is necessary, recall that $x_t P^*_\lambda x_0$ only if there is some sequence $\mathcal{S} = \big\{(t,a),(a,b),\ldots,(z,0)\big\}$ such that $k_{ij} \leq 0$, for $(i,j) \in \mathcal{S}$, and $-\sum_{(i,j)\in\mathcal{S}} k_{ij} > |\mathcal{S}|$. Since numbers $(-k_{ij})$ are integers, the above condition requires for the sum to be weakly greater than $|\mathcal{S}|+1$. In addition, $x_t \in B_0$ implies $k_{0t} \leq 0$. This allows us to construct a cycle $\mathcal{C} = \mathcal{S} \cup \big\{(0,t)\big\}$ with $k_{ij} \leq 0$, for all $(i,j) \in \mathcal{C}$, where the corresponding sum of integers $(-k_{ij})$ is weakly greater that $|\mathcal{C}| = |\mathcal{S}|+1$. However, this contradicts that $x_0 \in S_\lambda(B_0)$. We prove (ii) and (iii) analogously.

The conditions (i)–(iii) are necessary but *not* sufficient for a bundle x_0 to be an element of $S_\lambda(B_0)$. Consider a dataset $\mathcal{O}$ consisting of vectors $x_1 = (5,2)$ and $x_2 = (4,5)$ selected respectively from budget sets $B_t = \big\{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\big\}$, for $t = 1,2$, where $p_1 = (3,1)$ and $p_2 = (2,1)$. Note that, this two-element dataset satisfies Axiom 1 for $\lambda = 1.1$. Given a previously unobserved linear budget set B_0, defined as above for $x_0 = (5,3)$ and $p_0 = (1,5)$, we would like to determine the corresponding set $S_\lambda(B_0)$. Observe that, vector $x_0 \in B_0$ satisfies conditions (i)–(iii). However, it does not belong to $S_\lambda(B_0)$. In particular, neither x_1 nor x_2 is revealed preferred to x_0 with respect to P^*_λ or $\succ^*_\lambda$. Thus, properties (i) and (ii) hold trivially. Moreover, even though x_0 is an element of B_2, it is outside of $\lambda^{-2}B_2$. Therefore, condition (iii) holds as well. At the same time, it is possible to construct a cycle $\mathcal{C} = \big\{(0,1),(1,2),(2,0)\big\}$ such that $k_{01} = -3$, $k_{12} = 0$, $k_{20} = 0$. Since $-\sum_{(t,s)\in\mathcal{C}} k_{ts} \geq |\mathcal{C}|$, this contradicts that the extended dataset $\mathcal{O} \cup \big\{(B_0,x_0)\big\}$ satisfies Axiom 1 for $\lambda = 1.1$.

Whenever we restrict our attention to linear budget sets, the corresponding set of consistent choices admits a property that proves to be useful in applications. Formally, a budget set is *linear* if $B_t := \big\{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq m_t\big\}$, for some $p_t \in \mathbb{R}^\ell_{++}$ and $m_t > 0$.

Proposition 4. *Consider a dataset $\mathcal{O}$ in which B_t is linear, for all $t \in T$. If $\mathcal{O}$ obeys Axiom 1 for some $\lambda > 1$, then the corresponding set $S_\lambda(B_0)$ is convex, for any linear B_0.*

Proof. Whenever B_t is linear, then $k_{ts} := \inf\{k \in \mathbb{Z} : (p_t \cdot x_s)/m_t \leq \lambda^k\}$, for any x_s. Let x_0 be a convex combination of any x_0', x_0'' in $S_\lambda(B_0)$. Clearly, we have $x_0 \in B_0$. Since $p_t \cdot x_0 \geq \min\{p_t \cdot x_0', p_t \cdot x_0''\}$, for all $t \in T$, the above set violates the axiom only if either $\mathcal{O} \cup \{(B_0, x_0')\}$ or $\mathcal{O} \cup \{(B_0, x_0'')\}$ fails to satisfy the the restriction. □

In fact, linear budget sets imply that set $S_\lambda(B_0)$ is a polyhedron. However, since any of its elements must be outside of $\lambda^{-1}B_0$, in general, the set is not closed.

4 Interpreting noticeable differences

In the preceding section we established a necessary and sufficient conditions under which a set of observations is rationalisable by a semiorder P with a noticeable difference λ. Additionally, we provided a method of eliciting the utility function u that represented P in the sense specified in (2) in Section 2. In this part of the paper we claim that the semiorder rationalisation may be interpreted as maximisation of the utility u with thickened indifferences curves, where the thickness is determined by the parameter λ.

Take any $\lambda > 1$ and assume that preferences of a consumer can be represented by a continuous utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y. Consider a model of choice such that, for all $(B, x) \in \mathcal{O}$,

$$u(z) \geq u(x) \ \text{ implies } \ u(\lambda z) > u(y), \ \text{for all } \ y \in B \ \text{ and non-zero } \ z \in \mathbb{R}^\ell_+. \tag{4}$$

Even though the consumer determines the choice using her actual utility u, she is making a consistent error in evaluating which bundles are superior to x. In particular, although there might be an available option y such that $u(y) > u(x)$, the choice x remains optimal in the above sense unless there is some z such that $u(y) \geq u(\lambda z)$ and $u(z) \geq u(x)$. That is, the indifference curves intersecting bundles y and x must be sufficiently far apart for y to be chosen over x.[17] Roughly speaking, the consumer optimises as if her indifference curves were thick, where the thickness is determined by λ. For a graphical interpretation of this condition, recall Figure 1 in the Introduction. An important feature of the above model is that it is specified in ordinal terms. Therefore, cardinality of the utility function u plays no role in determining optimal choices.

[17] Let U_x and U_y denote indifference curves that contain x and y, respectively. Bundle x is optimal only if the value $\sup\{\theta > 0 : z \in U_x \text{ and } (\theta z) \in U_y\}$ is strictly lower than λ.

It is easy to show that if set $\mathcal{O}$ is rationalisable by a semiorder P with a noticeable difference λ, then it can be supported by the above model of consumer choice. Following our discussion in Section B.2 of the Online appendix, whenever the dataset is rationalisable in the above sense, the semiorder P is representable by a continuous, increasing utility function u that satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y. In addition, it must be that $(B, x) \in \mathcal{O}$ and $y \in B$ implies $u(x) + 1 \geq u(y)$. Whenever the utility u violates condition (4), there is some $(B, x) \in \mathcal{O}$, $y \in B$, and a non-zero z such that $u(y) \geq u(\lambda z) > u(z) + 1 \geq u(x) + 1$, which contradicts that $\mathcal{O}$ is rationalisable by P.

Interestingly, the converse implication also holds. Therefore, given a set of observations $\mathcal{O}$ specified as in Section 3.1, it is impossible to distinguish between the two forms of rationalisation. We state this observation formally in the following proposition.

Proposition 5. *Suppose that set $\mathcal{O}$ is rationalisable according to (4) for some $\lambda > 1$ and a continuous function $u : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ that satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y. Then, the set is rationalisable by a semiorder P with the noticeable difference λ.*

Proof. Define relation P by: xPy if and only if $u(x) \geq u(\lambda z)$ and $u(z) \geq u(y)$, for some non-zero z. First of all, since $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y, we obtain $u(\lambda' y) \geq u(\lambda y) > u(y)$, for any $\lambda' \geq \lambda$. Thus, we have $(\lambda' y)Py$.

Next, we argue that P is a semiorder. Note that xPx only if there is a non-zero z such that $u(x) \geq u(\lambda z)$ and $u(z) \geq u(x)$. As $u(\lambda z) > u(z)$, this implies that $u(x) > u(x)$, yielding a contradiction. Thus, P is irreflexive. To show that it satisfies $(S1)$, let xPy and $x'Py'$. By construction, there is some non-zero z, z' such that $u(x) \geq u(\lambda z)$ and $u(z) \geq u(y)$, as well as $u(x') \geq u(\lambda z')$ and $u(z') \geq u(y')$. Whenever $u(y) \geq u(y')$, then $u(x) \geq u(\lambda z)$ and $u(z) \geq u(y')$. Thus, we obtain xPy'. Otherwise, it must be that $x'Py$. Using an analogous argument, we show that property $(S2)$ is also satisfied.

To prove that P rationalises $\mathcal{O}$, take any observation (B, x) and $y \in B$. Whenever yPx, there is some non-zero z such that $u(y) \geq u(\lambda z)$ and $u(z) \geq u(x)$. However, this contradicts that utility u rationalises the data according to (4). □

Proposition 5 provides an alternative interpretation of the Main Theorem. Suppose that a set of observations $\mathcal{O}$ violates GARP, but satisfies Axiom 1 for some $\lambda > 1$. Clearly, there is no locally non-satiated utility function u that rationalises the data. However, the above result suggests that λ may be interpreted as the consistent error that the consumer is making while maximising her preferences. In particular, the parameter determines thickness of indifference curves that the agent is using when comparing alternatives. As the testable implications of this model are equivalent to those of semiorder maximisation, the two interpretations describe the same observable behaviour.

This result highlights the sense in which noticeable differences measure departures from rationality. Specifically, the closer λ is to 1, the thinner become the indifference curves that the consumer is using when making a comparison, and the closer are her choices to those of the utility maximiser. In fact, for λ arbitrarily close to 1, the model in (4) coincides with maximisation of the function u.[18] In the following section we explore the properties of noticeable differences as measures of GARP violations.

5 Measuring revealed preference violations

There is an extensive literature on various methods of evaluating severity of departures from rationality. For a detailed discussion see Apesteguia and Ballester (2015) or Chapter 5.1 in Echenique and Chambers (2016). In this section we explore the properties of the just-noticeable difference as an alternative measure of GARP violations.

Formally, let $\lambda_* \geq 1$ denote the infimum over all numbers $\lambda > 1$ for which a set of observations $\mathcal{O}$ satisfies Axiom 1. Henceforth, we refer to λ_* as the *just-noticeable difference*. In the remainder of this section, we find it convenient to focus on the inverse of λ_* as a possible measure of revealed preference violations. Nevertheless, any property established for $(1/\lambda_*)$ can be easily reformulated for λ_*.

Since $\lambda_* \geq 1$, the value of its inverse ranges from 0 to 1. In particular, Proposition 2 implies that the index is equal to 1 if and only if the dataset satisfies GARP. To better understand this measure, let $\theta_{ts} := \inf\big\{\theta > 0 : x_s \in \theta B_t\big\}$, for all t, $s \in T$. Therefore, the number θ_{ts} is the least value by which one can scale set B_t without excluding x_s. This may be interpreted as the relative distance of the bundle x_s from the upper bound of B_t. Notice that, for any $\lambda > 1$ it must be that $\lambda^{k_{ts}} \geq \theta_{ts} > \lambda^{k_{ts}-1}$, where integer k_{ts} is defined as previously. Given this and (3), we show in Section A.3 of the Appendix that

$$\lambda_*^{-2} \;\leq\; \min\left\{ \sqrt[|\mathcal{C}|]{\textstyle\prod_{(t,s)\in\mathcal{C}} \theta_{ts}} \;:\; \mathcal{C} \text{ is a cycle in } T\times T \right\} \;\leq\; \lambda_*^{-1}.$$

[18] Suppose that (4) holds for all $\lambda > 1$. Whenever there is some observation $(B, x) \in \mathcal{O}$ and a bundle $y \in B$ such that $u(y) > u(x)$, continuity and monotonicity of u guarantees existence of some λ sufficiently close to 1 for which $u(y) \geq u(\lambda x) > u(x)$. However, this violates our initial assumption.

Hence, the inverse of the just-noticeable difference is an approximation of the least geometric mean of numbers θ_{ts} over all cycles in the dataset. The index measures a violation of GARP by evaluating the minimal average distance of bundle x_s from the upper bound of the corresponding set B_t, along all observable cycles. To provide a better context, we discuss the remaining properties of our index by comparing it to two well-known measures of revealed preference violations: Afriat's efficiency index and the money-pump index. Specifically, we argue that our approach combines the desirable features of both indices, while responding to the main criticism addressed at those measures.[19]

5.1 Afriat's efficiency index and just-noticeable differences

First, we compare the inverse of the just-noticeable difference to a generalised version of Afriat's efficiency index, proposed in Afriat (1973). A dataset $\mathcal{O}$ is rationalisable by a locally non-satiated utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ for an efficiency index $e \in [0, 1]$ if

$$(B, x) \in \mathcal{O} \text{ and } y \in eB \;\text{ implies }\; u(x) \geq u(y). \tag{5}$$

Afriat's efficiency index, denoted by e_*, is the supremum over all efficiency indices e for which there exists a locally non-satiated utility function u rationalising $\mathcal{O}$.

Equivalently, a dataset $\mathcal{O}$ obeys GARP for an efficiency index $e \in [0, 1]$, if for any sequence $\mathcal{C} = \big\{(a, b), (b, c), \ldots, (z, a)\big\}$ in $T \times T$ with $x_s \in eB_t$, for all $(t, s) \in \mathcal{C}$, we have $x_s \in \partial(eB_t)$, for all $(t, s) \in \mathcal{C}$.[20] Therefore, Afriat's efficiency index e_* is the supremum over all efficiency indices for which $\mathcal{O}$ obeys GARP.

Similarly to Axiom 1, Afriat's efficiency index imposes a weaker restriction on a set of observations than GARP. In particular, a dataset $\mathcal{O}$ obeys the latter condition only if $e_* = 1$; however, the converse need not be true (see Example 1 below). Before discussing further distinctions between this measure and the inverse of the just-noticeable difference, we point out an important relation satisfied by the two indices.

Proposition 6. *Whenever dataset $\mathcal{O}$ obeys Axiom 1 for some $\lambda > 1$, then it satisfies GARP for any efficiency index $e \leq \lambda^{-1}$.*

[19] We select measures that are both commonly used in the literature and can be evaluated using efficient algorithms. For other notable indices see, e.g., Houtman and Maks (1985) and Dean and Martin (2016).

[20] That is, bundle x_s belongs to the upper bound of eB_t, for all $(t, s) \in \mathcal{C}$. The fact, that a dataset $\mathcal{O}$ is rationalisable as in (5) for some efficiency index $e \in [0, 1]$ if and only if it obeys GARP for e, follows from a modification of the argument in Forges and Minelli (2009, Section 1.2).

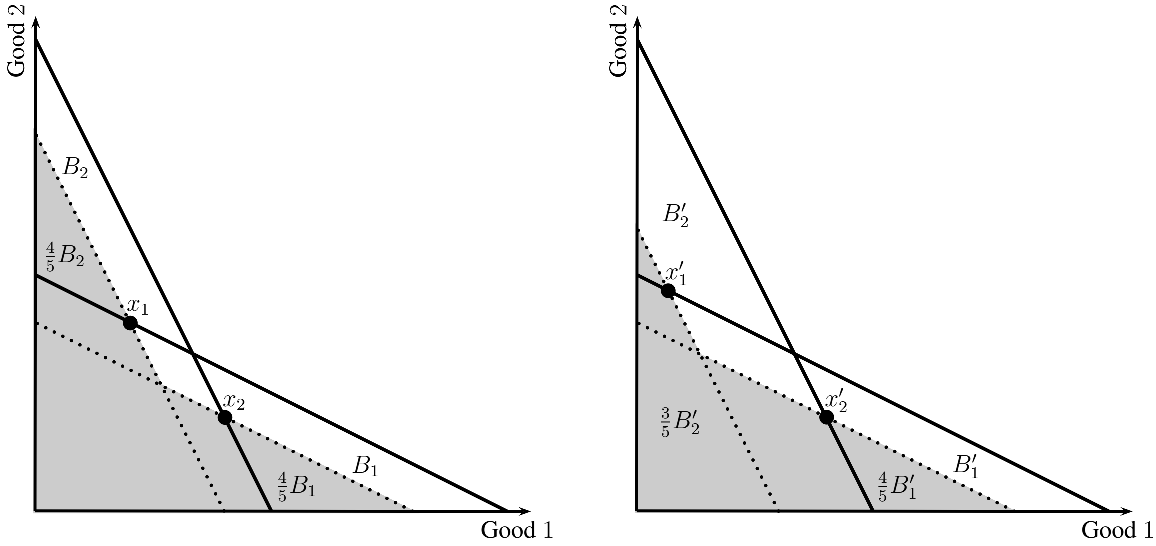

Figure 5: Sets $\mathcal{O} = \{(B_1, x_1), (B_2, x_2)\}$ and $\mathcal{O}' = \{(B_1', x_1'), (B_2', x_2')\}$ violate GARP. In either case, the corresponding Afriat's efficiency index is 4/5. However, the inverse of the just-noticeable difference is equal to 4/5 and $\sqrt{3/5}$, respectively.

Proof. Suppose that set $\mathcal{O}$ obeys Axiom 1 for some $\lambda > 1$. Whenever the dataset admits a sequence $\mathcal{C} = \{(a, b), (b, c), \dots, (z, a)\}$ in $T \times T$ with $x_s \in \lambda^{-1} B_t$, for all $(t, s) \in \mathcal{C}$, the corresponding integers k_{ts} must be less than -1, for all (t, s). This implies that

$$-\sum_{(t,s)\in\mathcal{C}} k_{ts} \geq |\mathcal{C}|,$$

which yields a contradiction. Thus, no such cycle is admissible. This suffices for the dataset to satisfy GARP for any efficiency index $e \leq \lambda^{-1}$. □

Proposition 6 implies that the inverse of the just-noticeable difference is always lower than Afriat's efficiency index. However, the opposite implication need not hold. That is, there are observation sets that obey GARP for some efficiency index e, but fail to satisfy Axiom 1 for $\lambda = e^{-1}$. For instance, recall the dataset in Figure 3. Notice that, the corresponding Afriat's efficiency index is equal to $e_* = 9/10$. At the same time, the inverse of the just-noticeable difference is $\lambda_*^{-1} = \sqrt{4/5} < \sqrt{81/100} = 9/10$.

The above property is implied by one crucial feature of Afriat's efficiency index. As noted in Echenique, Lee, and Shum (2011), this measure seeks to "break" a violation of GARP at its weakest link. Consider the example in Figure 5. Clearly, both sets $\mathcal{O}$ and $\mathcal{O}'$ violate GARP. Moreover, the inverse of the just-noticeable difference corresponding to the latter set is lower than the one evaluated for the former. This would suggest that the violation in $\mathcal{O}'$ is more severe than in $\mathcal{O}$, as it would require a larger value of the just-noticeable difference λ_* to rationalise $\mathcal{O}'$ in the sense discussed in Section 3. At the same time, Afriat's efficiency index does not discriminate between these two cases.

More generally, suppose that set $\mathcal{O}$ admits a sequence $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ in $T \times T$ with $x_s \in B_t$, for all (t,s) in $\mathcal{C}$. Clearly, we have $\theta_{ts} \leq 1$, for any pair of indices (t,s) in the cycle, where numbers θ_{ts} are specified as previously. In order to "break" the above cycle, it suffices to scale each budged set by a number strictly less than $\bar{\theta}_{\mathcal{C}} := \max\{\theta_{ts} : (t,s) \in \mathcal{C}\}$. In fact, Afriat's efficiency index is the minimal $\bar{\theta}_{\mathcal{C}}$ over all such sequences $\mathcal{C}$. Hence, it is the least value by which one has to scale each budget set B_t in order to break every cycle in the set of observations.

Unlike Afriat's efficiency index, the purpose of the inverse of the just-noticeable difference is not to "break" cycles. Rather, by approximating the geometric mean of numbers θ_{ts} along such sequences, our measure quantifies the severity of a GARP violation in each cycle revealed in the set of observations. We revisit an example from Echenique, Lee, and Shum (2011, p. 1209) to highlight the extent to which the same violation of revealed preference can be evaluated differently by these two measures.

Example 1. Take any $\epsilon \in (0,1)$ and consider a dataset $\mathcal{O} = \{(B_t, x_t) : t = 1,2,3\}$, where $x_1 = (\epsilon^2, 0)$, $x_2 = \big(\epsilon^2/(1+\epsilon^2), \epsilon^2/(1+\epsilon^2)\big)$, and $x_3 = (0,1)$, while

$$B_t = \{y \in \mathbb{R}^2_+ : p_t \cdot y \leq p_t \cdot x_t\},$$

for all $t = 1,2,3$, with $p_1 = (1/\epsilon, \epsilon)$, $p_2 = (\epsilon, 1/\epsilon)$, and $p_3 = (1,1)$. Note that, by construction of this example, we obtain $\theta_{12} = 1$, $\theta_{23} = 1$, and $\theta_{31} = \epsilon^2$.

Take any number $\lambda > 1$ and sequence $\mathcal{C} = \{(1,2),(2,3),(3,1)\}$. Evaluate the corresponding integers k_{ts} as in Axiom 1. In particular, recall that $\lambda^{k_{ts}} \geq \theta_{ts} \geq \lambda^{k_{ts}-1}$. Therefore, we have $k_{12} = k_{23} = 0$, while set $\mathcal{O}$ obeys the axiom only if $k_{31} > -3$, or

$$\sqrt[3]{\lambda^{k_{12}}\ \lambda^{k_{23}}\ \lambda^{k_{31}}} \;=\; \sqrt[3]{\lambda^{k_{31}}} \;>\; \lambda^{-1},$$

where $\lambda^{k_{31}} \geq \epsilon^2 > \lambda^{k_{31}-1}$. Specifically, this implies that $\lambda^{-1} < \sqrt{\epsilon}$. As ϵ tends to 0, the infimum over numbers λ under which the set $\mathcal{O}$ satisfies Axiom 1 tends to infinity, while its inverse approaches 0. At the same time, it is easy to verify that Afriat's efficiency index is equal to 1, irrespectively of the value of the parameter ϵ.

Varian (1990) modified the Afriat's measure by considering a vector $(e_t)_{t\in T}$ of efficiency indices, rather than a scalar. The objective is to choose numbers e_t in $[0,1]$, as close to 1 as possible (according to some metric), such that the dataset $\{(e_t B_t, x_t) : t \in T\}$ satisfies GARP. Varian's idea is similar to the original efficiency index, as numbers e_t are selected in order to eliminate all cycles in the dataset.[21] In fact, it is easy to show that the index is bounded from below by the Afriat's measure. Thus, given Proposition 6, it always dominates the inverse of the just-noticeable difference.

The interpretation of efficiency indices differs from the just-noticeable difference. The former can be perceived as a margin of error the agent is allowed to make in her choices in terms of tolerance for wasted expenditure. Analogously, the measure determines the size of consideration sets — the greater is the deviation from GARP, the smaller is the set of options considered by the decision maker. In contrast, our approach pertains to the consumer's inability to discern between bundles that are similar. By thickening the indifference curves that the agent is using while making a choice, we increase the number of binary comparisons in which the agent is indifferent.

An important feature of Afriat's (and Varian's) efficiency index is that it allows to infer consumer preference from the data. Specifically, by assuming that the observed bundle x is superior to every option in the perturbed set $e_* B$, one is able to construct a revealed preference relation that is consistent with any utility function u rationalising the observations over the restricted sets, as in (5).[22] Similarly, our approach provides not only a model of consumer behaviour that supports choices which fail to satisfy GARP, but also allows us to recover preferences from the observable data — both the semiorder P generating the observations and the "true" utility function u.

5.2 Money-pump index and just-noticeable differences

Echenique, Lee, and Shum (2011) introduce an alternative measure of revealed preference violations, called the *money-pump index*. As this notion refers to choices over linear budget sets, in this subsection we restrict our attention to datasets $\mathcal{O} = \{(B_t, x_t) : t \in T\}$ such that, for all $t \in T$ and some price $p_t \in \mathbb{R}^\ell_{++}$, we obtain

$$B_t = \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\}.$$

In such a case, the dataset satisfies GARP if for any sequence $\mathcal{C} = \{(a, b), (b, c), \ldots, (z, a)\}$

[21] However, the measure possesses some additional properties. See Apesteguia and Ballester (2015).

[22] See also Halevy, Persitz, and Zrill (2017) for an extensive discussion on how efficiency indices can be applied to recover preferences from observable data using parametric methods.

in $T \times T$ such that $p_t \cdot x_s \leq p_t \cdot x_t$, for all $(t, s) \in \mathcal{C}$, all the inequalities are binding.

Suppose that the dataset $\mathcal{O}$ admits a cycle $\mathcal{C}$ with $p_t \cdot x_s \leq p_t \cdot x_t$, for all $(t, s) \in \mathcal{C}$. The money-pump index $m_{\mathcal{C}}$ corresponding to such a sequence is given by

$$(1 - m_{\mathcal{C}}) \;:=\; \frac{\sum_{(t,s)\in\mathcal{C}} p_t \cdot (x_t - x_s)}{\sum_{(t,s)\in\mathcal{C}} p_t \cdot x_t}.$$ [23]

To obtain a single value of the measure, Echenique, Lee, and Shum propose to take the mean or the median of money-pump indices, over all cycles $\mathcal{C}$. Alternatively, Smeulders, Cherchye, De Rock, and Spieksma (2013) suggest to use the least or the greatest value of the index, as it reduces the computational burden of evaluating the measure. In order to guarantee that the value of the index is equal to 1 if and only if $\mathcal{O}$ satisfies GARP, we find it appropriate to use its mean or minimum.[24]

The money-pump index has an intuitive interpretation. Consider a sequence $\mathcal{C}$ specified as previously. Suppose there is a devious "arbitrageur" who is aware of the choices made by the consumer. In particular, by acquiring bundle x_s from the agent at prices p_t and reselling it at p_s, for any pair (t, s) in $\mathcal{C}$, the arbitrageur would make a positive profit equal to $p_a \cdot (x_a - x_b) + p_b \cdot (x_b - x_c) + \ldots + p_z \cdot (x_z - x_a)$. This value is the *money-pump cost* associated with the sequence. Hence, measure $(1 - m_{\mathcal{C}})$ is the money-pump cost normalised by the total expenditure in the cycle $\mathcal{C}$.

Similarly to the inverse of the just-noticeable difference, the purpose of the money-pump index is to evaluate the extent to which a cycle observed in the dataset is violating GARP. To show the distinction between our approach and the above measure, take any cycle $\mathcal{C} = \big\{(a, b), (b, c), \ldots, (z, a)\big\}$. Since we focus on linear budget sets, we obtain $\theta_{ts} = p_t \cdot x_s / p_t \cdot x_t$, for all $t, s \in T$. Once we let $r_t := p_t \cdot x_t / \sum_{(i,j)\in\mathcal{C}} p_i \cdot x_i$ to be the share of budget $p_t \cdot x_t$ in the total expenditure along the cycle $\mathcal{C}$, this allows us to reformulate the money-pump index associated with cycle $\mathcal{C}$ as

$$m_{\mathcal{C}} \;=\; \sum_{(t,s)\in\mathcal{C}} r_t \theta_{ts}.$$

Therefore, the value of $m_{\mathcal{C}}$ is equivalent the to the algebraic mean of scalars θ_{ts}, for all $(t, s) \in \mathcal{C}$, weighted by the shares r_t. Just like the inverse of the just-noticeable difference,

[23] The original definition of the money-pump index for sequence $\mathcal{C}$ is equivalent to $(1-m_{\mathcal{C}})$. We modified the notion to make it comparable with the other measures discussed in this paper.

[24] Notice that, it is possible for a set of observations to violate GARP, even though the median or the maximal value of the corresponding money-pump index is equal to 1.

rather than focusing on "breaking" cycles, the money-pump index measures GARP violations by evaluating the average deviation from rationality along every such sequence. Hence, apart from the obvious differences, the idea behind the two approaches is similar. Specifically, since both measures are sufficiently sensitive to changes in the dataset, they are not vulnerable to the critique of efficiency indices presented by Echenique, Lee, and Shum, or discussed in Figure 5 and Example 1.

Despite the similarities, there are several significant differences between the two approaches. First of all, since $m_{\mathcal{C}}$ depends on weights r_t, scaling prices $\{p_t\}_{t\in T}$ by different constants affects the value of the index. This is because absolute values of expenditures are relevant for establishing the money-pump cost associated with each cycle. In contrast, the just-noticeable difference (as well as Afriat's and Varian's efficiency indices) focus only on the actual budget sets. Thus, they are independent of such transformations. Please note that it is arguable which property is more desirable.

Second of all, unlike efficiency indices or the just-noticeable difference, the money pump-index does not provide a method of eliciting preference from the observable data. Even though it measures the extent to which the agent violates GARP, it does not propose an alternative model that would explain her behaviour.

Finally, suppose that we normalise prices in the dataset, so that $p_t \cdot x_t = 1$, for all $t \in T$. Clearly, such a transformation is irrelevant to whether the set of observations satisfies GARP. However, the corresponding money-pump index changes its interpretation. Namely, since $r_t = 1/|\mathcal{C}|$ for each $t \in T$, value $(1 - m_{\mathcal{C}})$ determines the mean arithmetic share of each budget set along the cycle that the consumer is losing to the arbitrageur. Thus, it is a money-pump in terms of fractions of expenditure losses. This interpretation is analogous to the inverse of the just-noticeable difference, which approximates the geometric mean of shares $\theta_{ts} = p_t \cdot x_s / p_t \cdot x_t$ along each cycle $\mathcal{C}$.

5.3 A significant just-noticeable difference

When measuring revealed preference violations, it is fundamental to determine the value of the index at which it is considered significant. In this subsection, we propose one simple method of establishing such a critical value by employing the notion of the just-noticeable difference. Our approach is not a formal statistical test; rather, it is an objective way of establishing which values of our measure can be regarded as "large".

Assume that consumer preference can be represented by a continuous utility function u that satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero $y \in \mathbb{R}^{\ell}_{+}$. Given our discussion in Section 3.1, any set of observations generated via maximisation of the above relation would satisfy GARP. However, whilst evaluating the available options, the agent is prone to a particular error. To be precise, for each $t \in T$, there is a number $\lambda_t > 1$ drawn from a cumulative probability distribution $F : (1, \infty) \to [0, 1]$ such that, for all $t \in T$:

$$u(z) \geq u(x) \;\text{ implies }\; u(\lambda_t z) > u(y), \;\text{ for all }\; y \in B_t \;\text{ and non-zero }\; z \in \mathbb{R}^{\ell}_{+}.$$

Thus, bundle x_t need not maximise u over B_t, as there might be some available option y such that $u(y) > u(x_t)$. Nevertheless, it remains optimal in the above sense as long as the indifference curves intersecting x_t and y are sufficiently close together. This model of consumer choice resembles the one introduced in Section 4. However, in this case, we do not assume that the agent is making a consistent error; rather, we require for λ_t to be drawn independently in each observation and vary between choices.

In the remainder of this subsection we assume that $\lambda_t = \exp|\epsilon_t|$, where ϵ_t is drawn independently from a normal distribution with mean 0 and a standard deviation σ, for each $t \in T$. Equivalently, this is to say that $F(z) = \Phi_\sigma(\log z)$, where Φ_σ denotes the half-normal cumulative distribution with mean $\sigma\sqrt{2/\pi}$.

Suppose we observe the numbers λ_t, for all $t \in T$, and let $\lambda := \max\{\lambda_t : t \in T\}$. In the null hypothesis of our test we assume that the realisations are drawn independently from the F. Our decision to reject the hypothesis is based on the statistic λ. Given the construction of the variable, its cumulative distribution is given by $G(\lambda) := [F(\lambda)]^{|T|}$.[25] We reject the null hypothesis if λ is strictly greater than the critical value $c_\alpha := G^{-1}(1-\alpha)$, where $\alpha \in [0, 1]$ is a predetermined confidence level. That is, we consider it unlikely for the set of observations to be generated by the above model if the probability of drawing a number greater than λ from G is lower than α.

The intuition of our test is the following. Suppose an agent is a utility maximiser, but in each observation t the choice is affected by a noticeable difference λ_t drawn independently from distribution F. Unless the maximal value of the observed realisations exceeds the $(1-\alpha)$'th quantile of the the corresponding distribution G, we have no basis for to reject the hypothesis that the set of observations was generated by the above model.

[25]This follows from the fact that $\text{Prob}(\lambda \leq z) = \text{Prob}(\lambda_t \leq z, \text{ for all } t \in T)$.

Otherwise, we consider it unlikely for the data to be generated by utility maximisation with the particular form of random error.

Unfortunately, the actual realisation of the statistic λ is unobservable. Nevertheless, we claim that it is always bounded below by the just-noticeable difference λ_*. In fact, it is easy to verify that, whenever a set of observations $\mathcal{O}$ is rationalisable in the above sense, then $u(z) \geq u(x_t)$ implies $u(\lambda z) > u(y)$, for all $y \in B_t$ and non-zero $z \in \mathbb{R}^\ell_+$, $t \in T$. Therefore, any such dataset can be supported by the model discussed in Section 4. Moreover, Proposition 5 and the argument in Section 4 guarantees that the set of observations satisfies Axiom 1 for λ. Hence, it must be that $\lambda_* \leq \lambda$.

Given the above observation, we employ λ_* as an alternative statistic.[26] Thus, we reject the null hypothesis if $\lambda_* > c_\alpha$. Introducing λ_* to our test has a significant drawback. Namely, whenever λ_* is strictly greater than c_α, the null hypothesis may be rejected with at least the imposed confidence level α. However, substituting the true value of λ with the just-noticeable difference leads to under-rejection of the null hypothesis, affecting the actual power of the test. This is because $\lambda_* \leq c_\alpha$ does not exclude the possibility that the actual realisation of λ is strictly greater that the critical value.

The result of our test depends crucially on the choice of the parameter σ that determines the cumulative probability distribution F.[27] In the following section we perform the test for several values of σ to provide ourselves with a better intuition of what could be regard as a significantly "large" value of the just-noticeable difference.

6 Implementation

We implement our test using household-level scanner panel data from the so-called Stanford Basket Dataset. The set of observations was collected by Information Resources Inc. and has been thoroughly analysed, among others, in Echenique, Lee, and Shum (2011) and Smeulders, Cherchye, De Rock, and Spieksma (2013). The panel contains grocery expenditure data for 494 households, aggregated monthly, with 26 purchase observations per household, and the total number of 375 available commodities. For further details, see Section IV of Echenique, Lee, and Shum (2011).

[26] Constructing a statistical test by bounding the true statistic from below has been performed in, e.g., Varian (1985) and Echenique, Lee, and Shum (2011).

[27] A similar issue was encountered in, e.g., Varian (1985), Echenique, Lee, and Shum (2011).

	Just-noticeable difference	Inverse of the just-noticeable difference	Afriat's efficiency index	Minimal money-pump index
Mean	**1.0898**	**.9210**	**.9440**	**.9064**
	[.0690]	[.0529]	[.0502]	[.0616]
Minimum	1.0042	.6868	.6868	.5990
First quartile	1.0436	.8946	.9260	.8726
Median	1.0707	.9340	.9563	.9203
Third quartile	1.1178	.9583	.9760	.9511
Maximum	1.4559	.9958	1.0000	.9951

Table 1: Descriptive statistics for the just-noticeable difference, its inverse, Afriat's efficiency index, and the minimal money-pump index evaluated for the available sample of households. The values within brackets indicate standard deviations of the variables. Recall that Afriat's efficiency index may take the value of 1 even if the set of observations violates GARP.

In our analysis we assume that, for each household, an observation set $\mathcal{O}$ consists of elements (B_t, x_t), for $t = 1, \ldots, 26$, where $x_t \in \mathbb{R}^\ell_+$ denotes a bundle of $\ell = 375$ consumption goods selected at observation t from a linear budget set

$$B_t = \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\},$$

given the observed shelf prices $p_t \in \mathbb{R}^\ell_{++}$ of all available commodities. In the preliminary analysis we established that, out of all 494 households, 396 violated GARP. The remainder of our analysis pertains to the latter subset of households.

6.1 Eliciting the just-noticeable difference

For each household, we determined the infimum over all numbers $\lambda > 1$ under which the corresponding dataset obeyed Axiom 1.[28] Thus, the just-noticeable difference λ_*. In the first column of Table 1 we present descriptive statistics of the empirical distribution of the parameter, evaluated for the analysed sample of households. The median of λ_* was approximately 1.07. Roughly speaking, this is to say that in order to rationalise the median set of observations by a semiorder P with a noticeable difference, we had to assume that a 7% increase in the size of a bundle was sufficient to perceive the change. For 65% of datasets, the just-noticeable difference did not exceed 1.10.

[28]Recall that, for a given value of $\lambda > 1$, it is possible to verify Axiom 1 in a polynomial time. We approximate the infimal value of the parameter using the *binary search* algorithm.

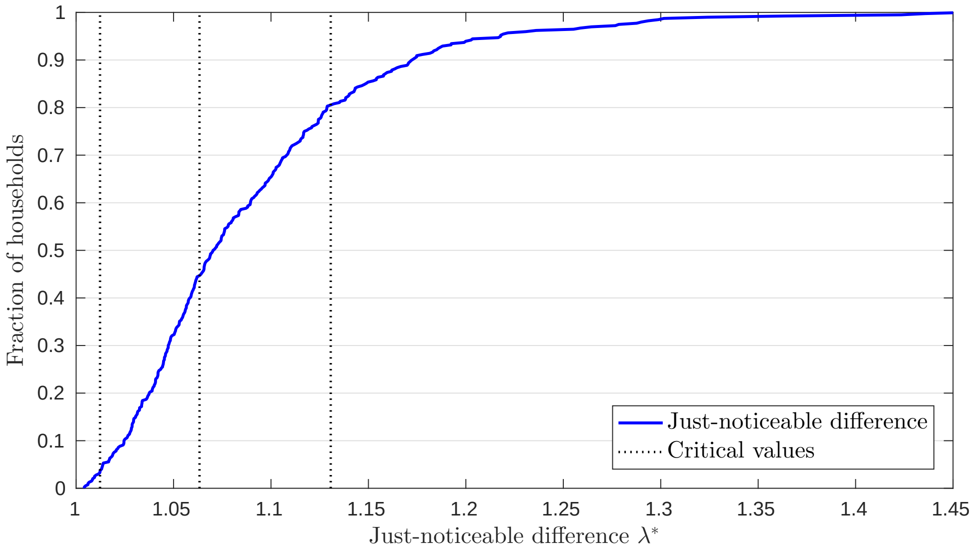

Figure 6: Empirical cumulative distribution of the just-noticeable difference λ_*. The vertical dotted lines mark the critical values of the variable evaluated for the parameter σ equal to .005, .025, and .05, respectively. See Table 2 for details.

The existing literature on sensory discrimination provides a variety of estimates for the so-called *Weber's constant*, for different types of stimuli. Recall that, the above value is defined as the ratio between magnitudes of two physical stimuli that allows a subject to discriminate between the sensations. See Laming (1997) for a detailed discussion on this and alternative measures of sensory discrimination. Depending on the type of the stimuli, the estimates of this ratio range from 1.017 for the perception of cold and temperature, to 1.25 when it concerns discriminating between bitter substances. See Tables 8.1-8.3 in Laming (1997) for a summary of estimates obtained in the literature. Our evaluation of the median/mean just-noticeable difference λ_* is included in this range. Nevertheless, one should be cautious with such comparisons. First of all, the estimates of Weber's constant were obtained for one-dimensional stimuli, rather than a combination of sensations. Second of all, these fractions were estimated in probabilistic terms. That is, they determined a threshold for which an agent correctly indicated the more intense stimuli with a predetermined frequency. At the same time, parameter λ_* represents the minimal value by which one should scale a bundle x in order to guarantee that the agent prefers $(\lambda_* x)$ to x, for *all* non-zero x. Finally, it is arguable whether preference over consumption bundles and physical stimuli are comparable notions.

In order to provide ourselves with an intuition whether the obtained values of the just-noticeable difference were significantly high, we employed the test introduced in

	Value of σ		
	.005	.025	.05
Critical value	1.0156	1.0804	1.1673
% of households	5.8 (24.5)	60.6 (68.4)	91.6 (93.3)

Table 2: Percentage of households for which the value of the just-noticeable difference was not significantly large, for the confidence level of .05. Values within the brackets indicate the total percentage of datasets, including the 98 that initially satisfied GARP.

Section 5.3. We performed our analysis for three values of the parameter σ: .005, .025, and .05. To put those numbers in to perspective, this implied that the theoretical mean value of the random variable λ_t was approximately 1.004, 1.020, and 1.041, respectively. Table 2 presents the evaluated critical values for the confidence level $\alpha = .05$ as well as the percentage of households for which we were unable to reject the null hypothesis. That is, for the indicated share of households we did not dismiss the possibility that the data was generated by maximisation of a locally non-satiated utility function with the particular random error. Equivalently, we concluded that choices of those households were not significantly violating GARP, conditional on the value of σ.

For the middle value of the parameter $\sigma = .025$, the critical value of our test was 1.0804. This is to say that, if one was willing to accept that the mean of the random error affecting consumer preferences was 1.020, i.e., on average the agent was able to notice a 2% increase in the size of a bundle, then the probability of observing a just-noticeable difference λ_* greater than 1.0804 was lower than $\alpha = .05$. Under this assumption, the null hypothesis had to be rejected for almost 40% of datasets. However, since the test generally under-rejected the the null hypothesis, the actual number of such subjects was possibly higher. Finally, in order to conclude that choices of all households were not significantly violating GARP, it was necessary to assume that σ was at least .1213. Thus, the theoretical mean of the variable λ_t had to be greater than 1.10.

6.2 Comparing measures of revealed preference violations

In the second part of our analysis we performed a comparison between different measures of revealed preference violations. More precisely, we focused on three indices that were discussed in Section 5. Namely, the inverse of the just-noticeable difference, Afriat's efficiency index, and the minimal money-pump index. Concerning the latter measure, we decided to restrict our attention to the minimum over money-pump indices for two reasons. First of all, unlike the median or the maximum, the minimum attains the value of 1 if and only if the corresponding set of observations satisfies GARP. Second of all, unlike the mean, determining the exact value of the minimum over all money-pump indices is computationally efficient, thus, it can be performed in a polynomial time. See Smeulders, Cherchye, De Rock, and Spieksma (2013) for details.

The remaining three columns of Table 1 provide descriptive statistics of the evaluated distributions of the indices. See also Figure 7. First of all, the distributions of all measures were statistically distinct.[29] In addition, for each household, the corresponding inverse of the just-noticeable difference and the minimum of the money-pump index were lower than Afriat's efficiency index. By Proposition 6, such a relation was expected from our measure, as it is always dominated by Afriat's efficiency index. However, given the discussion in Section 5, it is easy to show that the same property generally holds for the minimum of the money-pump index.[30] In addition, the distribution of the minimal money-pump index was stochastically dominated by the distribution of the inverse of the just-noticeable difference, even though there was no such relation between the two measures in the individual-level data.[31]

We found it instructive to determine correlations between the three measures of revealed preference violations. The correlation between the inverse of the just-noticeable difference and the minimal money-pump index was .9822. Thus, the values of the two measures seemed to be very closely related in the analysed dataset. In fact, this is even more visible in Figure 8 (on the right). Simultaneously, the two indices were less correlated with Afriat's efficiency index, although values of the corresponding coefficients remained high — respectively .8514 and .8614 for the inverse of the just-noticeable difference and the minimal money-pump index. Nevertheless, following the discussion in Section 5, this observations should not be surprising. Recall that, by construction, the inverse of the just-noticeable difference and the money-pump index are more sensitive

[29] In each pairwise comparison, we rejected the null hypothesis of the Anderson-Darling test that the random variables were drawn from the same distribution, for all conventional confidence levels.

[30] By construction of the efficiency index, there is a cycle $\mathcal{C}$ such that $e_* = \max\{\theta_{ts} : (t,s) \in \mathcal{C}\}$. Hence, we have $m_{\mathcal{C}} = \sum_{(t,s)\in\mathcal{C}} r_t \theta_{ts} \leq e_*$. In particular, this holds for the minimal money-pump index.

[31] To be precise, for 378 out of 396 households the minimal money-pump index was strictly lower that the inverse of the just-noticeable difference. Values of the two measures were never equal.

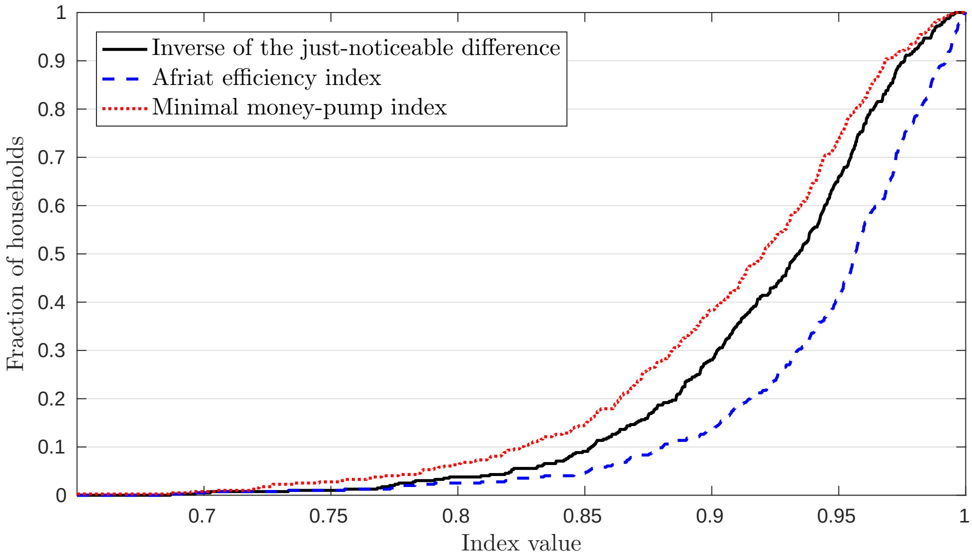

Figure 7: Empirical cumulative distributions of the inverse of the just-noticeable difference, Afriat's efficiency index, and the minimal money-pump index.

to the distribution of observations in a dataset than Afriat's efficiency index. Therefore, they should respond similarly to particular violations of GARP.

Finally, we analysed differences in how each measure ordered individual datasets using Spearman's rank correlation. In case of the just-noticeable difference and the money-pump index the correlation was .9731. Hence, the two approaches ordered datasets in a similar manner. At the same time, these rankings were less consistent with the one induced by Afriat's efficiency index. The corresponding correlations were .7953 and .8227 for the just-noticeable difference and the money pump index, respectively.

7 Conclusion

We provided the testable implications for a model of consumer choice in which an agent was unable to discriminate between similar options. Analogously to GARP, which is a special case of this condition, the purpose of our axiom was to exclude a specific type of cycles from the dataset. Furthermore, the method allowed us to elicit the just-noticeable difference which was sufficient for the agent to discern between bundles as well as the "true" underlying preference that generated the observations (i.e., as is perfect discrimination were possible). Finally, we showed that the evaluated noticeable difference provided an informative measure of deviations from rationality in terms of utility maximisation with thickened indifference curves.

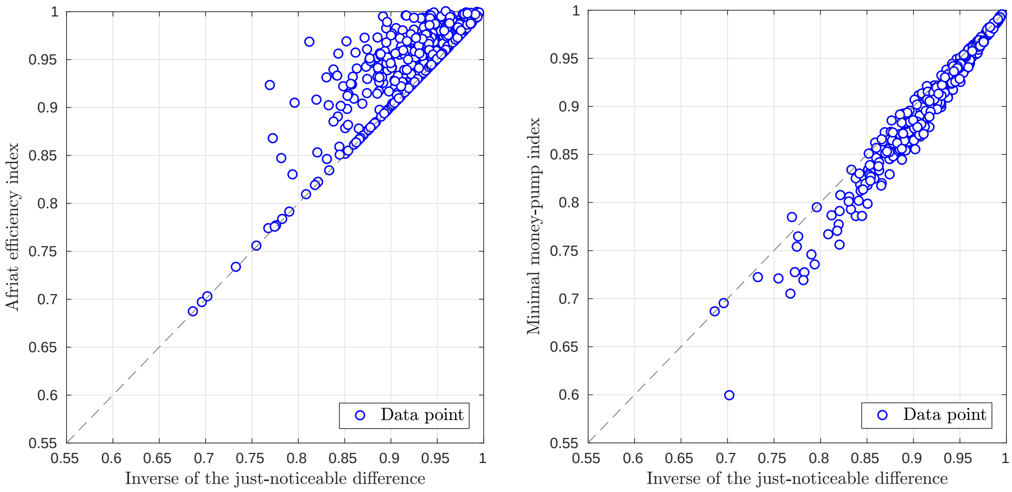

Figure 8: Inverse of the just-noticeable difference against Afriat's efficiency index (on the left) and the minimal money-pump index (on the right).

In the empirical analysis performed on a scanner panel data of food purchases we evaluated the just-noticeable difference for individual households. The median value of the parameter was approximately 1.07. That is, a 7% increase in the bundle size was sufficient for the median consumer to notice the difference.

A Appendix

We begin this section by presenting one auxiliary result to which we refer in the main body of the paper. In the second part of the appendix we prove Proposition 1 and the property of the just-noticeable difference to which we referred to in Section 5.

A.1 Auxiliary result

Take any real numbers $\delta > 0$ and q_{ts}, for all indices t, s that belong to some finite set T. Consider the following proposition.

Proposition A.1. *Suppose that for any sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ such that $q_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, we have $-\sum_{(t,s)\in\mathcal{C}} q_{ts} < |\mathcal{C}|\delta$. There are real numbers $\{\phi_t\}_{t\in T}$ and $\mu > 1$ such that $q_{ts} \leq 0$ implies $\phi_s - \delta \leq \phi_t + \mu q_{ts}$, for all t, s in T.*

Proof. We construct those numbers recursively. First, choose any $\mu > 1$ such that, for any cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ with $q_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, we have $-\mu\sum_{(t,s)\in\mathcal{C}} q_{ts} \leq |\mathcal{C}|\delta$. Clearly, this is always possible.

With a slight abuse of the notation, we shall assume that $T = \{1, 2, \ldots, T\}$. Moreover, let $\mathcal{S}_{az} = \big\{(a,b),(b,c),\ldots,(y,z)\big\}$ denote a sequence in $T \times T$ satisfying $q_{ts} \leq 0$, for all $(t,s) \in \mathcal{S}_{az}$. Take any number ϕ_1. For an arbitrary $t \geq 2$, let ϕ_t satisfy

$$\phi_s + \mu \sum_{(i,j)\in\mathcal{S}_{st}} q_{ij} + |\mathcal{S}_{st}|\delta \ \geq \ \phi_t \ \geq \ \phi_r - \mu \sum_{(i,j)\in\mathcal{S}_{tr}} q_{ij} - |\mathcal{S}_{tr}|\delta,$$

for all $\mathcal{S}_{st}$, $\mathcal{S}_{tr}$ and s, $r < t$. First, we show that such ϕ_t always exists. If not, then

$$\phi_s + \mu \sum_{(i,j)\in\mathcal{S}_{st}} q_{ij} + |\mathcal{S}_{st}|\delta \ < \ \phi_r - \mu \sum_{(i,j)\in\mathcal{S}_{tr}} q_{ij} - |\mathcal{S}_{tr}|\delta,$$

for some $\mathcal{S}_{st}$, $\mathcal{S}_{tr}$ and s, $r < t$. Consider three cases. Whenever (i) $s < r$, then

$$\phi_s + \mu\left(\sum\nolimits_{(i,j)\in\mathcal{S}_{st}} q_{ij} + \sum\nolimits_{(i,j)\in\mathcal{S}_{tr}} q_{ij}\right) + \big(|\mathcal{S}_{st}| + |\mathcal{S}_{tr}|\big)\delta \ < \ \phi_r.$$

However, as $\mathcal{S}_{st}$ and $\mathcal{S}_{tr}$ form a sequence $\big\{(s,a),\ldots,(z,t),(t,a'),\ldots,(z',r)\big\}$, this would be inconsistent with the construction of ϕ_r. Analogously, if (ii) $s > r$, then

$$\phi_s \ < \ \phi_r - \mu\left(\sum\nolimits_{(i,j)\in\mathcal{S}_{st}} q_{ij} + \sum\nolimits_{(i,j)\in\mathcal{S}_{tr}} q_{ij}\right) - \big(|\mathcal{S}_{st}| + |\mathcal{S}_{tr}|\big)\delta,$$

which would be inconsistent with the construction of ϕ_s. Finally, if (iii) $s = r$, then $-\mu\big(\sum_{(i,j)\in\mathcal{S}_{st}} q_{ij} + \sum_{(i,j)\in\mathcal{S}_{tr}} q_{ij}\big) > \big(|\mathcal{S}_{st}| + |\mathcal{S}_{tr}|\big)\delta$. Given that $\mathcal{S}_{st}$ and $\mathcal{S}_{tr}$ form a cycle $\big\{(s,a),\ldots,(z,t),(t,a'),\ldots,(z',r)\big\}$, this would be inconsistent with the choice of μ.

Take any such numbers $\{\phi_t\}_{t\in T}$ and μ. Note that $q_{ts} \leq 0$ implies $\phi_s - \delta \leq \phi_t + \mu q_{ts}$, for all t, s in T. Take any t, s in T such that $q_{ts} \leq 0$. In particular, there is a one-element sequence $\mathcal{S}_{ts} = \big\{(t,s)\big\}$. Suppose that $t < s$. By construction of ϕ_t, ϕ_s, it must be that

$$\phi_t + \mu q_{ts} + \delta \ = \ \phi_t + \mu \sum_{(i,j)\in\mathcal{S}_{ts}} q_{ij} + |\mathcal{S}_{ts}|\delta \ \geq \ \phi_s.$$

Analogously, we can show that the inequality holds for $t > s$. Finally, whenever $t = s$, then $\phi_s - \delta \leq \phi_s + \mu q_{ts} = \phi_t + \mu q_{ts}$, since $-\mu q_{ts} \leq \delta$. □

A.2 Proof of Proposition 1

In this part of the appendix we present the postponed proof of Proposition 1. The notation employed below corresponds directly to the one introduced in the article. Before we proceed, we find it convenient to state two preliminary results.

Lemma A.1. *If either (i) xPy, yPx', and $x'Iy'$; (ii) xPy, yIx', and $x'Py'$; or (iii) xIy, yPx', and $x'Py'$; then xPy', for any x, y, x', y' in X.*

Proof. To show that (i) implies xPy', notice that, by condition (S2), if xPy and yPx' then either xPy' or $y'Px'$. As $x'Iy'$, it must be that xPy'. Next, suppose that (ii) is true. Given xPy and $x'Py'$, condition (S1) requires that either xPy' or $x'Py$. Since yIx', we have xPy'. Finally, we show that (iii) implies xPy'. Assuming that yPx' and $x'Py'$, condition (S2) implies either yPx or xPy'. Given xIy, we conclude that xPy'. □

The following result provides a useful property of semiorderings.

Lemma A.2. *For any sequence $\mathcal{S} = \{(x_i, x_{i+1})\}_{i=1}^m$ with $x_i R x_{i+1}$, for all $i = 1, \dots, m$, let $\mathcal{P}_\mathcal{S}$ and $\mathcal{I}_\mathcal{S}$ denote sets containing pairs (x_i, x_{i+1}) such that $x_i P x_{i+1}$ and $x_i I x_{i+1}$, respectively. Whenever (i) $|\mathcal{P}_\mathcal{S}| > |\mathcal{I}_\mathcal{S}| \geq 1$; or (ii) $|\mathcal{P}_\mathcal{S}| = |\mathcal{I}_\mathcal{S}| > 2$ is satisfied, there is a sequence $\mathcal{S}' = \{(y_j, y_{j+1})\}_{j=1}^n$ with $y_j R y_{j+1}$, for all $j = 1, \dots, n$, such that $y_1 = x_1$ and $y_{n+1} = x_{m+1}$, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_\mathcal{S}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_\mathcal{S}| - 1$.*

Proof. Take any such sequence $\mathcal{S} = \{(x_i, x_{i+1})\}_{i=1}^m$. First, we claim that if (i) or (ii) holds, there is some $i = 1, \dots, (m-2)$ such that either (a) $x_i I x_{i+1}$, $x_{i+1} P x_{i+2}$, and $x_{i+2} P x_{i+3}$; (b) $x_i P x_{i+1}$, $x_{i+1} I x_{i+2}$, and $x_{i+2} P x_{i+3}$; or (c) $x_i P x_{i+1}$, $x_{i+1} P x_{i+2}$, and $x_{i+2} I x_{i+3}$.

We prove the claim by contradiction. Suppose that $|\mathcal{P}_\mathcal{S}| = k$. By assumption, we have $k \geq 2$. If there is no $i = 1, \dots, (m-2)$ that obeys (a), (b), or (c), then it must be $|\mathcal{I}_\mathcal{S}| \geq 2(k-1)$.[32] Therefore, if $|\mathcal{P}_\mathcal{S}| > |\mathcal{I}_\mathcal{S}|$ then $k < 2$. On the other hand, $|\mathcal{P}_\mathcal{S}| = |\mathcal{I}_\mathcal{S}|$ implies $k \leq 2$. In either case, we reach a contradiction.

Given the above observation, take any $i = 1, \dots, (m-2)$ that satisfies (a), (b), or (c). By Lemma A.1, this implies that $x_i P x_{i+3}$. Consider the following sequence:

$$\mathcal{S}' := \{(x_1, x_2),\ \dots\ ,\ (x_{i-1}, x_i),\ (x_i, x_{i+3}),\ (x_{i+3}, x_{i+4}),\ \dots\ ,\ (x_m, x_{m+1})\}.$$

Clearly, for any $(y, y') \in \mathcal{S}'$, we have yRy', while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_\mathcal{S}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_\mathcal{S}| - 1$. □

We proceed with the argument supporting Proposition 1.

Proof of Proposition 1. Take any sequence $\mathcal{S} = \{(x_i, x_{i+1})\}_{i=1}^m$ such that $x_i R x_{i+1}$, for all $i = 1, \dots, m$. Moreover, let $\mathcal{P}_\mathcal{S}$ and $\mathcal{I}_\mathcal{S}$ denote sets of pairs (x_i, x_{i+1}) satisfying $x_i P x_{i+1}$ and $x_i I x_{i+1}$, respectively. First, we show that $|\mathcal{P}_\mathcal{S}| = |\mathcal{I}_\mathcal{S}| = k$ implies $x_1 \succ x_{m+1}$. Our argument is inductive on the cardinality of sets $\mathcal{P}_\mathcal{S}$ and $\mathcal{I}_\mathcal{S}$. By conditions (W1) and (W2), the result holds for $k = 1$. Suppose that $k = 2$, hence, $m = 4$. By Lemma A.1,

[32] In such a case, any two pairs (x, y), (x', y') such that xPy and $x'Py'$ must be separated by at least two other elements (u, v), (u', v') for which uIv and $u'Iv'$.

if there is some $i = 1, 2$ such that (a) $x_i I x_{i+1}$, $x_{i+1} P x_{i+2}$, and $x_{i+2} P x_{i+3}$; (b) $x_i P x_{i+1}$, $x_{i+1} I x_{i+2}$, and $x_{i+2} P x_{i+3}$; or (c) $x_i P x_{i+1}$, $x_{i+1} P x_{i+2}$, and $x_{i+2} I x_{i+3}$; then $x_i P x_{i+3}$. In particular, there is a sequence $\mathcal{S}' = \{(y_1, y_2), (y_2, y_3)\}$ such that $y_1 = x_1$ and $y_3 = x_5$, while $y_1 P y_2$ and $y_2 I y_3$, or $y_1 I y_2$ and $y_2 P y_3$. In either case, (W1) and (W2) imply that $x_1 \succ x_5$. Alternatively, if there is no $j = 1, 2$ that obeys (a), (b), or (c), then

$$x_1 P x_2,\ x_2 I x_3,\ x_3 I x_4,\ \text{and}\ \ x_4 P x_5.$$

By definition of $\succeq$, we have $x_1 \succ x_3$ and $x_3 \succ x_5$, which implies $x_1 \succ x_5$.

Suppose our initial claim is true for some $k \geq 2$. If $|\mathcal{P}_\mathcal{S}| = |\mathcal{I}_\mathcal{S}| = k + 1$, Lemma A.2 guarantees that there is a sequence $\mathcal{S}' = \{(y_j, y_{j+1})\}_{j=1}^n$ with $y_j R y_{j+1}$, for $j = 1, \ldots, n$, such that $y_1 = x_1$ and $y_{n+1} = x_{m+1}$, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_\mathcal{S}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_\mathcal{S}| - 1$. Given that $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{I}_{\mathcal{S}'}| = k$, by our initial assumption $x_1 = y_1 \succ y_{n+1} = x_{m+1}$.

Next, we show that $|\mathcal{P}_\mathcal{S}| > |\mathcal{I}_\mathcal{S}| = k$ implies $x_1 P x_{m+1}$. Our argument is inductive on the cardinality of the set $\mathcal{I}_\mathcal{S}$. By transitivity of P, the result holds for $k = 0$. Suppose the claim is true some $k \geq 0$. Whenever $|\mathcal{P}_\mathcal{S}| > |\mathcal{I}_\mathcal{S}| = k + 1$, Lemma A.2 guarantees that there is a sequence $\mathcal{S}' = \{(y_j, y_{j+1})\}_{j=1}^n$ with $y_j R y_{j+1}$, for $j = 1, \ldots, n$, such that $y_1 = x_1$ and $y_{n+1} = x_{m+1}$, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_\mathcal{S}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_\mathcal{S}| - 1$. Given that $|\mathcal{P}_{\mathcal{S}'}| > |\mathcal{I}_{\mathcal{S}'}| = k$, our initial assumption guarantees that $x_1 = y_1 P y_{n+1} = x_{m+1}$. Since $x_1 P x_{m+1}$ implies $x_1 \succ x_{m+1}$, this concludes the proof. □

A.3 Just-noticeable difference as a bound of revealed cycles

In this subsection we argue that the just-noticeable difference λ_* corresponding to a set of observations $\mathcal{O} := \{(B_t, x_t) : t \in T\}$ satisfies

$$\lambda_*^{-2} \ \leq\ \min\left\{ \sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \theta_{ts}} \ :\ \mathcal{C} \text{ is a cycle in } T \times T \right\} \ \leq\ \lambda_*^{-1},$$

where we denote $\theta_{ts} := \inf\{\theta > 0 : x_s \in \theta B_t\}$.

In order to prove the first inequality, take any $\lambda > 1$ for which set $\mathcal{O}$ obeys Axiom 1 and evaluate the corresponding integers $k_{ts} := \inf\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\}$, for all t, $s \in T$. In particular, we have $\lambda^{k_{ts}-1} \leq \theta_{ts}$, for all t, $s \in T$. Given condition (3), this implies that

$$\sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \theta_{ts}} \ \geq\ \sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \lambda^{k_{ts}-1}} \ =\ \lambda^{-1} \sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \lambda^{k_{ts}}} \ >\ \lambda^{-2},$$

for any cycle $\mathcal{C}$. We obtain the desired inequality by taking the supremum with respect to λ over the right hand side of the above condition.

We prove the second inequality by contradiction. Since $\lambda^{k_{ts}} \geq \theta_{ts}$, for all $t, s \in T$, whenever it is violated, there is a number $\lambda > 1$ such that, for any cycle $\mathcal{C}$, we obtain

$$\lambda_*^{-1} \;<\; \lambda^{-1} \;<\; \sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \theta_{ts}} \;\leq\; \sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \lambda^{k_{ts}}}.$$

However, by condition (3), this would imply that set $\mathcal{O}$ obeys Axiom 1 for $\lambda < \lambda_*$, thus, contradicting that the latter is the just-noticeable difference.

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Eliciting the just-noticeable difference: Online appendix

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October 2017

Abstract

We gather some extensions and additional results on the testable implications of consumer choice with noticeable differences presented in Dziewulski (2017). It is advised to read these notes in conjunction with the main paper.

Keywords: revealed preference, testable restrictions, semiorder, just-noticeable difference, GARP, Afriat's efficiency index, money-pump index
JEL Classification: C14, C60, C61, D11, D12

B.1 Overview

We introduce results that extend the analysis in Dziewulski (2017). In Section B.2 we provide an alternative proof to the Main Theorem of the original paper. Apart from arguing that Axiom 1 implies existence of a semiorder P with a noticeable difference $\lambda > 1$, we show that the weak order $\succeq$ induced by P is representable by a continuous and increasing utility u that satisfies $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y.

The subsequent Section B.3 focuses on the testable implications of locally non-satiated semiorders. We show that GARP is both a necessary and sufficient condition for rationalisation within this class of preferences. Therefore, such a model of consumer choice is empirically indistinguishable from locally non-satiated utility maximisation.

Finally, in Section B.4 we investigate the testable implications of a wider class of semiorders. We determine a necessary and sufficient condition under which a set of

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observations $\mathcal{O}$ can be rationalised by a semiorder P that satisfies:

$$\text{if} \;\; y \in \Gamma(x) \;\; \text{then} \;\; yPx,$$

for a particular correspondence $\Gamma : \mathbb{R}^\ell_+ \to \mathbb{R}^\ell_+$. This approach provides tools for the analysis of a wider class of preferences. At the same time, the observable restriction for such models are analogous to those presented in Section 3 of the main paper.

B.2 Alternative proof of the Main Theorem

In this section we present an alternative proof to the sufficiency part of the main result. Suppose that set $\mathcal{O}$ obeys Axiom 1 of the original paper for some $\lambda > 1$. We argue that this is equivalent to existence of a semiorder P with a noticeable difference λ that rationalises the set $\mathcal{O}$. Moreover, the relation induces a weak order $\succeq$ that is representable by a continuous and increasing utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$, satisfying $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y. If set B is co-convex, for all $(B, x) \in \mathcal{O}$, then the function is also quasiconcave. Throughout, we employ the notation introduced in the original paper.

Take any dataset $\mathcal{O} := \{(B_t, x_t) : t \in T\}$ specified as in Section 3.1 of the main paper. For each $t \in T$, define function $\gamma_t : \mathbb{R}^\ell_+ \to \mathbb{R}_+$ by

$$\gamma_t(y) := \inf \{\theta > 0 : y \in \theta B_t\}. \tag{B.1}$$

We refer to γ_t as the *gauge function* of set B_t. Roughly speaking, value $\gamma_t(y)$ determines the scalar θ for which bundle y belongs to the upper bound of set θB_t. In particular, any such function satisfies $\gamma_t(\theta y) > \gamma_t(y)$, for all $\theta > 1$ and non-zero y. By assumptions imposed on sets B_t, the following result by Forges and Minelli (2009) holds true.

Proposition B.1. *For all $t \in T$, function γ_t is continuous, increasing, and homogeneous of degree one. Moreover, we have $B_t = \{y \in \mathbb{R}^\ell_+ : \gamma_t(y) \leq 1\}$.*

We proceed with the following lemma.

Lemma B.1. *For any $\lambda > 1$, $\epsilon > 0$, and $t \in T$, there is a continuous and increasing function $h_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfies the following properties:*

(i) $y \in B_t$ if and only if $h_t(y) \leq 0$;

(ii) $\theta > 1$ implies $h_t(\theta y) > h_t(y)$, for all non-zero y;

(iii) $\lambda' \geq \lambda$ implies $h_t(\lambda' y) \geq h_t(y) + 1$, for all non-zero y;

(iv) $k_{ts} \geq h_t(x_s) > k_{ts} - \epsilon$*, where* $k_{ts} := \inf\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\}$*, for all* $s \in T$*.*

Moreover, if the complement of set B_t *is convex then* h_t *is quasiconcave.*

Proof. Take any $\lambda > 1$, $\epsilon > 0$, and $t \in T$. Denote $k_{ts} := \inf\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\}$, for each $s \in T$. Choose any number $v_* > 1$ such that $v_* \leq \min\{\gamma_t(x_s)/\lambda^{k_{ts}-1} : s \in T\}$, where the gauge function γ_t is defined as in (B.1). Given that $\lambda^{k_{ts}} \geq \gamma_t(x_s) > \lambda^{k_{ts}-1}$, such a scalar always exists. Moreover, we have $v_* \leq \lambda$. Define function $f : [1, \lambda] \to \mathbb{R}$ by

$$f(y) := \min\{a(y-1),\ b(y-\lambda)+1\},$$

for any strictly positive numbers a and b such that $a > (1-\epsilon)/(v_*-1)$ and $b < \epsilon/(\lambda - v_*)$. It is straightforward to verify that the function is continuous and strictly increasing, while $f(\lambda) = 1$ and $f(1) = 0$. Hence, we have $f(y) - f(z) \geq -1$, for all y, $z \in [1, \lambda]$, where the inequality is binding only if $z = \lambda$ and $y = 1$. Finally, notice that $1 \geq f(v_*) > 1 - \epsilon$.

We extend function f to $\mathbb{R}_+$ as follows. Let $g : \mathbb{R}_+ \to \mathbb{R}$ be given by

$$g(y) := f\big(y/\lambda^k\big) + k, \quad \text{for} \quad y \in \big(\lambda^k, \lambda^{k+1}\big] \quad \text{and} \quad k \in \mathbb{Z},$$

which is continuous. To prove that it is strictly increasing, take any $y > z$. Whenever y, $z \in \big(\lambda^k, \lambda^{k+1}\big]$, for some $k \in \mathbb{Z}$, the result follows from monotonicity of f. Alternatively, let $y \in \big(\lambda^k, \lambda^{k+1}\big]$ and $z \in \big(\lambda^l, \lambda^{l+1}\big]$, for some k, $l \in \mathbb{Z}$ with $k \geq l + 1$. Recall that, for any a, $b \in [1, \lambda]$, we have $f(a) - f(b) \geq -1$, where the inequality is binding only if $a = 1$ and $b = \lambda$. In particular, if $f\big(y/\lambda^k\big) - f\big(z/\lambda^l\big) = -1$ then $y = \lambda^k$ and $z = \lambda^{l+1}$. Since $y > z$, this holds only if $k \geq l + 2$. Therefore, we obtain

$$g(y) - g(z) \;=\; f\big(y/\lambda^k\big) + k - f\big(z/\lambda^l\big) - l \;=\; f\big(y/\lambda^k\big) - f\big(z/\lambda^l\big) + (k-l) \;>\; 0.$$

Next, we argue that $\lambda' \geq \lambda$ implies $g(\lambda' y) \geq g(y) + 1$, for any $y > 0$. Suppose that $y \in \big(\lambda^k, \lambda^{k+1}\big]$. This implies that λy belongs to $\big(\lambda^{k+1}, \lambda^{k+2}\big]$. Therefore, we have

$$\begin{aligned} g(\lambda' y) - g(y) &\geq g(\lambda y) - g(y) \\ &= f\big(\lambda y/\lambda^{k+1}\big) + (k+1) - f\big(y/\lambda^k\big) - k \\ &= f\big(y/\lambda^k\big) - f\big(y/\lambda^k\big) + 1 \\ &= 1, \end{aligned}$$

where the first inequality follows from monotonicity of function g.

Finally, notice that $g(y) \leq 0$ if and only if $y \leq 1$. In order to show this, recall that $g(1) = f(1) = 0$. Since function g is strictly increasing, this proves the claim.

Define function $h_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ by $h_t(y) := (g \circ \gamma_t)(y)$. Clearly, it is continuous. Moreover, as g and γ_t are increasing, so is h_t. In order to show that the function h_t satisfies condition (i), recall from Proposition B.1 that $y \in B_t$ is equivalent to $\gamma_t(y) \leq 1$. By the previous observation regrading function g, we have $y \in B_t$ if and only if

$$h_t(y) \;=\; (g \circ \gamma_t)(y) \;\leq\; g(1) \;=\; 0.$$

Next, we show that h_t satisfies condition (ii). Recall from Proposition B.1 that function γ_t is homogeneous of degree one. Therefore, for any $\theta > 1$ and non-zero y, we have $\gamma_t(\theta y) = \theta\gamma_t(y) > \gamma_t(y)$. By monotonicity of g, this implies that $h_t(\theta y) > h_t(y)$.

In order to prove property (iii), take any $\lambda' \geq \lambda$ and a non-zero y. Let $z := \gamma_t(y)$. Homogeneity of γ_t implies that $\gamma_t(\lambda' y) = \lambda'\gamma_t(y) = \lambda' z$. Thus, we obtain

$$h_t(\lambda' y) \;=\; (g \circ \gamma_t)(\lambda' y) \;=\; g(\lambda' z) \;\geq\; g(z) + 1 \;=\; (g \circ \gamma_t)(y) + 1 \;=\; h_t(y) + 1.$$

We argue that h_t obeys condition (iv). Recall the construction of the number v_*. In particular, it satisfies $1 < v_* \leq \gamma_t(x_s)/\lambda^{k_{ts}-1}$, for all $s \in T$. As $\gamma_t(x_s) \in \big(\lambda^{k_{ts}-1}, \lambda^{k_{ts}}\big]$, while values of function f range from 0 to 1, we obtain

$$k_{ts} \;\geq\; f\big(\gamma_t(x_s)/\lambda^{k_{ts}-1}\big) + (k_{ts} - 1) \;=\; (g \circ \gamma_t)(x_s) \;=\; h_t(x_s).$$

On the other hand, recall that $f(v_*) > 1 - \epsilon$. Since function f is monotone and takes values between 0 and 1, while $v_* \leq \gamma_t(x_s)/\lambda^{k_{ts}-1}$, we obtain

$$h_t(x_s) \;=\; (g \circ \gamma_t)(x_s) \;=\; f\big(\gamma_t(x_s)/\lambda^{k_{ts}-1}\big) + (k_{ts} - 1) \;\geq\; f\big(v_*\big) + (k_{ts} - 1) \;>\; k_{ts} - \epsilon.$$

Finally, suppose that the complement of set B_t is convex, for all $t \in T$. First, we claim that function γ_t is quasiconcave. Take any y and y' in $\mathbb{R}^\ell_+$ and suppose that $\gamma_t(y) = \theta$, while $\gamma_t(y') = \theta'$. Without loss of generality, we may assume that $\theta' \geq \theta$. Moreover, let $z := (\theta/\theta')y'$. Clearly, we have $y' \geq z$ and $\gamma_t(z) = \theta$. In particular, as the complement of set B_t is convex, for any $\delta \in [0, 1]$, element $\delta y + (1 - \delta)z$ does not belong to the interior of set θB_t. Thus, $\gamma_t\big(\delta y + (1 - \delta)z\big) \geq \theta$. By monotonicity of γ_t:

$$\min\big\{\gamma_t(y), \gamma_t(y')\big\} \;=\; \theta \;\leq\; \gamma_t\big(\delta y + (1 - \delta)z\big) \;\leq\; \gamma_t\big(\delta y + (1 - \delta)y'\big).$$

Hence, function γ_t is quasiconcave. Since h_t is a strictly monotone transformation of γ_t, it preserves this property. This concludes our argument. □

In the following lemma we present an alternative characterisation of Axiom 1.

Lemma B.2. *Take any set $\mathcal{O}$ and $\lambda > 1$. There exist functions $h_t : \mathbb{R}^\ell_+ \to \mathbb{R}$, for $t \in T$, such that $\mathcal{O}$ obeys Axiom 1 for λ if and only if, for any cycle $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ with $h_t(x_s) \leq 0$, for all $(t,s) \in \mathcal{C}$, we have*

$$-\sum_{(t,s)\in\mathcal{C}} h_t(x_s) < |\mathcal{C}|.$$

Moreover, function h_t is continuous, increasing, and satisfies properties (i)–(iii) from Lemma B.1; if B_t is co-convex, then h_t is quasiconcave, for all $t \in T$.

Proof. Consider a set $\mathcal{O}$ and $\lambda > 1$. Take any $\epsilon \leq 1/|T|$ and an arbitrary function $h_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ specified as in Lemma B.1, for $t \in T$. Thus, any such function is continuous, increasing, and satisfies properties (i)–(iv). Moreover, if the complement of set B_t is convex, then function h_t is quasiconcave, for all $t \in T$.

To prove ($\Rightarrow$), suppose that set $\mathcal{O}$ satisfies Axiom 1 for λ. Take an arbitrary sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$. By property (i) of Lemma B.1, this is equivalent to $h_t(x_s) \leq 0$, for all such pairs (t,s). Let number k_{ts} be defined as in the axiom. By assumption, it must be that $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$. In particular, as $|\mathcal{C}|$ and k_{ts} are integers, for all $(t,s) \in \mathcal{C}$, this implies that

$$|\mathcal{C}| + \sum_{(t,s)\in\mathcal{C}} k_{ts} \;\geq\; 1 \;\geq\; \tfrac{|\mathcal{C}|}{|T|} \;\geq\; |\mathcal{C}|\epsilon,$$

and so $|\mathcal{C}| + \sum_{(t,s)\in\mathcal{C}}(k_{ts} - \epsilon) \geq 0$. Given that h_t satisfies property (iv) in Lemma B.1, for all $t \in T$, we have $h_t(x_s) > k_{ts} - \epsilon$, for each pair $(t,s) \in \mathcal{C}$. Then it must be that

$$0 \;\leq\; |\mathcal{C}| + \sum_{(t,s)\in\mathcal{C}} (k_{ts} - \epsilon) \;<\; |\mathcal{C}| + \sum_{(t,s)\in\mathcal{C}} h_t(x_s).$$

Next, we show ($\Leftarrow$). Consider a sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ with $h_t(x_s) \leq 0$, for all $(t,s) \in \mathcal{C}$. Since h_t obeys property (i) from Lemma B.1, for all $t \in T$, this is equivalent to $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$. By condition (iv) in Lemma B.1, we have $k_{ts} \geq h_t(x_s)$, for any pair of indices (t,s) in $\mathcal{C}$. Therefore, we obtain

$$0 \;<\; |\mathcal{C}| + \sum_{(t,s)\in\mathcal{C}} h_t(x_s) \;\leq\; |\mathcal{C}| + \sum_{(t,s)\in\mathcal{C}} k_{ts},$$

which implies that $\mathcal{O}$ obeys Axiom 1 for λ. This concludes our argument. □

Notice that, the proofs of Lemmas B.1 and B.2 are constructive. In particular, we show explicitly how to obtain functions h_t employed in the two results, for $t \in T$.

Lemma B.3. *Set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$ only if there are real numbers $\{\phi_t\}_{t\in T}$ and $\mu > 1$ such that $h_t(x_s) \leq 0$ implies $\phi_s - 1 \leq \phi_t + \mu h_t(x_s)$, for all t, $s \in T$, where function h_t is specified as in Lemma B.2, for all $t \in T$.*

This result follows directly from Lemma B.2 above and Proposition A.1 in the original paper. Given the above results, we proceed with the construction of a semiorder rationalising the set $\mathcal{O}$. For any $t \in T$, take a function h_t specified in Lemma B.2, and numbers $\{\phi_t\}_{t\in T}$, μ satisfying inequalities in Lemma B.3. Take any $\nu > 1$ such that $h_t(x_s) > 0$ implies $\phi_s - 1 \leq \phi_t + \nu h_t(x_s)$, for all t, $s \in T$. Define $v_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ by

$$v_t(y) := \begin{cases} \phi_t + \mu h_t(y) & \text{if } \ h_t(y) \leq 0, \\ \phi_t + \nu h_t(y) & \text{otherwise.} \end{cases}$$

Moreover, let function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ be given by $u(y) := \min \big\{ v_s(y) : s \in T \big\}$. Since h_t is continuous and increasing, so is u. Moreover, it is easy to show that $u(\theta y) > u(y)$, for all $\theta > 1$ and non-zero y. In addition, as in the proof of Lemma 2 in the main paper, we may argue that $\lambda' \geq \lambda$ implies $u(\lambda' y) > u(y) + 1$, for all non-zero y, while $(B, x) \in \mathcal{O}$ and $y \in B$ imply $u(x) + 1 \geq u(y)$. Finally, whenever the complement of set B is convex, for all $(B, x) \in \mathcal{O}$, Lemma B.1 implies that function h_t is quasiconcave, for all $t \in T$. Since the min operator preserves quasiconcavity, this suffices for u to be quasiconcave.

Take any function u constructed as above and define a semiorder P such that xPy if and only if $u(x) > u(y) + 1$. As in Section 3.3 of the main paper, the relation admits a noticeable difference λ and rationalises $\mathcal{O}$. Moreover, we claim in the following proposition that the utility function u represents the weak order $\succeq$ induced by P.

Proposition B.2. *For some $\lambda > 1$, let $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ be a continuous function such that $\theta > 1$ implies $u(\theta y) > u(y)$, while $\lambda' \geq \lambda$ implies $u(\lambda' y) > u(y) + 1$, for any non-zero $y \in \mathbb{R}^\ell_+$. Whenever semiorder P is defined by: xPy if and only if $u(x) > u(y) + 1$, then the weak order $\succeq$ induced by P satisfies $x \succeq y$ if and only if $u(x) \geq u(y)$.*

Proof. Suppose that $\succeq$ is a weak order induced by P. Clearly, if $x \succ y$ then $u(x) > u(y)$.[1] To prove the converse, we show that there is some $z \in \mathbb{R}^\ell_+$ such that either xPz and zRy, or xRz and zPy. This suffices for $x \succ y$. First, assume that x is non-zero. Take a number θ such that $u(x) = u(\theta x) + 1$. Since u is continuous, while $u(x) > u(\lambda^{-1} x) + 1$, such a scalar exists. Given that $u(\theta x) + 1 = u(x) > u(y)$, we can always find some $\theta' < \theta$,

[1]This follows directly from the definition of the induced weak order $\succeq$.

arbitrarily close to θ, such that

$$u(x) \;=\; u(\theta x) + 1 \;>\; u(\theta' x) + 1 \;\geq\; u(y).$$

Denote $z := \theta' x$. Clearly, this guarantees that xPz and zRy. Whenever y is non-zero, we apply an analogous argument to show that there is some $z \in \mathbb{R}^\ell_+$ such that xRz and zPy. It follows that $x \sim y$ is equivalent to $u(x) = u(y)$. □

Since the utility function u represents the induced weak order $\succeq$, by our previous observation this must imply that the relation is continuous, increasing, while $\theta > 1$ implies $(\theta y) \succ y$, for all non-zero y. Moreover, whenever the complement of B is convex, for all $(B, x) \in \mathcal{O}$, quasiconcavity of u implies convexity of $\succeq$.

The latter observation is not novel in the literature of revealed preference analysis. In fact, in his original result, Afriat (1967) stated that a set of observations $\mathcal{O}$ with linear budget sets is rationalisable by a locally non-satiated weak order if and only if it is rationalisable by a continuous and concave utility function. Thus, in such a framework continuity and concavity are not testable properties of the utility function.

However, notice that our result implies that the function u is quasiconcave but *not* concave. In fact, as we show in the following example, even if we consider choices over linear budget sets, there might be no convex function that rationalises the data.

Example B.1. Consider a set $\mathcal{O} = \big\{(B_t, x_t) : t = 1, 2, 3\big\}$, where $x_1 = (5, 5)$, $x_2 = (5, 2)$, and $x_3 = (6, 0)$, while $B_t := \big\{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\big\}$, for $t = 1, 2, 3$, where $p_1 = (2, 1)$, $p_2 = (4, 2)$, and $p_3 = (8, 2)$. The above set obeys Axiom 1 for $\lambda = 1.05$. Hence, there is a continuous, increasing, and quasiconcave function u such that

$$(B, x) \in \mathcal{O} \text{ and } \; y \in B \;\text{ implies }\; u(x) + 1 \geq u(y),$$

for all $t = 1, 2, 3$, while $\lambda' \geq \lambda$ implies $u(\lambda' y) > u(y) + 1$, for all non-zero y.

Suppose that the function u is concave, rather than quasiconcave. Note that $k_{12} = -4$ and $k_{23} = 0$, which implies that $\big(\lambda^4 x_2\big) \in B_1$ and $x_3 \in B_2$. Therefore, it must be that $u(x_2) \geq u(x_3) - 1$, since x_3 is an element of B_2, as well as

$$u(x_1) \;\geq\; u\big(\lambda^4 x_2\big) - 1 \;>\; u(x_2) + 3 \;\geq\; u(x_3) + 2.$$

Construct bundle $y = (2/3)x_1 + (1/3)x_2 = (5, 4)$ and notice that $y \in B_3$. If function u is concave, this implies that $u(y) \geq (2/3)u(x_1) + (1/3)u(x_2)$. However, as $y \in B_3$,

$$u(x_3) \geq u(y) - 1 \geq \tfrac{2}{3}u(x_1) + \tfrac{1}{3}u(x_2) - 1$$
$$> \tfrac{2}{3}\big[u(x_3) + 2\big] + \tfrac{1}{3}\big[u(x_3) - 1\big] - 1 = u(x_3),$$

which yields a contradiction. Thus, function u can not be concave. In fact, the above example shows that the function can not be concave even along each dimension.[2]

B.3 Locally non-satiated semiorders

In this section we focus on testable implication of locally non-satiated semiorders. We show that a set of observations $\mathcal{O}$ is rationalisable by such a relation if and only if it obeys GARP. Therefore, the observable restrictions for such a model of consumer choice are indistinguishable from maximisation of a locally non-satiated utility.

A semiorder P is *locally non-satiated* if for any $x \in \mathbb{R}^\ell_+$ and its neighbourhood U_x, there is some $y \in U_x$ such that yPx. For example, consider a semiorder P over $\mathbb{R}^2_+$ in which $x'_1 > x_1$ implies $(x'_1, x_2)P(x_1, x_2)$, for any x_2, and there is some $\lambda > 1$ such that $(x_1, x'_2)P(x_1, x_2)$ only if $x'_2 \geq (\lambda x_2)$, for all x_1. Thus, the agent is able to perfectly discriminate between amounts of good 1, but discerns volumes of commodity 2 only up to the noticeable difference λ. We formalise our claim in the following proposition.

Proposition B.3. *A set of observations $\mathcal{O}$ is rationalisable by a locally non-satiated semiorder P if and only if it satisfies GARP.*

Proof. As any weak order is a semiorder, the results by Afriat (1967) or Varian (1982) imply that GARP is sufficient for rationalisation by a locally non-satiated semiorder.

We prove the converse by contradiction. Suppose that set $\mathcal{O}$ is rationalisable by a locally non-satiated semiorder P, but does not satisfy GARP. In particular, there is a sequence $\mathcal{C} = \big\{(a,b),(b,c),\dots,(z,a)\big\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, but $x_s \notin \partial B_t$, for some (t,s). Take any such pair of indices and define

$$\mu_* := \min\big\{\|y - x_s\| : y \in \partial B_t\big\}$$

where $\|\cdot\|$ denotes the norm on $\mathbb{R}^\ell_+$. As B_s is compact and the norm is continuous, the number μ_* is well-defined and strictly positive. Take any strictly positive number

[2] That is, it is not true that function $v(x_i) := u(x_i, x_{-i})$ is concave on $\mathbb{R}$, for all $x_{i-1} \in \mathbb{R}^{\ell-1}_+$.

$\epsilon < \mu_*/|\mathcal{C}|$. Clearly, it exists. By local non-satiation of P, there is a sequence of vectors $\{y_i\}_{i=1}^{|\mathcal{C}|}$ such that $y_1 \in U(x_s, \epsilon)$, $y_2 \in U(y_1, \epsilon)$, ... , and $y_{|\mathcal{C}|} \in U(y_{|\mathcal{C}|-1}, \epsilon)$ such that $y_1 P x_s$, $y_2 P y_1$, ... , and $y_{|\mathcal{C}|} P y_{|\mathcal{C}|-1}$, where $U(z, \epsilon)$ denotes a ball of radius ϵ centred at z. Moreover, it must be that $y_{|\mathcal{C}|} \in B_t$, thus, $x_t R y_{|\mathcal{C}|}$. Construct a sequence

$$\big\{(x_a, x_z), (x_z, x_y), \ \dots \ , (x_t, y_{|\mathcal{C}|}), (y_{|\mathcal{C}|}, y_{|\mathcal{C}|-1}), \ \dots \ , (y_2, y_1), \\ (y_1, x_s), (x_s, x_r), \ \dots \ , (x_c, x_b), (x_b, x_a)\big\}.$$

Clearly, for each of the $2|\mathcal{C}|$ elements (y, y') of the sequence, we have yRy'. Moreover, for at least $|\mathcal{C}|$ elements we have yPy'. Therefore, by Proposition 1 of the main paper, it must be that $x_a \succ x_a$, where $\succeq$ denotes the weak order induced by P. Since $\succ$ is irreflexive, this yields a contradiction. The proof is complete. □

The above proposition suggests that with local non-satiation of preferences non-transitive indifferences play no role in observable implications of semiorder maximisation. In fact, the model is empirically equivalent to locally non-satiated utility maximisation. However, the two models are generally *not* equivalent.

It is worth pointing out that locally non-satiated semiorders do not admit the utility representation specified as in (2) in Section 2 of the original paper. In fact, it is easy to show that those relations violate Axiom A1 in Beja and Gilboa (1992), which is necessary for a semiorder to be representable in such a way. Nevertheless, as we argued in the preceding proposition, whenever set $\mathcal{O}$ obeys GARP, there exists a locally non-satiated weak order $\succeq$ (thus a semiorder) that can be represented by a utility function u in an even stronger sense, i.e., $x \succeq y$ if and only if $u(x) \geq u(y)$.

B.4 General approach to semiorder rationalisation

In this section we generalise the method developed in the main paper to a wider class of semiorders. A *property* is a well-defined correspondence $\Gamma : \mathbb{R}^\ell_+ \to \mathbb{R}^\ell_+$ satisfying (i) $x \notin \Gamma(x)$ and (ii) if $y \in \Gamma(x)$ then $\Gamma(y) \subseteq \Gamma(x)$. A relation P admits Γ if

$$y \in \Gamma(x) \ \text{ implies } \ yPx.$$

In other words, the graph $\big\{(y, x) : y \in \Gamma(x)\big\}$ of the above correspondence determines those comparisons that are necessarily included in P. Notice that, as the above implication needs to be satisfied only in one direction, the property Γ does not define the relation

P. That is, it need not be that yPx implies $y \in \Gamma(x)$. Clearly, whenever P is irreflexive and transitive, restrictions (i) and (ii) are trivially satisfied.

Throughout this section we assume that P is a semiorder. Moreover, we focus only on those properties Γ that satisfy the additional two conditions specified below.

Assumption 1. Let $\Gamma(x) \subseteq \{y \in \mathbb{R}^\ell_+ : y \geq x\}$, for all $x \in \mathbb{R}^\ell_+$.

Assumption 2. For all x, y such that $y \in \Gamma(x)$, any sequence $\{z_i\}$ in $\Gamma(x)$ satisfying $z_{i+1} \in \Gamma(z_i)$ and $y \in \Gamma(z_i)$, for all i, is finite.

Assumption 1 imposes a form of monotonicity on the semiorder. Even thought it is rather arbitrary, we find it crucial for our analysis. At the same time, the second assumption corresponds to Axiom A1 in Beja and Gilboa (1992), which is necessary for a semiorder to admit a utility representation specified as in Section 2 of the main body of the paper. The axiom requires that for any x and an infinite sequence $\{z_i\}$, if $z_i P z_{i+1}$, for all i, then xPz_n, for some n, and if $z_{i+1}Pz_i$, for all i, then z_nPx, for some n. It is easy to verify that Assumption 2 is implied by the above condition. As utility representation plays a significant role in our argument, we find this assumption to be indispensable. Consider two examples of correspondences Γ that obey both assumptions.

Example B.2. The class of semiorders that was discussed in the main body of the paper can be expressed in the above terms. Take any $\lambda > 1$ and define $\Gamma(x) := \{\lambda' x : \lambda' \geq \lambda\}$, for all non-zero $x \in \mathbb{R}^\ell_+$. Clearly, the property satisfies Assumptions 1 and 2.

Example B.3. Consider functions $\delta_i : \mathbb{R}_+ \to [\epsilon_i, \infty)$ for some $\epsilon_i > 0$, for all $i = 1, \ldots, \ell$. Define property $\Gamma : \mathbb{R}^\ell_+ \to \mathbb{R}^\ell_+$ by

$$\Gamma(x) := \Big\{y \in \mathbb{R}^\ell_+ : y \geq x \text{ and } y_i \geq x_i + \delta_i(x_i), \text{ for some } i = 1, \ldots, \ell\Big\}.$$

Therefore, the agent prefers bundle y to x if $y \geq x$ and the difference between the consumed amounts of at least one commodity i exceeds $\delta_i(x_i)$. It is straightforward to show that this property also satisfies Assumptions 1 and 2.

The main purpose of this section is to provide a necessary and sufficient condition under which a set of observations $\mathcal{O}$, defined as in Section 3.1 of the original paper, can be rationalised by a semiorder that admits a particular property Γ.

Axiom 1(∗). *Set $\mathcal{O}$ satisfies the axiom for property Γ whenever for an arbitrary sequence $\mathcal{C} = \{(a,b),(b,c),\dots,(z,a)\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, we have*

$$-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|,$$

where $(-k_{ts})$ denotes the greatest integer k for which there exists a sequence $\{y_i\}_{i=1}^{k}$ in $\Gamma(x_s) \cap B_t$ such that $y_{i+1} \in \Gamma(y_i)$, for all $(t,s) \in \mathcal{C}$.[3]

For $\Gamma(x) := \{\lambda' x : \lambda' \geq \lambda\}$, for some $\lambda > 1$, we have $k_{ts} := \inf\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\}$, for all $t,\ s \in T$. Thus, the number $(-k_{ts})$ is the greatest integer k for which $\lambda^k x_s \in B_t$. Equivalently, it is the cardinality of the longest sequence $\{y_i\}_{i=1}^{k}$ in $\Gamma(x_s) \cap B_t$ such that $y_{i+1} \in \Gamma(y_i)$. Therefore, Axiom 1 of the main paper is a special case of Axiom 1(∗).

Similarly to Axiom 1, once we evaluate numbers k_{ts}, Axiom 1(∗) can be verified efficiently using the minimal cost-to-time ratio algorithm. Nevertheless, note that the total complexity of the algorithm depends on the imposed property Γ.

Main Theorem (∗). *A set of observations $\mathcal{O}$ is rationalisable by a semiorder P that admits property Γ if and only if satisfies Axiom 1(∗) for Γ.*

In order to prove the necessity part of the above theorem, suppose that set $\mathcal{O}$ is rationalisable by a semiorder P that admits property Γ. Take any cycle $\mathcal{C}$ specified as in the axiom and evaluate numbers k_{ts}, for all $(t,s) \in \mathcal{C}$. Clearly, they are all negative.

Take any $(t,s) \in \mathcal{C}$. By construction, there is a sequence $\{y_i\}_{i=1}^{-k_{ts}}$ in $\Gamma(x_s) \cap B_t$ such that $y_{i+1} \in \Gamma(y_i)$. By definition of Γ, we have $y_1 P x_s$, $y_2 P y_1$, … , $y_{-k_{ts}} P y_{-(k_{ts}+1)}$. Since P rationalises $\mathcal{O}$ and $y_{-k_{ts}} \in B_t$, it must be that $x_t R y_{-k_{ts}}$. Construct a sequence

$$\mathcal{S}_{ts} = \{(x_t, y_{-k_{ts}}), (y_{-k_{ts}}, y_{-(k_{ts}+1)}), \dots, (y_2, y_1), (y_1, x_s)\},$$

of length $(1 - k_{ts})$ such that yRy' for all pairs (y, y'), while the number of elements (y, y') satisfying yPy' is at least $(-k_{ts})$. Note that, whenever $(-k_{ts})$ is infinite, we can construct an arbitrarily long sequence $\mathcal{S}_{ts}$ that satisfies the above properties.

Given that the above observation holds true for any $(t,s) \in \mathcal{C}$, combining all sequences $\mathcal{S}_{ts}$ allows us to construct another sequence $\{(z_i, z_{i+1})\}_{i=1}^{m}$, where $m = |\mathcal{C}| - \sum_{(t,s)\in\mathcal{C}} k_{ts}$, while $z_i R z_{i+1}$, for all $i = 1,\dots,m$, and $z_1 = z_{m+1} = x_a$. Moreover, the total number of pairs (z_i, z_{i+1}) for which $z_i P z_{i+1}$ is greater than $-\sum_{(t,s)\in\mathcal{C}} k_{ts}$. By Proposition 1 of the

[3] Whenever $\Gamma(x_s) \cap B_t = \emptyset$, let $k_{ts} = 0$.

main paper, this requires that $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$. Otherwise, it would have to be that $x_a = z_1 \succ z_{m+1} = x_a$, contradicting that the induced relation $\succ$ is irreflexive.

Just like in the proof of the main result of the original paper, the above argument highlights the fact that whenever we observe a sequence $\mathcal{S} := \{(a,b),(b,c),\dots,(y,z)\}$ of index pairs in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{S}$, then

$$-\sum_{(t,s)\in\mathcal{S}} k_{ts} \geq |\mathcal{S}| \text{ implies } x_a \succ x_z \text{ and } -\sum_{(t,s)\in\mathcal{S}} k_{ts} > |\mathcal{S}| \text{ implies } x_a P x_z.$$

Therefore, any sequence satisfying one of the above conditions reveals that the bundle x_a is strictly superior to x_z — either with respect to the induced weak ordering $\succ$ or the original semiorder P. Clearly, as both relations are irreflexive, this excludes the possibility that $x_a = x_z$. In fact, the exact role of Axiom 1(∗) is to guarantee that the set of observations $\mathcal{O}$ excludes any such cycles.

Notice that we never referred in the above argument to Assumptions 1 and 2. In fact, Axiom 1(∗) provides a necessary condition for a dataset to be rationalised within an arbitrary class of semiorders that admit a well-defined property Γ.

Next, we prove the sufficiency part of the theorem. First, for all $t \in T$, define function $g_t : \mathbb{R}^\ell_+ \to \mathbb{Z}$ as follows. Whenever $y \in B_t$, then $g_t(y) = -k$, where k is the greatest integer for which there is a sequence $\{z_i\}_{i=1}^k$ in $\Gamma(y) \cap B_t$ such that $z_{i+1} \in \Gamma(z_i)$. Otherwise, let $g_t(y) = k+1$, where k is the greatest integer for which there is a sequence $\{z_i\}_{i=1}^k$ outside of B_t, satisfying $z_i \in \Gamma(z_{i+1})$ and $y \in \Gamma(z_i)$, for all i.[4]

Lemma B.4. *For all $t \in T$, function g_t is well-defined and satisfies (i) $y \in B_t$ if and only if $g_t(y) \leq 0$; and (ii) $y \in \Gamma(x)$ implies $g_t(y) \geq g_t(x) + 1$, for all x, $y \in \mathbb{R}^\ell_+$.*

Proof. First, we show that is it well-defined. We prove it by contradiction. Take any $x \in B_t$ and suppose that $g_t(x)$ is not well-defined. By construction of g_t, Assumption 1, and the fact that B_t is downward comprehensive, this can happen only if there is an infinite sequence $\{z_i\}$ in $\Gamma(x) \cap B_t$ such that $z_{i+1} \in \Gamma(z_i)$, for all i. Since $z_i \notin \Gamma(z_i)$ and $\Gamma(z_i) \subset \{y \in \mathbb{R}^\ell_+ : y \geq z_i\}$, for all i, the sequence must be strictly increasing and bounded below by x. By compactness of B_t, the sequence converges to some $z \in B_t$. Hence, there are bundles x, z such that $z \in \Gamma(x)$, with an infinite sequence $\{z_i\}$ in $\Gamma(x)$, satisfying $z_{i+1} \in \Gamma(z_i)$ and $z \in \Gamma(z_i)$, for all i. This contradicts Assumption 2.

Whenever $x \notin B_t$, the result can be shown analogously. In particular, whenever $g_t(x)$ is not well-defined, there is an infinite, strictly decreasing sequence $\{z_i\}$ outside of B_t,

[4] If no such sequences exist, let $k = 0$.

converging to some $z \in \mathbb{R}^\ell_+$, that satisfies $z_{i+1} \in \Gamma(z_i)$ and $x \in \Gamma(z_i)$, for all i. Since $x \in \Gamma(z)$ and $z_i \in \Gamma(z)$, for all i, this would violate Assumption 2.

Property (i) follows directly from the definition of the function. To show (ii), take some x and y in $\mathbb{R}^\ell_+$ such that $y \in \Gamma(x)$. The condition holds trivially if $x \in B_t$ and $y \notin B_t$. Take some x, $y \in B_t$ and let $g_t(y) = -k$. Number k is the greatest integer for which there is a sequence $\{z_i\}_{i=1}^k$ in B_t such that $z_1 \in \Gamma(y)$, $z_2 \in \Gamma(z_1)$, ... , $z_k \in \Gamma(z_{k-1})$. Since $y \in \Gamma(x)$, it must be that $g_t(x) \leq -(k+1)$. Therefore $g_t(y) = -k \geq g_t(x) + 1$. We prove the result analogously for x, $y \notin B_t$. □

Similarly to the case discussed in the main body of the paper, Axiom 1(∗) implies existence of a solution to a particular system of linear inequalities.

Lemma B.5. *Set $\mathcal{O}$ obeys Axiom 1(∗) for Γ only if there are numbers $\{\phi_t\}_{t\in T}$ and $\mu > 1$ such that $g_t(x_s) \leq 0$ implies $\phi_s - 1 \leq \phi_t + \mu g_t(x_s)$, for all t, $s \in T$.*

Given the definition of g_t, for $t \in T$, and Lemma B.4, set $\mathcal{O}$ satisfies the axiom only if, for any sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(y,z),(z,a)\big\}$ in $T \times T$ such that $g_t(x_s) \leq 0$, we have $-\sum_{(t,s)\in\mathcal{C}} g_t(x_s) < |\mathcal{C}|$. The rest follows from Proposition A.1 of the original paper.

The above result proposes an alternative way of verifying Axiom 1(∗) by solving a system of linear inequalities. In particular, this implies that, given values $g_t(x_s)$, for t, $s \in T$, this method is computationally efficient. Nevertheless, the general efficiency of the algorithm depends on complexity of the function g_t at points x_s, for $s \in T$.

Similarly to the case discussed in the main body of the paper, the above result becomes a basis for the construction of a semiorder that is consistent with property Γ.

Lemma B.6. *Whenever there is a solution to the system of inequalities specified in Lemma B.5, there exists a utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that $y \in \Gamma(x)$ implies $u(y) > u(x) + 1$, while $(B, x) \in \mathcal{O}$ and $y \in B$ implies $u(x) + 1 \geq u(y)$.*

The proof is analogous to the one supporting Lemma 2 in the main body of the paper. Hence, it is omitted. To finish our argument, define semiorder P such that

$$xPy \text{ if and only if } u(x) > u(y) + 1.$$

Given Lemma B.6 it is straightforward to show that it rationalises the set of observations, while $y \in \Gamma(x)$ implies yPx. This concludes our proof.

Notice that, at the imposed level of generality, we are unable to determine any additional properties of the function u. Unlike in Section B.2 of this Online appendix, we cannot say whether there exists a continuous, monotone, or quasiconcave utility that induces a semiorder P rationalising the dataset. Since all such features depend on the choice of the correspondence Γ, they have to be determined independently.

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