REDISTRIBUTION FROM THE CRADLE TO THE GRAVE: A UNIFIED APPROACH TO HETEROGENEITY IN AGE, INCOME, AND WEALTH

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ABSTRACT. This paper studies the macroeconomic and cross-sectional consequences of redistributive fiscal policy, with a focus on pensions. Evidence suggests that transfers crowd out private savings heterogeneously across households in different income, wealth, and age groups. These changes cumulate to dynamic effects on the wealth distribution, which must be taken into account for policymakers with distributional goals. To quantify these channels, I build an overlapping generations heterogeneous agent model based on continuous time methods, joining canonical mechanisms of lifecycle behavior and precautionary savings. Despite its parsimony, the model yields empirically realistic distributions of savings and of the cross-sectional impact of pension reform. I use it to make two main contributions. First, I quantify the cross-sectional impact of a pension reform on household savings. Adjustment is concentrated among workers in lower-middle wealth groups. Richer households are indifferent about transfers, while the poorest are constrained. Thus, the equilibrium real rate stays largely unchanged, in support of previous efforts studying these effects in partial equilibrium. Second, in a transition experiment I show that raising social security benefits leads wealth inequality to fall in the short run, but to grow past its original level after fifteen years – even if the accompanying tax increase is progressive. This follows from lower-middle workers reducing savings most strongly. Means-testing amplifies this effect. Progressive transfers to young workers have similar impact, but through different channels. Transfers encourage riskier portfolios, however crowding out is weaker since it is easier to save than to borrow.

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1. Introduction

Most countries have seen large changes to fiscal policy over the past 50 years, with fluctuations in tax rates and increases in social insurance expenditure. Often – and notably in the United States – these changes were followed by a rise in wealth inequality. There is an ongoing debate about whether these issues are linked. Key to this question is understanding how household savings respond to fiscal changes, and what this implies for the distribution of wealth.

In this paper, I provide a quantitative model of the dynamic effects of fiscal reform on household savings, the wealth distribution, and macroeconomic aggregates.

Public transfers can crowd out private savings: they serve as external insurance to households against fluctuations in their income, alleviating the motive to accumulate wealth as self-insurance. The extent of this crowding-out varies across households with different wealth or income, or at different points in their lifecycle. As a result, a reform to social insurance has heterogeneous impact on the distribution of household savings. Over time, this leads to changes in the distribution of household wealth.

A simple example illustrates these dynamic effects on the wealth distribution. Suppose a policymaker expands a program of (possibly means-tested) transfers to senior citizens, such as the Supplemental Social Insurance program existing in the United States. The short-run effect of this reform is likely to be a small reduction in wealth inequality, as poor seniors receive more wealth. However, over the long-run, working households incorporate this transfer schedule into their consumption–savings decision. Poorer workers prefer to consume more during working life and rely on these extra transfers for retirement consumption, thus accumulating less wealth over their lifecycle. By contrast, these transfers do not substantially affect the savings of the rich, being small relative to their assets. Thus, this reform – designed to increase social insurance – will lead to higher wealth inequality over the long-run, an idea from Hubbard et al. (1995). Understanding how – and how fast – we get from a slight decrease in inequality to a longer-term increase is one of the key questions of this paper, which I address by means of a quantitative model of the dynamics of the wealth distribution.

In light of increasing wealth inequality, it is important to be able to quantify these effects. Firstly, economists are debating the causes of this increase – especially in the United States – weighing the contribution of structural changes (skills-biased technological change leading to rising income inequality) and fiscal changes (with a focus on declining taxes on the rich since the 1960s).[1] In this paper, I find that it takes around 25 years for the full effect of social insurance on wealth inequality to materialize, emphasizing a larger contribution of social insurance than previously thought. Indeed, Pham-Dao (2016) finds that differences in social insurance account for 70% of the variation in wealth inequality across Eurozone countries.

Rising wealth inequality highlights a second question. What – if anything – is the appropriate policy response? Redistributive transfers may reduce inequality in the short term but are likely to *increase* it over the long-run. Further, social insurance is generally welfare improving at the margin, so that the accompanying wealth inequality can be seen as a side-effect to ignore. Indeed, despite popular backlash against wealth inequality (a

1. Recent discussions include Kaymak and Poschke (2016), Heathcote et al. (2017), and Hubmer et al. (2016).

natural result of political economy considerations), the policymaker would do better to focus on welfare measures, permanent income inequality, or consumption inequality. However, regardless of the policymaker's objective function, a tool for predicting the distributional impact of fiscal polciy remains useful.

To this effect, I build a macroeconomic framework with which to analyze these distributional effects, nesting the standard incomplete markets model and canonical lifecycle mechanisms. With this model, I make five contributions.

First, I quantify the cross-sectional reaction of household savings to social security reform in the model. The response is highly heterogeneous, with 45–65 year-olds in lower wealth groups dissaving the most. These patterns are in line with a growing body of empirical evidence on how public pensions crowd out private wealth.[2] Crowding-out of savings is concentrated strongest at the bottom of the wealth distribution. As a result, the change in aggregate wealth is small, so that the quantitative effect on the equilibrium interest rate is negligible. By addressing this question with a dynamic general equilibrium model, I thus find additional support for a large applied and empirical literature which analyzes the effects of social insurance on consumption and savings from a structural partial equilibrium standpoint.[3]

Second, I compute the dynamics of the wealth distribution resulting from the change in household savings. I find (in line with the theoretical results of Hubbard et al. (1995)) that more social insurance leads to higher wealth inequality. In fact, these dynamics are slow: it takes roughly 25 years for the full distributional effect to materialize. The reason for this is twofold. First, it takes time for changes in savings to cumulate to changes in the stock of wealth. Second, there are temporary cohort effects if the extra transfers are financed contemporaneously with taxes, as early generations receive transfers which they have not fully paid for.

Accordingly, I analyze how the reactions of savings and wealth vary for different kinds of transfer programs. Making transfers more redistributive strengthens the distributional effects: more progressive subsidies lead to even *higher* wealth inequality, by increasingly crowding-out specifically the savings of poorer households. I also consider means-tested transfer programs aimed at younger workers. These insure households against income shocks rather than lifecycle variation in earnings. Subsidies to workers have a weaker effect on aggregates than transfers to seniors, as households find it easier to save against future taxation than to borrow against future transfers. These transfers also tend to increase wealth inequality, by reducing the savings of households with low earnings.[4] Also, better insured against income risk, households are willing to take on riskier portfolios, further increasing wealth inequality.

2. Bottazzi et al. (2006), Attanasio and Rohwedder (2003), Aguila (2011), Feng et al. (2011), and Lachowska and Myck (2018) look at various pension reforms in the 1990s in Italy, the United Kingdom, Mexico, China and Poland respectively, finding evidence for relatively strong crowding-out effects using a difference-in-difference design. By contrast Chetty et al. (2014) use a regression discontinuity approach around a reform based in Denmark and find much weaker effects. I discuss this literature in more detail in Section 6.4.

3. See, for example, Deaton et al. (2002), Storesletten et al. (2004), Heathcote et al. (2009b, 2014, 2017), and Huggett et al. (2011).

4. Income is positively correlated with wealth, especially early in the lifecycle.

Third, I analyze the cross-sectional effect of an increase in the risk-free rate in the model. Typically, we would expect lenders to gain and borrowers to be worse off. However, in the model as in the data, most agents spend their working years as net borrowers, using leverage to buy a risky asset (mainly housing). Lenders tend to be better off and older – however, in their youth many of them were also borrowers. Thus over the long run, even lenders will have paid the higher borrowing costs, ending up with less wealth and consumption. Wealth inequality drops in response to the rate rise, a result of portfolio rebalancing and of these heterogeneous wealth patterns in bond holdings.

Fourth, I quantify the aggregate contribution of lifecycle savings to household wealth, by comparing the model to a version in which lifecycle motives are disabled. I find that lifecycle behavior accounts for 20% of aggregate household wealth, depressing the risk-free rate by a further 150 basis points. Thus, lifecycle savings exacerbate the oversaving problems in models with uninsured income risk first identified by Huggett (1993) and Aiyagari (1994), speaking to asset pricing puzzles which ask why safe interest rates are so much lower than predicted by standard models. Population aging strengthens these effects: lengthening lifespans amplify the motive to save for retirement.

Finally, a methodological contribution of this paper is in the solution method: using new tools based on continuous time, I write a unified model of heterogeneity in age, income and wealth. The model is parsimonious by design, the union of a canonical lifecycle model with a standard incomplete markets model of heterogeneity in income and wealth. Lifecycle motives come from uncertain lifespans – determined from actuarial data on survival rates – and a hump-shaped profile of average income over the lifecycle, obtained from the data. Households face uninsured idiosyncratic shocks to income at each age.[5] With the extra computational space, I use the approach of enlarging the state space in the lifecycle and income–wealth dimensions rather than introducing extra features. The model yields empirically realistic lifecycle patterns and distributions of income and wealth, despite its parsimony.

Literature. This paper contributes to several strands of the literature.

There exists a large empirical and applied literature which analyzes the effect of social insurance on household consumption and savings, using partial equilibrium models of lifecycle behavior and incomplete markets.[6] This paper builds on that literature by analyzing the resulting dynamic distributional effects and the accompanying general equilibrium changes. I find that the reaction of the interest rate is quantitatively small, providing support for studies with a partial equilibrium approach.

Several papers investigate the effects of fiscal policy on the wealth distribution. Kaymak and Poschke (2016), Hubmer et al. (2016), and Heathcote et al. (2017) focus on the relationship of top income tax rates with tail wealth inequality. By contrast, this paper emphasizes the distributional effects of social insurance, first argued by Hubbard et al. (1995) to increase long-run wealth inequality. Using a partial equilibrium structural

5. The model does not feature aggregate risk for technical reasons, although Ahn et al. (2017) provide promising results in that direction.

6. See for example Gourinchas and Parker (2002), Storesletten et al. (2004), Kaplan (2012), Deaton and Paxson (1994), Cocco et al. (2005), Ziliak (2003), Deaton et al. (2002), Heathcote et al. (2009b, 2014, 2017), Huggett et al. (2011), and Hubbard et al. (1995).

model, Pham-Dao (2016) finds that social insurance can explain 70% of the variation in wealth inequality across the Eurozone.

The approach of this paper is to build an overlapping generations heterogeneous agent (OLGHA) model to investigate the distributional effects of social insurance. Early uses of OLGHA models include İmrohoroglu et al. (1995), Huggett (1996), and Ríos-Rull (1995), who augment large-scale OLG models in the vein of Auerbach, Kotlikoff, et al. (1987) with uninsured household income risk, which they solve in steady state. More recent studies analyze equilibrium transition dynamics or aggregate fluctuations in this setting.[7] By focusing on the distributional consequences of fiscal reform, this paper differs in emphasis from these studies.

A technical contribution of this paper to this literature is to provide an alternative solution method for these models, using continuous time tools, in the style of Achdou et al. (2017) and Kaplan et al. (2016). Rather than a large 80-period system, the model reduces to a system of two differential equations called a *mean-field game* which has been well studied in mathematics and physics. As a result, efficient numerical solution tools can be deployed. Further, the relative parsimony of the system allows for analytic results about the model to be derived, such as the asymptotics of policy functions or the shape of the tail of the wealth distribution. Achdou et al. (2017) derive these tools for solving models with heterogeneity in income and wealth. In this paper, I extend the approach in order to solve an overlapping generations model with lifecycle behavior. A straightforward application of those tools to this problem runs into the *curse of dimensionality*, in which the four state variables of household income, wealth, age, and date of birth render the method computationally prohibitive. Instead, I solve the system for each cohort separately and splice these back together to find an equilibrium. This method allows for a parallelized numerical solution algorithm.

Finally, I take the approach in this paper of building a parsimonious model nesting canonical mechanisms of lifecycle behavior and incomplete markets. The idea is to use the extra computational power afforded by the new solution method to enlarge the household state space, and to see how well a standard model matches the data in such a setting. As a result, there are mechanisms not explicitly captured in the model that may be relevant to this question. First is health risk, a form of uncertainty displaying cross-sectional and lifecycle patterns that has been argued to be important for household savings (Gruber and Yelowitz 1999; Maynard and Qiu 2009; De Nardi et al. 2016). Health expenditure could be introduced to the model as a form of negative income risk to seniors. Faced with this risk, households have a larger requirement for insurance. Public transfers may thus crowd out their savings less strongly, so that the elasticities in this paper can be seen as an upper bound.

Another important determinant of household savings behavior is the bequest motive.[8] These are important for matching certain facts about the level of savings. However, when analyzing *changes* in saving (i.e. crowding out), the effect of bequests seems to cancel out. For this reason, I incorporate bequests into the model only in Appendix A.1.

7. These include Krueger and Ludwig (2007), Conesa and Krueger (1999), Castaneda et al. (2003), Kindermann and Krueger (2015), Heathcote et al. (2009a), and Carroll et al. (2017) and the Penn Wharton Budget Model.

8. See, for example, Castaneda et al. (2003), Benhabib et al. (2011), and Straub (2017).

The remainder of this paper is laid out as follows. Section 2 explains the model. Section 3 lays out the parameterization of the model and evaluates its ability to fit the data. Section 4 quantifies the importance of lifecycle behavior for aggregate savings. Section 5 demonstrates the dynamic distributional impact of changes in the interest rate. Section 6 sets out the distributional effects of increased social insurance, and Section 7 concludes.

2. A Unified Model of Heterogeneity in Age, Income, and Wealth

The model runs in continuous time. A finite mass of agents is born at each point in time. Agents have uncertain lifetimes: at age t they face a probability $\lambda_t \, \mathrm{d}t$ of dying before age $t + \mathrm{d}t$.[9] I will calibrate λ_t to age-specific mortality rates from actuarial tables. Constant λ_t would correspond to a perpetual youth model.

The *age structure,* or density function for agents' ages in the economy, is denoted by $D(t)$, so that $\int_0^\infty D(t)\,\mathrm{d}t$ is the total population size. I assume the entry rate is exogenous, and can depend on the age structure.

In an overlapping generations model, an agent's behavior depends on his age t. Decisions can vary separately with calendar time τ (or equivalently with date of birth $s = \tau - t$) if there are period (or cohort) effects. To avoid cumbersome notation, I suppress this dependence as far as possible.

2.1. **Agent decisions.**

In addition to length of life, agents face idiosyncratic risk to productivity and to returns on wealth. Lifecycle behavior has two principal sources: uncertain lifespans, and an age-varying profile of average income that displays a hump shape. Labor is exogenous, and takes the form of endowment income at each age.

2.1.1. *Markets.* They choose between consumption and investment in three assets: a risk-free bond b, a risky asset k, and an annuity contract. Agents take as given the return r_t^b of the risk-free bond at each age, which is an equilibrium outcome. Following Benhabib et al. (2011, 2015, 2016), we assume the risky asset carries *idiosyncratic* risk.[10] The idea is that close to half of US household wealth is tied up in housing capital or private business projects, both of which carry a large idiosyncratic component of risk. Assuming no portfolio adjustment, the risky asset k_t pays a return given by

$$\frac{\mathrm{d}k_t}{k_t} = r^k \,\mathrm{d}t + \sigma \,\mathrm{d}B_t. \tag{2.1}$$

The average return is r^k, with volatility σ. (Process B_t is a Brownian motion particular to each agent.) The supply of the asset is exogenous and infinitely elastic. Note that in certain applications, such as in Section 4, I do not allow agents to buy r^k. This way, we can understand r_t^b as the rate that equilibrates all borrowing and saving, and not just the bond market.

As in Blanchard (1985), agents participate in an annuity market. Each of these instantaneous contracts pays out π_t to agents aged t, in exchange for a claim of \$1 on

9. Formally, agents die at the first arrival time of an idiosyncratic Poisson point process with rate $(\lambda_t)_{t\geq 0}$.

10. Aggregate risk is technically difficult to implement in these models. Ahn et al. (2017) provide promising results in this direction.

the agent's assets were he to die at that moment. If an agent chooses to buy p_t worth of these annuities at age t, then his estate would be worth

$$q_t = w_t - p_t/\pi_t \tag{2.2}$$

were he to die, where $w_t = k_t + b_t$ is his current net worth. Agents are allowed to buy or sell annuities, with the latter case corresponding to a purchase of life insurance, but they will typically buy annuities.

In a competitive market $\pi_t = \lambda_t$. Instead, I allow for an imperfect annuity market, in order to match evidence on consumption patterns in old age.[11] For simplicity, the profits and losses go to the government, as do all accidental bequests.

Rather than p_t, I treat q_t, the total estate the agent would leave if he were to die at age t, as the agent's choice variable. In Appendix A.1, I discuss how to introduce a bequest motive by giving households *joy of giving* (or *warm glow*) utility from q_t. Absent a bequest motive, households optimally choose $q_t = 0$ always, annuitizing all their wealth.

2.1.2. *Income.* Agents receive exogenous income $y_t z_t$, with an age-dependent wage profile y_t common to all agents that exogenously captures lifecycle trends in labor income. This profile could be microfounded using a model of human capital. Productivity z_t carries idiosyncratic risk. Following Kaplan et al. (2016) and Guvenen et al. (2015) I assume that log productivity is mean reverting but subject to two sources of jump shocks: one small, frequent and impersistent; the other larger, rarer and more persistent.

$$\log z_t = \zeta_t^1 + \zeta_t^2; \qquad \mathrm{d}\zeta_t^i = -\zeta_t^i \,\mathrm{d}t + \mathrm{d}J_t^i, \tag{2.3}$$

where J_a^i are (independent) jump shocks, compound Poisson processes with stochastic jump sizes.

The tax and transfer schedule is captured by function $\Theta(t, w, z)$, which can depend on age and be asset- and income-tested. This general formulation captures cash transfers, income taxes, wealth taxes, and taxes to capital returns. Agents internalize the transfer schedule. Typically, Θ will include income taxes, social security, and some progressive transfers to unproductive agents.

2.1.3. *Maximization.* Agents choose lifecycle paths of consumption c_t, bond holdings b_t, risky capital holdings k_t to maximize expected discounted utility

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t - \int_0^t \lambda_s \,\mathrm{d}s} u(c_t)\,\mathrm{d}t\right] \tag{2.4}$$

where $\rho > 0$ is the agent's time discount rate. Although the specification works in general, I assume the utility functions u for consumption has constant relative risk aversion:

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}. \tag{2.5}$$

Households maximize equation (2.4) subject to the budget constraint, expressed in terms of net financial wealth $w_t = b_t + k_t$:

$$\mathrm{d}w_t = (y_t z_t + \Theta(t, w_t, z_t) + r_t^b w_t + (r^k - r_t^b)k_t - c_t + \pi_t w_t)\,\mathrm{d}t + \sigma k_t \,\mathrm{d}B_t, \tag{2.6}$$

11. Mortality rates λ_t exceed 20% for 90+ year-olds, implying counterfactually high asset returns for elderly households if $\pi_t = \lambda_t$. Annuity contracts do exist in real life; it remains puzzling that they are so rarely taken up.

and to the dynamics of productivity, equation (2.3).

Agents are not allowed to hold short positions on the risky asset, so $k_t \geq 0$, a constraint motivated by informational considerations. They also face a borrowing constraint $b_t \geq \Phi$, where $\Phi \leq 0$. The borrowing constraint can be thought of as a limit on leverage for risky capital, leading to an upper bound on capital holdings $k_t \leq w_t - \Phi$.

I solve the agent's problem recursively. Denote by $V(t, w, z)$ the optimal value of having wealth w and productivity z at age t,

$$V(t, w, z) = \max_{(c_s, k_s)_{s \geq t}} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t) - \int_t^s \lambda_\xi \, \mathrm{d}\xi} u(c_s) \, \mathrm{d}s, \tag{2.7}$$

maximizing subject to the dynamics for wealth and productivity outlined above, the borrowing constraints, and to productivity and net worth at age t being equal to the given values z and w.

The value function $V(t, w, z)$ satisfies the *Hamilton–Jacobi–Bellman equation*:[12]

$$\begin{aligned}(\rho + \lambda_t) V(t, w, z) = \max_{c,k} \, & u(c) + \lambda_t \tilde{u}(q) + \partial_t V \\ & + \partial_w V \big[y_t z + \Theta(t, w, z) + r_t^b w + (r^k - r_t^b) k - c + \pi_t w \big] + \partial_{ww} V \frac{\sigma^2}{2} k^2 \\ & + \partial_z V \mu^z(z) + \eta \int_{-\infty}^{\infty} (V(t, w, z') - V(t, w, z)) \phi^z(z') \, \mathrm{d}z' \end{aligned} \tag{HJB}$$

where the maximum is taken over all $c \geq 0$ and $k \in [0, w - \Phi]$. See Appendix B.1 for a heuristic derivation and a proof.[13] The final integral term captures the expected effect of the compound shock to productivity combining the two components, where η is the aggregate arrival rate of the jumps and ϕ^z the density function.[14]

In addition, function V obeys a number of boundary conditions derived from the constraints on w, k, z. These state-constraint boundary conditions are detailed in Appendix B.1.

In a model with no lifecycle (such as perpetual youth), this equation has a stationary solution, where V is independent of age t. Here, however, we need to solve the full time-dependent version of the HJB equation above. No analytic solution typically exists.

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12. For a variable x, denote by ∂_x the partial differentiation operator $\partial/\partial x$.

13. It can be helpful to think in terms of the *generator* $\mathcal{A}_t$ of the joint process (w_t, z_t), defined as

$$\mathcal{A}_t f(w, z) = \lim_{\epsilon \to 0+} \frac{E[f(w_{t+\epsilon}, z_{t+\epsilon}) \mid w_t = w, \, z_t = z] - f(w, z)}{\epsilon}. \tag{2.8}$$

Ito's lemma implies that $\mathbb{E} \, df(w_t, z_t) = (\partial_t f + \mathcal{A}_t f) \, \mathrm{d}t$, so that $\mathcal{A}_t f$ captures the expected change in $f(w_t, z_t)$ that comes from dynamics of (w_t, z_t). If the policy functions are optimally chosen, we can thus rewrite the HJB equation as

$$(\rho + \lambda_t) V(t, w, z) = u(c) + \lambda_t \tilde{u}(q) + \underbrace{\partial_t V + \mathcal{A}_t V}_{\mathbb{E} \, \mathrm{d}V / \mathrm{d}t}. \tag{2.9}$$

So we can understand the last two lines of equation (HJB) as continuation value coming from changes in the two state variables w_t and z_t.

14. In computations, I follow Kaplan et al. (2016) by replacing the two-dimensional productivity process (ζ_t^1, ζ_t^2) with a one-dimensional approximation, in order to reduce the size of the state space.

Instead, I will solve the equation numerically using a finite difference method explained in Appendix C.1.

Agents' policy functions satisfy simple static first order conditions, with constraints only binding on the boundary of the state space. The usual Euler equations also hold:

Proposition 1 (First order conditions)**.** *The agent's optimal policy functions $c(t,w,z)$, $k(t,w,z)$ for consumption, wealth at death, and capital holdings at age t, wealth level w, and productivity z, are characterized by the first order conditions*

$$c(t,w,z) = (u')^{-1}\big(\partial_w V(t,w,z)\big) \tag{FOC c}$$

$$k(t,w,z) = \min\left\{-\frac{\partial_w V(t,w,z)}{\partial_{ww} V(t,w,z)}\frac{r^k - r_t^b}{\sigma^2}, w - \Phi\right\}, \tag{FOC k}$$

for any (t,w,z) such that $t > 0, w > \Phi$.

Proposition 2 (Euler equations)**.** *Let (w_t, z_t) denote the optimal paths of wealth and productivity given an initial value and the draws of the shock processes. Then the path of consumption $c_t = c(t, w_t, z_t)$ satisfies*

$$(\rho - r_t^b + \pi_t - \lambda_t - \partial_w \Theta(t, w_t, z_t))\, dt = E\frac{du'(c_t)}{u'(c_t)} + \frac{dk_t}{k_t}\frac{du'(c_t)}{u'(c_t)}$$

away from the constraint, i.e. whenever $w_t > \Phi$.

This Euler equations is the continuous time version of the familiar expression. For example, $du'(c_t)/u'(c_t)$ is the growth rate of marginal utility, and $dk_t/k_t \cdot du'(c_t)/u'(c_t)$ is a covariance between this and the return on capital. Note that when annuity markets are competitive $\pi_t = \lambda_t$ then agents fully smooth consumption against their age-specific mortality risk λ_t. Asset-tested transfers (where $\partial_w \Theta \neq 0$) distort households' consumption decision.

2.2. **Distribution of wealth and productivity within age groups.** Due to uncertainty in productivity and returns to capital, there is a nontrivial distribution of wealth and productivity within each age group.

After solving the household's problem, Proposition 1 gives us the optimal policy functions. Letting

$$s(t,w,z) = y_t z + \Theta(t,w,z) + r_t^b w + (r^k - r_t^b)k(t,w,z) - c(t,w,z) + \pi_t w \tag{2.10}$$

denotes optimal savings, write the budget constraint equation (2.6) as

$$\mathrm{d}w_t = s(t, w_t, z_t)\,\mathrm{d}t + \sigma k(t, w_t, z_t)\,\mathrm{d}B_t. \tag{2.11}$$

Equation (2.11) writes the dynamics of individual wealth in terms of a deterministic trend $s(t, w_t, z_t)$ and a random component coming from the risky return on capital. Using standard results in stochastic analysis, we can use this decomposition to write a law of motion for the joint density wealth and productivity within an age group, $g(w, z|t)$. The density g satisfies the *Kolmogorov Forward equation*

$$\begin{aligned}\partial_a g(w,z|t) = &-\lambda_t g - \partial_w[sg] + \partial_{ww}\left[\frac{(\sigma k)^2}{2}g\right]\\ &- \partial_z[\mu^z g] + \eta\int_{-\infty}^{\infty}\big(g(w,z'|t) - g(w,z'|t)\big)\phi^z(z')\,\mathrm{d}z',\end{aligned} \tag{KF}$$

with boundary condition $g(w,z|t) = g_0(w,z)$ for some initial distribution. Appendix B.2 outlines a derivation of equation (KF). Details of the numerical solution method can be found in Appendix C.1.[15]

Thinking of $g(w,z|t)$ as a joint density of wealth and productivity conditional on age, we can multiply it by the density of ages $D(t)$ to get the joint density of wealth, productivity, and ages in the economy. Denote the associated probability measure by $m(\mathrm{d}t, \mathrm{d}w, \mathrm{d}z)$. This is the main distributional object in the model. It lets us to work out household aggregates. For example, aggregate capital is given by

$$\begin{aligned} K &= \int k(t,w,z)\, m(\mathrm{d}t, \mathrm{d}w, \mathrm{d}z) \\ &= \int_{t\geq 0} \int_{w\in\mathbb{R}} \int_{z\in\mathbb{R}} k(t,w,z) g(w,z|t) D(t)\, \mathrm{d}t\, \mathrm{d}w\, \mathrm{d}z. \end{aligned}$$

2.2.1. *Initial distribution.* I assume newborns enter with a productivity level randomly drawn from the ergodic distribution of z_t. Newborn wealth is drawn from a fixed distribution $g_0(w)$.

In Appendix A.1.1 I show how to relate the distribution of newborn wealth to inheritance in a model with bequests. This makes the initial wealth distribution endogenous, which is computationally non-trivial. I argue in the appendix that although intergenerational transmission of inequality is an important channel, this operates at a larger time scale than the transmission experiments considered in the main text of this paper. As a result, I show that the model predictions are robust to removing this channel.

2.3. **Government.** Let τ denote calendar time.

The schedule of taxes and subsidies in place at time τ is denoted $\Theta_\tau(t,w,z)$. This can depend on age, financial wealth, and productivity. Section 3.2 sets out the precise fiscal policies setup used in the numerical experiments.

Government consumption c^g_τ is fixed exogenously and may be negative. Finally, any budget deficit or surplus is held in bonds b^g_τ.

The government budget constraint is

$$\frac{\mathrm{d}b^g_\tau}{\mathrm{d}\tau} = r^b_\tau b^g_\tau - c^g_\tau - \int \Theta_\tau \, \mathrm{d}m_\tau. \tag{2.12}$$

Here m_τ is the joint distribution of age, income and wealth at calendar time τ, so that last term represents the net receipts to the government of the tax/transfer schedule.

2.4. **Equilibrium.** Recall that in the presence of period or cohort effects, agents' policy functions will also vary with calendar time τ (or equivalently their date of birth $s = \tau - t$). I reintroduce this dependence into the notation using subscript τ's.

Recalling that $w = b + k$, aggregate household bond holdings at calendar date τ must equal

$$b^h_\tau = \int \big(w - k(t,w,z)\big)\, m_\tau(\mathrm{d}t, \mathrm{d}w, \mathrm{d}z).$$

15. I exploit a deep symmetry between equations (HJB) and (KF), by which equation (KF) can be written in terms of the L^2-adjoint operator of $\mathcal{A}_t$, namely $\partial_t g = \mathcal{A}^*_t g$. Solving (HJB) yields the operator $\mathcal{A}_t$, which makes it easy to solve (KF).

The government also participates in the bond market, so the bond market clearing condition is

$$b^g_\tau + b^h_\tau = 0 \qquad \text{at all times } \tau. \tag{EQM}$$

An equilibrium of this model is defined as a collection of policy functions for consumption, and investment for agents of each cohort together with a path for the risk-free interest rate and for government policy, so that the policy functions solve each agent's HJB equation and so that bonds markets always clear.

A *stationary equilibrium* is one in which none of the policy functions, prices, or distribution functions vary with calendar date τ. This is the appropriate stochastic steady concept for the model, corresponding to an economy with no period or cohort effects.

Note that the market for risky capital is not in zero net supply. Rather, supply of risky capital is unbounded and infinitely elastic. In Section 4, we will want to read extra meaning into the equilibrium rate r^b, so I shall disable the risky asset k ensuring that aggregate *wealth* is in zero net supply.

Transition equilibrium. In transition, calendar time τ, agent age t and date of birth s jointly determine *two* important state variables (since $\tau = s + t$).

A transition equilibrium requires

(1) a solution $V(t, w, z; s)$ to the agent's problem for each cohort with date of birth s
(2) a joint distribution m_τ of wealth, productivity and age at each time τ such that the density function $g(w, z|t; \tau - t)$ of wealth and productivity conditional on age for each cohort s satisfies the relevant Kolmogorov Forward equation
(3) a path of prices (r^b_τ) at each time such that the bond market always clears,

$$\int b(t, w, z; \tau - t) m_\tau(\mathrm{d}t, \mathrm{d}w, \mathrm{d}z) + b^g_\tau = 0, \qquad \text{for all } \tau$$

2.5. **Solution method.** I provide a bird's eye view of the numerical algorithm for solving the model. More details can be found in the appendices.

The main components of the model are equations (HJB), (KF) and (EQM). Together, these define a *mean field game:* a type of system of integral and differential equations described in Lasry and Lions (2006, 2007). To solve it, I adapt the framework for numerical solution presented in Achdou et al. (2017).

Note that agent lifespans are uncertain and have unbounded support in the model. The solution method requires a bounded grid on lifespans. I assume that after the maximal age (150 years), the agent's problem becomes stationary (i.e. perpetual youth). To solve the agent's problem, I first solve this stationary problem, and then solve the lifecycle model backwards in time from there.[16]

The following sketches the solution algorithm for a transition:

(1) Make a guess of the path of interest rates (r^b_τ).

16. We could also kill the agents once they reach 150, by setting the value function at that age to zero and then solving backwards. In practice, it does not really matter economically what you choose for end of life if the maximal age is large enough. This stationary-at-the-end approach has the advantage of numerical stability since policy functions in the last period of life is well-behaved.

(2) Given (r^b_τ), solve the agent's problem for each cohort born at time $s \leq 0$ as follows. First, solve the stationary version forward in time for agents of age 150. Next, use a fully implicit finite difference method to solve backwards in time from the stationary solution. This involves finding the solution of a high dimensional nonlinear equation at each time step, using a Newton method. (This is fast when exploiting the sparsity of the system which results from the continuous time approach.) Use this to work out the policy functions for agents of all cohorts and ages.
(3) Given policy functions, solve equation (KF) for each cohort forward in time, to get the cohort-conditional joint distribution. As with Achdou et al. (2017), I exploit the adjointness of the KF and HJB equations to solve this very quickly (see Appendix B.2). Integrate the age- and cohort-specific wealth density against the demographic structure function to get the joint distribution m_τ of age, productivity, and wealth at each time τ.
(4) Use this distribution to calculate excess bond holdings as the difference between aggregate capital holdings and aggregate wealth (equation (EQM)) at each time τ.
(5) Update guess of (r_τ) in accordance with the signs of the excess bond holdings (a kind of tâtonnement). This can be done simply with binary search or more efficiently with a quasi-Newton method.
(6) Repeat until the bond market clears.

Strong results (Barles and Souganidis 1991; Lasry and Lions 2007; Achdou et al. 2014) guarantee that numerical schemes of this kind will converge to a solution. Existence and uniqueness results exist for stationary problems, with encouraging results for the time dependent case. Appendix B.2 contains more discussion on this topic.

3. Model Parameterization and Fit

The time scale is annual. As far as possible, I calibrate the model economy to the US in 2013.

3.1. **Household calibration.** Households are born economically at age $t = 25$. Age-specific mortality rates λ_t are taken from life tables for the US in 2015, from the Human Mortality Database. Specifically, I interpolate the actuarial measure q_x, the probability of surviving from age x to $x + 1$. This data is only available until age 110; I assume λ_t is constant after that age.

When solving the model numerically, I have to impose an upper bound to lifespans, which I set at 150 years (agents have an ex ante probability around 10^{-12} of reaching that age in the model).

I take the age structure $D(t)$ from the US Census Bureau for 2015, normalizing population size to 1 so that model aggregates correspond to per capita numbers.

3.1.1. *Preferences.* I set the intertemporal elasticity of substitution to $\gamma = 1.5$, and calibrate the household discount rate ρ internally to give an equilibrium interest rate $r^b = 2\%$ p.a. This corresponds to $\rho = 5.6\%$.

3.1.2. *Bequests and initial wealth.* The wealth of a newborn household is randomly drawn from a log-normal distribution, chosen to match the mean ($27,000) and coefficient of variation (13.1) for households aged 25 and below in the 2016 SCF.

3.1.3. *Assets.* I calibrate r^k to 5.8% and σ to 0.2. There is no standard way of setting the idiosyncratically risky asset k. Recall that it captures *idiosyncratic* returns on housing and private business equity. This entails looking at micro data on returns and cleaning them of aggregate fluctuations. Benhabib et al. (2011) report studies of individual return on housing and business equity in the PSID and SCF respectively, and estimate a return r^k between 6.5 and 9 percent and volatility σ around 0.20.

An alternative approach is to use the following result, which relates the average return r^k to the shape of the tail of the long-run wealth distribution:

Proposition 3. *Under certain conditions, the steady state marginal density of wealth* g_w *is asymptotically Pareto in the right tail, i.e. there exists* $\alpha > 0$ *such that*

$$\lim_{w\to\infty} \frac{g_w(w)}{\alpha \cdot w^{-\zeta-1}} = 1,$$

where the tail exponent ζ *satisfies*

$$\zeta = \gamma \left(\frac{2\sigma^2(\rho - r^b)}{(r^k - r^b)^2} - 1 \right). \tag{3.1}$$

This result was derived by Achdou et al. (2017) for a model with no lifecycle. I adapt their proof for this model in Appendix B.1.1.

Empirically, the Pareto parameter for the tail of the US wealth distribution is $\zeta \approx 1.5$. Assuming $\sigma = 0.20$, equation (3.1) implies value $r^k = 5.8\%$, slightly below estimates reported by Benhabib et al. (2011).

The borrowing constraint, which relates to short term unsecured debt, is set to -$10,000.

Few retirees buy annuities in the data (see Brown (2007) for example). Many explanations have been proposed for this long-standing puzzle, such as adverse selection (Mitchell et al. 1999), bequest motives (Friedman and Warshawsky 1990), self-insurance and liquidity requirements (De Nardi et al. 2006), and public insurance (Brown 2007). These channels are outside the scope of the model. As a result, I allow for a small amount of life insurance by setting the return on the annuity to a fraction of the mortality rates,

$$\pi_t = 0.1 \cdot \lambda_t.$$

Model fit and elasticities are quite robust to changes in λ_t/π_t, as long as this ratio does not exceed around a half. The reason is that mortality probabilities λ_t are very large after age 90. If these agents were to have exclusive access to an asset paying such a high return (above 10%), the wealth distribution would be counterfactually skewed in their favor.

3.1.4. *Income process.* As discussed, log-productivity is the sum of two components,

$$\log z_t = \zeta_t^1 + \zeta_t^2,$$

following Kaplan et al. (2016). Each process ζ^i is a mean-reverting process subject to jump shocks arriving at different frequencies, which pick a new productivity level

TABLE 3.1. Estimated parameters of productivity process (annualized).

Parameter	Component 1	Component 2
Arrival rate	0.320	0.028
Mean reversion	0.761	0.009
SD of jumps	1.740	1.530

Source: Kaplan et al. (2016)

FIGURE 3.1. Age-specific mortality λ_t (US 2015, Human Mortality Database) and wage y_t (mean of wage income in thousands of dollars, Survey of Consumer Finances), annualized.

from a normal distribution. Kaplan et al. (2016) estimate the shock parameters to match several moments of the distribution of income and income changes, described by Guvenen et al. (2015). I use their estimates, reproduced in Table 3.1.[17] The first component captures frequent but short-lived variation, while the other has low-frequency persistent movements of productivity.

I set the age profile of wages y_t to the mean of pre-transfer wage income in the 2013 Survey of Consumer Finances.[18] The maximal age in the SCF data is 95, where y_t is 0. I also set y_t to 0 for agents older than 95.

The calibrated mortality and wage profiles are plotted in Figure 3.1.

17. When solving the model numerically, I follow Kaplan et al. (2016) in using simulation to make a one-dimensional approximation to this process, in order to avoid making productivity a two-dimensional state variable.

18. In particular, I look at question `X5702`, which asks "In total, what was your (family's) annual income from wages and salaries in 2006, before deductions for taxes and anything else"

3.1.5. *Numerical solution.* The solution algorithm requires grids for wealth and productivity with upper and lower bounds. I choose the lower bound for wealth below the wealth constraint, so that it never binds. I pick the upper bound high enough so that at least 99.99% of the probability mass of wealth lies below it. For productivity, I take the 33-point grid from the discretized process in Kaplan et al. (2016).

3.2. **Fiscal policy.** Fiscal policy matters as this model is highly non-Ricardian, with borrowing constraints, finite lifespans, and heterogeneity within and between age groups.

3.2.1. *Social security.* For public pension transfers I use a simple lump-sum to over 65's of $10,000 per year, received continuously. This simple formulation captures the redistributive features of US social security while abstracting from the forced saving component.

3.2.2. *Taxes and subsidies to labor income.* I model a progressive labor income tax/transfer schedule using a power law. This simple parameterization is increasingly popular in the literature due to its remarkably good fit to the US tax system (Bénabou 2000; Kindermann and Krueger 2015; Kaymak and Poschke 2016; Heathcote et al. 2017; Straub 2017):

$$\text{(post-fiscal earnings)} = \theta_0 \cdot \text{(pre-fiscal earnings)}^{1-\theta_1}. \tag{3.2}$$

Post-fiscal earnings are composed of market labor income $y_t z_t$ plus working-life transfers net of income taxes. (I do *not* include social security transfers, which will be described below.)

Parameter θ_1 controls the progressivity of the schedule: $\theta_1 \to 1$ corresponds to the limiting most progressive setup in which everyone gets the same after-tax income. Parameter θ_0 shifts the level; it is useful for maintaining government budget balance.

Kaymak and Poschke (2016), Heathcote et al. (2017), and Straub (2017) all estimate the progressivity parameter θ_1 in the US using various methods.[19] They find a range of estimates between 0.08 and 0.2. I use the most recent estimate in the literature: Straub (2017) uses PSID data from 2013 to find a θ_1 of 0.16.

When the level θ_0 adjusts to keep government budget balance, it is typically around 1.5. In that case, low incomes receive a positive income subsidy.

3.3. **Model fit and analysis.** In this section I verify the ability of the model to match empirical key facts on inequality, on lifecycle, and on the intersection of these two axes.

3.3.1. *Inequality and cross-section.* Table 3.2 compares moments of the wealth distribution in the model and in the 2013 SCF. The model does a good job at matching the empirical Gini coefficient; however, it is missing some mass deep in the right tail (after the top 10%). For this reason, it slightly underestimates the mean-to-median ratio. The reason is twofold. Firstly, Achdou et al. (2017) and Benhabib et al. (2015) showed that income shocks (even leptokurtic) cannot alone deliver a fat Pareto right tail to the wealth distribution. The intuition is that for top asset holders, labor income is negligible

19. Kaymak and Poschke (2016) target the average income tax paid by the top 1% of earners to match data from Piketty and Saez (2007), while Heathcote et al. (2017) and Straub (2017) feed PSID data into the NBER tax simulator to back out θ_1.

TABLE 3.2. Fit of wealth distribution

Moment	Data (SCF 2013)	Model
Gini coefficient	0.85	0.84
90/50 ratio	11.6	11.9
99/50 ratio	97	71
Mean/median ratio	6.5	5.7
Wealth/income ratio	3.1	3.4
Fraction of hand-to-mouth*	12-20%	16%

* Data is from Kaplan and Violante (2014), who define hand-to-mouth (in the net worth sense) as households with negative net worth.

compared to capital. As a result, they have a negligible precautionary savings motive, and do not save enough to make the wealth tail fat.

Benhabib et al. (2011) argued that idiosyncratic risk to *capital returns* can yield a fat tailed stationary distribution of wealth. However, Weil (2017) argues that this mechanism is too slow in a lifecycle model to match the data on changes in inequality; it operates on the level of generations.

A fix would be to have a faster mechanism, as argued in Gabaix et al. (2016), such as superstar behavior in returns on wealth, or a positive correlation between returns and wealth (an established idea, with support from the Alstadsæter et al. (2017) data on tax evaders).

Alternatively, the wealth distribution of newborns could be made more leptokurtic, capturing other forms of inequality in permanent income stemming from human capital, education, and talent. Indeed, Huggett et al. (2011) and Storesletten et al. (2004) find that these are important determinants of wealth and income inequality.

Importantly, the model does well at matching the fraction of hand-to-mouth agents in the economy. These agents are defined as those with a negative net worth. As we shall see, these agents are very important for understanding the aggregate and cross-sectional response of savings to fiscal policy, and thus for wealth inequality.

Kaplan and Violante (2010) and Kaplan et al. (2014) stress the importance of the so-called *wealthy hand-to-mouth* agents in the data, who have large net worth but no liquidity. They estimate that around 30% of US households exhibit this behavior. I do not model wealthy hand-to-mouth behavior (which requires another asset) in this paper for simplicity and computational speed, but this could easily be added.

3.3.2. *Lifecycle.* Figure 3.2 plots average lifecycle profiles among different productivity subgroups. Consumption is hump-shaped, but flatter than income since agents partially self-insure through savings. Agents save during working life and dissave during retirement. Savings peak at the same time as income, in the mid-50s.

Consumption grows in early life for two reasons. Firstly, the borrowing constraint is more likely to bind among the young, who are generally less well off. As a result, they are not fully able to smooth consumption by borrowing against expected future income. Secondly, as agents age, they make their way through their permanent income.

FIGURE 3.2. Average profiles by productivity quantile, in thousands of dollars. For example, if the Ω are different quantiles of the marginal distribution of z_t, the top right plots $\int c(w,z,t)g(w,z|t)1_{z\in\Omega}\, m(\mathrm{d}w,\mathrm{d}z,\mathrm{d}t)$ against age.

As a result, future income becomes less risky, since there is less of it. The precautionary savings motive weakens, allowing agents to increase consumption.

Consumption peaks in the late fifties and then gradually declines. This can be ascribed to uninsured mortality risk. The consumption Euler equation (Proposition 2) shows us that agents further discount old age consumption by the uninsured mortality probability, pushing down the slope of consumption. As a result, consumption and wealth slowly decline to zero.

Especially important are the average profiles of consumption and income, plotted against the data in Figure 3.3. Consumption is not captured in the SCF, so I calculate profiles in the 2016 Consumer Expenditure Survey.

The model does quite well at matching the average consumption profile in the CEX during working life. There is some overconsumption in retirement; however, this may be a data issue. The right-hand panel plots profiles of income after tax/transfers. Income is higher in the model than in the CEX, which could explain the overconsumption. It is still, however, below the empirical profile in the SCF. To resolve this issue, I would require a data source with sufficiently good quality income data to calibrate the income process which also provides figures for consumption.

Nevertheless, it has traditionally been hard to get a hump shape in consumption that peaks in the mid 50s, as it does in the data and in the model (Thurow 1969; Bütler 2001; Fernández-Villaverde and Krueger 2007; Hansen and İmrohoroğlu 2008). The empirical literature also discusses the decline in consumption in retirement, for which

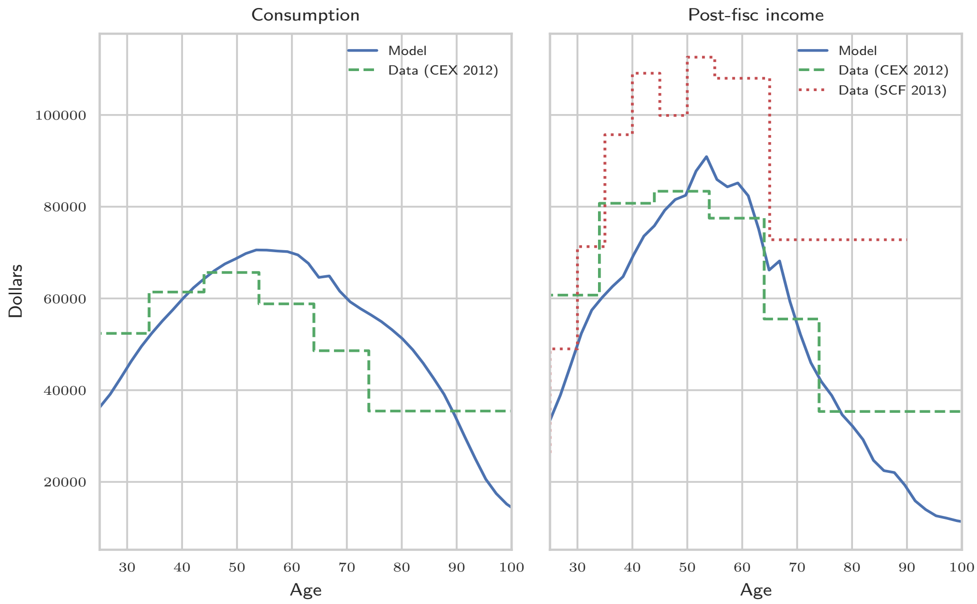

Figure 3.3. Hump-shaped profiles of mean consumption and income over the lifecycle, model and data.

see Gourinchas and Parker (2002), Feigenbaum (2008), Hurst (2008), and Olafsson and Pagel (2018).

3.3.3. *Inequality over the lifecycle.* Figure 3.4 plots Gini coefficients for wealth and income inequality by age group in the model and in the 2013 SCF, as computed by Kuhn and Rios-Rull (2016).

The series for income inequality use pre-tax earnings. In the model, labor income at each age consists of an age-specific wage multiplying idiosyncratic probability. Since agents are born with productivity randomly drawn from the ergodic distribution of process z_t, the distribution of productivity within every age group is the same. As a result, the profile of pre-tax labor income inequality is flat across ages in the model. In the data, income inequality increases with age. This may be ascribed to a more compressed initial distribution of skills in the data, or perhaps to patterns in wealth and income which permit high earners to stay in the workforce.

Note also the high level of pre-transfer earnings inequality among households 65 and over in the data. This may be ascribed to the particular characteristics of seniors who choose to stay in the labor force.

The model matches the empirical profile of wealth inequality by age relatively well. Note that in the data, the Gini coefficient for households with head aged 25 and below is very high, due to the large number of households with negative net worth. The model does not capture the full extent of inequality among these young households – there may be selection issues in the decision to form an independent household.

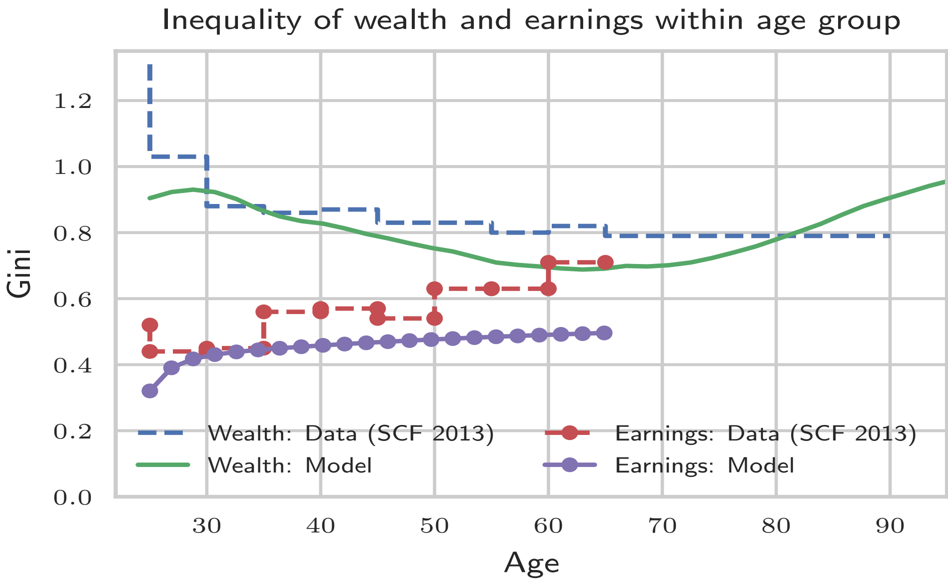

FIGURE 3.4. Gini coefficients of wealth within each age group. Dashed lines use 2013 data from the Survey of Consumer Finances. Note: Data series are binned over 5 year age groups. Leftmost point aggregates ages ≤ 25 and rightmost ages > 65.

4. The Lifecycle Component of Savings

In this section, I examine the effects of lifecycle behavior on savings. I do this by introducing lifecycle behavior to a canonical heterogeneous agent model.

Foundational papers in heterogeneous agent macro highlight the importance of the precautionary savings motive, showing how uninsured income risk leads to an inefficiently high level of savings. This leads to over-accumulation of capital (Aiyagari 1994) and a depressed risk-free rate (Huggett 1993).

I find that adding lifecycle exacerbates these effects. Lifecycle agents, facing a hump-shaped profile of expected income, will save when they work in order to finance consumption in retirement by dissaving. How savings vary over the lifecycle will also differ for different income and wealth groups. It is therefore *a priori* ambiguous whether introducing lifecycle will increase or decrease aggregate household savings. I show that in the calibrated model, aggregate lifecycle savings are large and positive.

Lifecycle then adds to over-saving relative to a complete market baseline. The motive to smooth consumption against lifecycle patterns in income is another form of precautionary savings that exacerbates over-accumulation in these economies.

This section asks how large is this effect quantitatively. Further, introducing a lifecycle will affect the savings behavior of rich and poor agents differently. How do savings change in the cross-section, and what are the implications for wealth inequality?

Intuition suggests that inequality between age groups should increase mechanically, reflecting heterogeneity in savings behavior over the lifecycle. However, the retirement savings motive turns out to decrease wealth inequality *within* age groups.

Below, I solve a version of the model of Section 2 disabling lifecycle effects, and compare it to the full specification. I find that introducing the lifecycle increases aggregate savings and wealth in partial equilibrium by around 20%. The equilibrium interest rate correspondingly increases from 1% to 3%. This finding speaks to the the risk-free rate puzzle (Weil 1989), which points out a theoretical difficulty in explaining the low risk-free rates prevailing in the data. Surprisingly, despite the large amount of extra heterogeneity in the form of age, overall wealth inequality is not much higher in the lifecycle model, as the extra savings serve as a buffer to income shocks, reducing within-cohort inequality.

4.1. **Setup.** In this section I analyze the role of introducing a lifecycle. I do this by solving a version of the model from Section 2 in which I kill off all lifecycle effects, and comparing this to the full model. This means that instead of an age profile y_t of labor income, agents receive a constant wage y multiplying their productivity. Lifespans remain uncertain: I switch to a perpetual youth setup where the flow mortality probability is the same at every age $\lambda_t \equiv \lambda$. I set y to the average of the profile y_t calibrated above, and the mortality rate λ is chosen to match life expectancy at age 25 in the calibrated model,

$$\frac{1}{\lambda} = \int_0^\infty e^{-\int_0^t \lambda_s \, \mathrm{d}s} \, \mathrm{d}t \approx 55. \tag{4.1}$$

I use perpetual youth rather than infinitely lived agents in this experiment for two reasons. Firstly, perpetual youth models do not feature lifecycle in any real sense: households of every age have the same policy functions. Age only matters insofar as it gives older agents more time potentially to accumulate wealth. There is no notion of saving for lifecycle reasons under perpetual youth, so this is an appropriate assumption.

Secondly, perpetual youth allows me to retain comparability to the lifecycle model. There, uninsured mortality risk is an important factor in households' consumption-savings decision; this would be absent with infinitely lived agents.

I make a few other changes in this section relative to the calibration of Section 3.

First, I turn off the risky asset k in both the lifecycle and perpetual youth economies. This means that initial conditions and income shocks are the only source of inequality within age groups. The equilibrium rate r^b adjusts to keep aggregate household wealth equal to government borrowing. By shutting off the exogenous returns coming from k, we can read more meaning into r^b as the interest rate that balances borrowing and saving. This gives us a rich tool for comparing the lifecycle and perpetual youth economies. In other sections we are not as interested in the meaning of equilibrium r^b, so I re-enable k, preferring better to replicate empirical wealth inequality.

Second, I impose a fixed exogenous level of government debt, set to US public debt per capita in 2016 ($60,000) in both economies. Otherwise, aggregate household wealth in the model would have to be zero in equilibrium, which can imply a negative equilibrium interest rate.[20]

20. Negative interest rates cause technical difficulties, especially with boundary conditions.

TABLE 4.1. Effect of lifecycle: model aggregates and inequality. Aggregates in thousands of dollars.

	Lifecycle	Non-lifecycle	
		Partial eq.	General eq.
Interest rate	1.0%	1.0%	2.4%
Agg. household wealth	60	49	60
Agg. consumption	56.0	56.1	56.5
Agg. savings	1.6	1.4	2.0
Income Gini	0.90	0.85	0.85
Within-age wealth ineq.	2.26	2.31	2.20
Between-age wealth ineq.	0.16	0.03	0.05

Note: Wealth inequality is measured using Theil coefficients.

Finally, fiscal policy in the full model redistributes wealth across income groups and thus implicitly across ages. This can mask the full impact of lifecycle variation in income. I minimize this effect by disabling fiscal policy as much as possible in both settings.[21]

4.2. **Comparison.** Table 4.1 lists important aggregate moments in the lifecycle and perpetual youth economies. Saving for retirement leads aggregate wealth and savings to be significantly higher in the lifecycle model. Fixing the interest rate (column 2), we see that aggregate savings and wealth are both 15-20% higher in the lifecycle model. As a result, the market clearing rate interest rate falls from 2.4% to 1% in the lifecycle model. The over-saving effects documented by Aiyagari (1994) and Huggett (1993) are exacerbated.

Income inequality is higher in the lifecycle model due to age variation in the wage. Wealth inequality overall is also higher in the lifecycle model. The table reports wealth inequality using the Theil index of inequality, a measure that lets us decompose wealth inequality into inequality between and within age groups.[22]

21. Seniors in the lifecycle model still need to receive a positive income, in order to be able to consume something at every wealth level (including negative). As a result, I maintain small social security transfers paid for by a linear income tax in the lifecycle model. Because this fiscal policy smooths out lifecycle variation in income, I am understating the true effects of the lifecycle.

22. For a probability measure $\mu(\mathrm{d}x)$ with positive support, the Theil index is

$$\int_0^\infty \frac{x}{\mathbb{E}_\mu(X)} \log\left(\frac{x}{\mathbb{E}_\mu(X)}\right) \mu(\mathrm{d}x). \tag{4.2}$$

It is decomposable as follows. Denote by $\mathcal{A}$ the set of agents. For any $A \in \mathcal{A}$ denote by $T(A) = \int \frac{w}{\mathbb{E}(W|A)} \log\left(\frac{w}{\mathbb{E}(W|A)}\right) m(\mathrm{d}w|A)$, the Theil index for wealth of agents in A, where $m(\mathrm{d}w|A)$ is the marginal probability of w conditional on A. Further, denote by $s(A) = P(A)\frac{\mathbb{E}(W|A)}{\mathbb{E}(W)}$ the share of total wealth held by agents in A. If Π is a partition of $\mathcal{A}$ into disjoint subsets, then

$$T(\mathcal{A}) = \underbrace{\sum_{A\in\Pi} s(A)T(A)}_{\text{within}} + \underbrace{\sum_{A\in\Pi} s(A)\log\left(\frac{E(W|A)}{E(W)}\right)}_{\text{between}}. \tag{4.3}$$

Figure 4.1. Average savings and wealth inequality by age group in the lifecycle and perpetual youth economies (fixing r^b – partial equilibrium). Savings in thousands of dollars per year.

There is very little between-cohort wealth inequality in the perpetual youth model. The only source of variation in wealth by age in that case comes from deterministic growth in wealth, which occurs when $r^b - \rho \neq 0$. This depends on r^b, but is quite small compared to wealth fluctuations that stem from lifecycle savings.

By contrast – and rather surprisingly – disparities within age groups are higher in the perpetual youth model. We can understand this by looking at Figure 4.1, which plots the Theil within- coefficient of inequality for each age group in both economies, together with average savings. The two plots mirror each other: wealth inequality tends increase in periods of the lifecycle where agents want to save.

This is due to constrained agents, who are consuming their income hand-to-mouth and saving almost zero. There are hand-to-mouth agents of every age. In periods of the lifecycle when agents want to save (during working life), those who have enough for a baseline level of consumption today will increase savings and accumulate more wealth. Constrained agents, however, cannot do so adequately. As a result, better off agents accumulate more wealth while the poorest cannot, leading to an increase in wealth inequality. Conversely, in periods of dissaving (retirement), most agents decumulate wealth but the poorest are hand-to-mouth and cannot, tightening the wealth distribution.

The overall impact of lifecycle patterns on aggregate wealth inequality thus depends on the relative masses of savers, dissavers, and hand-to-mouth agents over the lifecycle.

The first term is an average of inequality inside each subgroup, weighted by their share of wealth, measuring within-subgroup inequality. The second term measures inequality in the subgroup means, which captures inequality between subgroups.

Quantitatively, this sums up to a *negative*: lifecycle motives lead to an increase in within-cohort inequality.

However, overall wealth inequality increases. The decrease in lower within-cohort inequality is dominated by the higher inequality between cohorts, stemming from lifecycle fluctuations in income.

This change in the makeup of wealth inequality is important. For example, through self-insurance, the more income varies over the lifecycle, the lower is wealth inequality among seniors.

As for the general equilibrium response, the risk-free rate r^b falls by 1.4 percentage points when lifecycle is introduced. As a result, we see an increase in aggregate household savings and consumption, together with an in wealth inequality. These mechanisms were discussed in Section 5.

To wrap up, this section strives to explain the contribution of lifecycle to savings behavior. In a sense, I broaden the exercises in Aiyagari (1994) and Huggett (1993) to include the role of the lifecycle savings motive. I find that it is an important source of extra savings and can help explain the risk-free rate puzzle. Indeed, Section 6 provides a concrete example of how understanding these two savings motives matters for fiscal policy.

5. Cross-sectional Impact of an Interest Rate Change

In this section I discuss the distributional impact of a permanent exogenous change in the equilibrium risk-free rate r^b. This experiment elucidates many of the mechanisms in the model.

In response to an increase in the risk-free rate, the model predicts a long-run decline in consumption, savings, and household wealth. The richer and older the household, the stronger the effect. As a result, wealth inequality within age groups falls, especially so among retirees.

The key to understanding this reaction is the portfolio decision of households. An overwhelming majority of households are indebted, using leverage to maximize their investment in the high-yielding risky asset. The higher cost of borrowing reduces their total return on investment, and thus their disposable income.

Typically after an increase in the interest rate, lenders gain and borrowers lose out. However, lenders in the model tend to be well-off retirees. They also were net borrowers of bonds earlier in their life. As a result, their gain is outweighed by the losses they suffered from higher borrowing costs in their youth.

5.1. **Experiment.** I take the economy calibrated as in Section 3, and permanently increase government consumption from 0 to \$30,000 per capita, financed by extra government debt. As a result, the market-clearing rate for bonds increases from 2% to 3%.

Figure 5.1 plots comparative statics for three different wealth groups, the difference between lifecycle profiles in the low r^b and high r^b steady states. Disposable income, consumption, savings, and household wealth drop for all agents. The richer and older the household, the stronger the effect.

The key to understanding this reaction is the portfolio decision of households. The two left-hand panels of Figure 5.2 illustrate that lenders in the economy are overwhelmingly

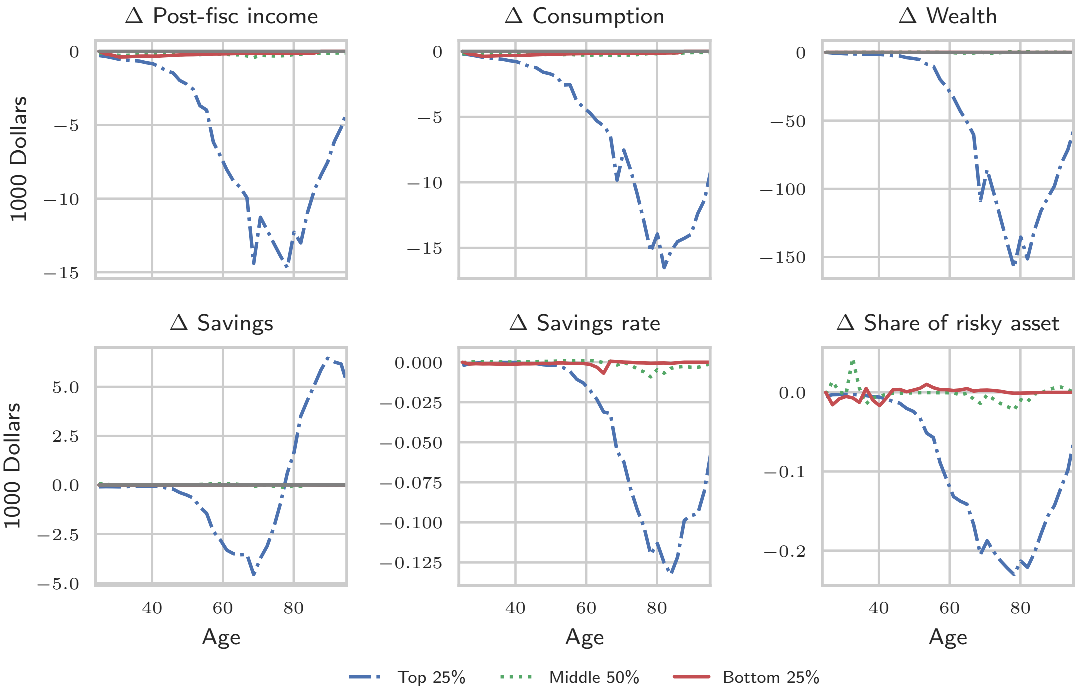

FIGURE 5.1. Change in long term lifecycle profiles in response to 1 percentage point increase in interest rate r^b, by wealth group.

older and at the top of the wealth distribution. An overwhelming majority of households are indebted, using leverage to maximize their investment in the high-yielding risky asset despite positive net worth. The higher cost of borrowing reduces their total return on investment, and thus their disposable income. This income effect reduces consumption and savings across the board.

This is felt especially strongly by wealthier households, with a high capital share of income. Since wealth – and thus capital income – are determined by cumulated savings, the reaction strengthens through the lifecycle of these households, as visible in Figure 5.1.

Wealth inequality within age groups declines as a result, as evident in the right-hand panel of Figure 5.2. The compounded effect of lower returns reduces the wealth of top asset holders more than lower wealth groups, compressing inequality.

There are two other important ways in which the increase in the risk-free rate affects wealth inequality in the model. These have to do with households adjusting their portfolios.

The bottom right panel of Figure 5.1 shows that when r^b increases, households find bonds more attractive and rebalance in their favor, away from risky capital. By simple virtue that they are holding fewer assets with idiosyncratic risk, wealth inequality decreases. This is the standard argument linking low safe rates with inequality.

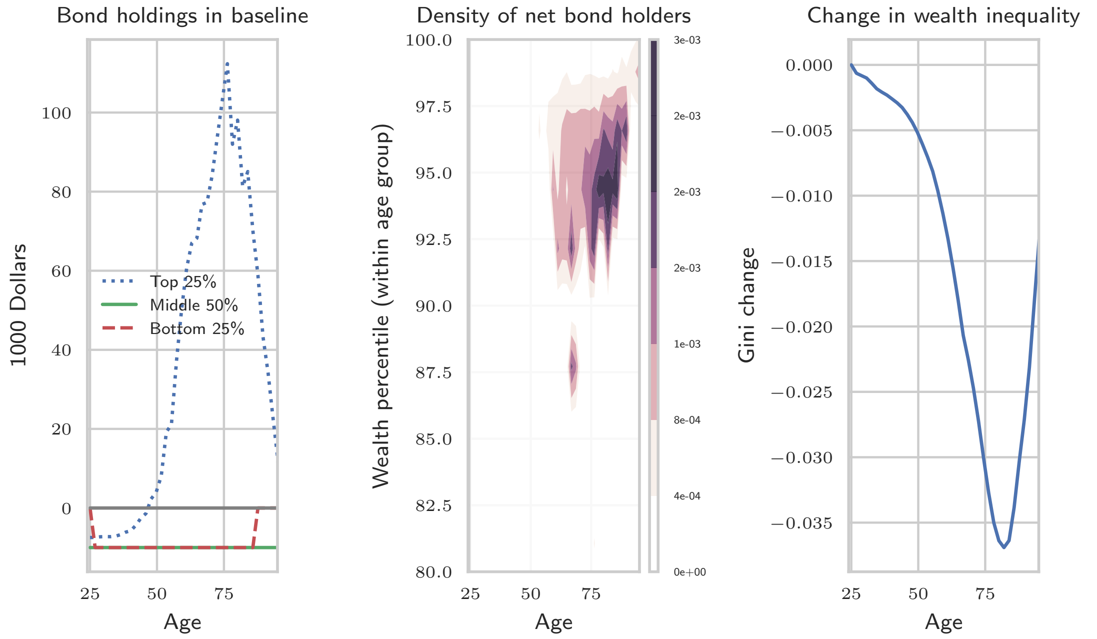

FIGURE 5.2. Lifecycle bond holdings in the baseline, and change in wealth inequality within age groups in response to 1 percentage point increase in interest rate r^b.

Indeed, we can see this in the model by rewriting the first-order condition for capital (Proposition 1) as

$$\frac{k}{w} = \frac{1}{\epsilon}\frac{1}{\gamma}\frac{r^k - r^b}{\sigma^2}, \tag{5.1}$$

where

$$\epsilon = \frac{\partial_w c}{c/w}$$

is the elasticity of consumption to financial wealth.[23] The optimal portfolio share of capital in financial wealth is given by the risk premium discounted twice: by the coefficient of relative risk aversion and by this elasticity. The more fluctuations in wealth (and thus in k) pass through to consumption, the more agents dislike risk and shy away from k.

This first mechanism simply comes from a decrease in the numerator of the right-hand-side of equation (5.1). However, another channel that emerges from the model is a change in effective risk aversion, through ϵ. Higher borrowing costs reduce the return on wealth and lead households to accumulate less wealth. Households in the model are more risk-loving at lower levels of financial wealth; especially so for retirees.

This may sound paradoxical but is in fact optimal. Consumption is determined by total wealth, financial and human. For richer retirees who have little remaining permanent income, financial wealth w is the most important factor. The elasticity ϵ for

23. Substitute the first order condition for consumption (and its derivative) into the condition for capital.

these agents is close to one. By contrast, households that are poorer and younger face borrowing constraints and anticipate future income. Their elasticity of consumption to wealth is below one. In other words, safe in the knowledge that they have other sources of income, poorer and younger households are relatively more willing to accept risk.

As a result, households have an incentive to invest further into risky capital. However, numerically this channel is dominated by the increase in the numerator, and overall households choose to deleverage.

Estimates of these reactions in the data vary. Ameriks and Zeldes (2004) do not find evidence of age variation in risky portfolio shares. Blundell et al. (1994) estimate that the intertemporal elasticity of substitution increases with age and income, in contrast with the predictions of my model. This can be ascribed to the simple household preference structure; extending the model to include non-homotheticities such as age-dependent elasticity of substitution would help match this fact (see Straub (2017) for a discussion).

Finally, a word on lenders. As discussed and charter in the middle panel of Figure 5.2, lenders in the model tend to be retired and in the top end of the wealth distribution. Lenders typically stand to gain from an interest rate rise. Here, however, even they are worse off in the long run. These wealthy retirees have had to go through a lifecycle in which they will have spent much of their lives as borrowers. As a result, they will have paid the higher borrowing rates and suffered the costs of the rate rise. This scarring effect dominates, keeping their wealth, consumption, and savings below the baseline levels despite the increased return on their lending.

This thinking illustrates the importance of considering the lifecycle alongside heterogeneity in income and wealth. A cruder analysis of an interest rate change which looks at the impact on borrowers and lenders without acknowledging that most savers were once borrowers, will miss this crucial channel and arrive at an incomplete answer.

6. Distributional Effects of Public Pension Reform

In the Section 4 we examined the impact on savings and wealth – both in aggregate and distributionally – of the introduction of lifecycle motives. Instead, in this section I discuss at the marginal effect of lifecycle saving, by marginally increasing public insurance against lifecycle fluctuations. This takes the form of an incremental reform to social security.

In this section I discuss the impact of fiscal policy reform on the distribution of consumption and savings. In particular, I will look at a variety of pension reforms, focusing on a small increase in social security transfers paid for by an increased income tax. These unconditional transfers act as partial public insurance against lifecycle variation.

I look at how this change in transfers affects consumption and saving across age, wealth and income groups in the model, and how these effects vary over time. Responses are very heterogeneous: some households essentially do not react, while the savings of others react with very high elasticities. The magnitude and distribution of these reactions agree with the literature that estimating reactions of savings to a range of fiscal policy changes.

The cross-sectional reaction leads to large and surprising effects on wealth inequality. In particular, more generous social security leads to increased wealth inequality, especially among seniors.

This experiment generalizes the textbook exercise in which pay-as-you-go social security is introduced to an overlapping generations economy. There, the two takeaways are that (a) there are winners and losers (the current old get a free ride) and (b) this policy can be Pareto improving by remedying dynamic inefficiency. Here, the answer to (a) becomes more nuanced: the current old still get a free ride, but many current workers also become better off by the provision of this insurance (despite the higher payroll tax).

The full effect of the mechanism takes the length of one career to be felt. When the policy is announced, workers rethink their consumption and retirement savings decision. Faced with more safe retirement income, they want to save less and consume more now. The extent of this change depends on their wealth and income. Asset-rich households plan to use their savings to consume a lot at retirement; transfers comprise a small part of their retirement income. As a result, the change in transfer does not significantly impact their savings decision.

Less well-off workers, however, find themselves with more permanent income coming later in life. They prefer to sacrifice saving for more consumption today, accumulating less wealth. The poorest workers in the model are hand-to-mouth consumers. They already are not and cannot not save.

As a result, we see a highly nonlinear reaction of savings across the wealth cross-section. Households in the top end of the wealth distribution are not significantly affected by the policy change, while those in the middle and lower end of the distribution accumulate less wealth. As a result, wealth inequality increases.

There are further effects to consider. When the pension increase is paid for by higher taxes, these affect wealth groups differently. Taxes aside, other mechanisms also link these transfers to inequality, such as a portfolio choice channel and interaction with buffer-stock saving. Finally, those who were already alive at the time of the policy announcement will find themselves with different wealth patterns and elasticities as later generations, leading to medium term dynamic effects. I discuss all of these lower down.

Later in this discussion, I compare the reaction to this flat social security increase to the case of an asset-tested program. I find that progressivity of the reform greatly amplifies the distributional effects, with important policy implications. I also run similar experiments for transfer programs targeting workers. There, the reactions of aggregates are less strong, as workers anticipate future tax liability. In Appendix A.2 I detail further policy experiments, to demonstrate the robustness of the results in this section.

6.1. **Setup.** In the initial equilibrium, social security consists of a universal lump-sum transfer of \$10,000/year to all agents over 65. This transfer is paid for by a progressive income tax, as described in Section 3.2.2. In the main experiment, fiscal policy changes as follows: the social security transfers increase by 10%, and the level of the income tax θ_0 increases in order to keep fiscal balance, keeping the progressivity parameter θ_1 fixed. The fiscal reform is a surprise, announced and implemented at calendar date $\tau = 0$.

In order to tease out the effects of the fiscal change, I decompose the reaction into three parts:

(1) the partial equilibrium with higher transfers but *without* the associated tax hike (in which the extra transfers arrive for free, like manna from heaven);

TABLE 6.1. Change of aggregates over pension wealth change, with and without corresponding tax increase

Variable	No Tax	Tax (Partial Eq.)	Tax (General Eq.)
Wealth	-54%	-67%	-69%
Savings (*flow*)	-2.7%	-3.1%	-3.2%
Cons.	2.5%	-0.5%	-0.8%
Wealth Gini*	0.0279	0.0281	0.0263

* Elasticity

(2) the partial equilibrium with higher transfers and the higher income tax;
(3) the general equilibrium response to both higher transfers and higher taxes, when r^b adjusts.

Importantly, I turn the risky asset k back on for these experiments in order to get a better picture of wealth inequality in these economies. The incremental effect of part (3) is identical to the discussion of Section 5. For this reason, I focus on (1) and (2) above.

The empirical literature focuses on the coefficient of substitution of private savings for expected pension wealth. I define pension wealth in the model as the expected present value of pension benefits,

$$PW(t, w, z) = \mathbb{E}_t \left[\int_t^\infty \Theta(t', w_{t'}, z_{t'}) 1_{t' \geq 65} e^{-\int_{t'}^{\xi} (r_s^b + \lambda_s)\, \mathrm{d}s} \, \mathrm{d}t' \mid w_t = w, z_t = z \right]. \tag{6.1}$$

Here $\Theta(t', w, z) 1_{t' \geq 65}$ are the transfers received after age 65. Appendix B.3 explains how to calculate this expectation efficiently using a Feynman–Kac equation.

6.2. **Comparative statics.** I begin with looking at the long term effects of increasing social security, by comparing the high-social-security and low-social-security steady states. Since the fiscal reform hits agents of all ages, it takes a long time to converge to the new steady state after the reform is announced, on the order of a lifetime. I discuss transition dynamics in the next section.

Table 6.1 lists the reaction of model aggregates and inequality to the increase in social security, computed as the difference from the baseline aggregates divided by the change in pension wealth.

The first column lists elasticities in the case of higher social security transfers without matching tax increase. There is a large amount of crowding out: an extra dollar of pension wealth reduces household financial wealth by 54 cents. This is achieved by a drop in savings, smoothed out over working life. As a result, consumption increases. I discuss the distributional impact lower down.

The second column has the transfer increase together with the tax increase, but in partial equilibrium (where the interest rate r^b is fixed at the baseline level and the bond market does not clear). Relative to the No Tax scenario, the wealth effect of the income tax hike leads to stronger negative reactions of aggregate savings and wealth. These feed through to consumption, which also falls.

Finally, the third column outlines the full general equilibrium response with transfers and tax hike. The bond market clearing interest rate r^b increases from 2% to 2.04%;

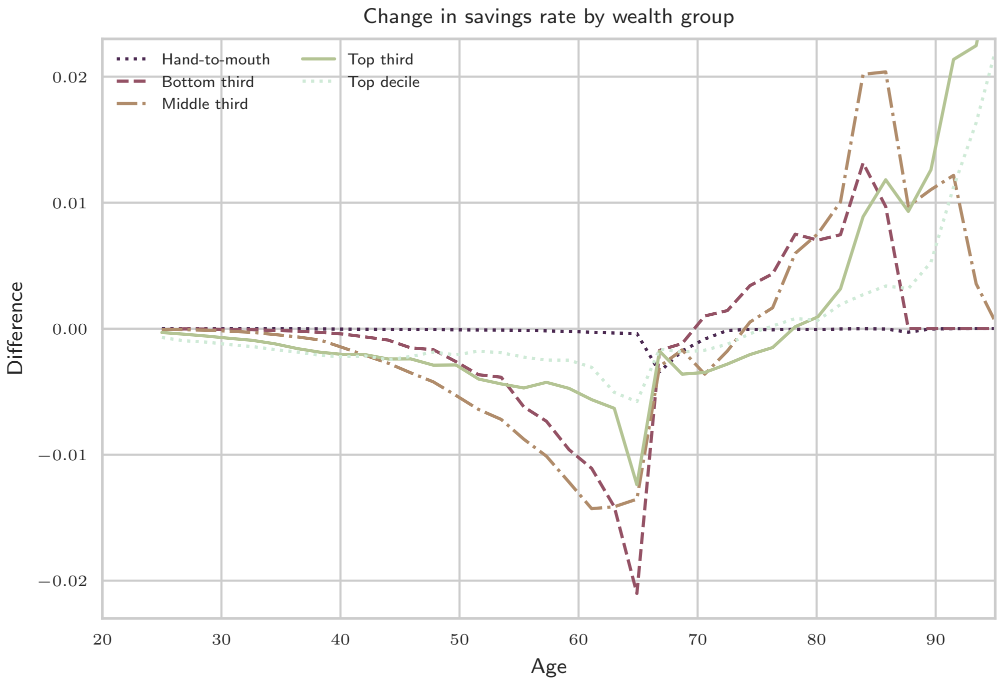

FIGURE 6.1. Change in savings rate (savings divided by total disposable income) over the lifecycle in response to a 10% increase in social security, for different wealth groups. (Averaged over income groups, *without* accompanying tax hike.)

an elasticity of 0.20. Relative to the second column, savings and wealth fall further, along with consumption. The wealth distribution becomes less unequal, as we saw in Section 5.

We can tease out a fuller picture by looking at cross sections. Figure 6.1 plots the change in savings rate for each age group, defined as savings over total post-fiscal income. Agents save less during working life, especially so right before retirement where they smooth using savings and anticipate the extra transfers. Those in the middle and bottom third of the wealth distribution dissave the most. For them, these extra transfers are an important component of anticipated retirement income, replacing their need to self-insure. By contrast, the hand-to-mouth and the top 10% do not change their savings substantially, either because they cannot or because for them the extra social security is insignificant.

At age 65, the savings rate of all wealth groups is a few basis points lower than in the baseline. As they age, the picture is almost a mirror image of the graph before retirement, with poorer agents increasing their savings rate the most. This is for a reason. Agents who reduced their retirement savings during working life now enter retirement with less financial wealth. As a result, their capital income in retirement is necessarily lower. This pushes up their savings rate for a given level of savings. Although the level of savings

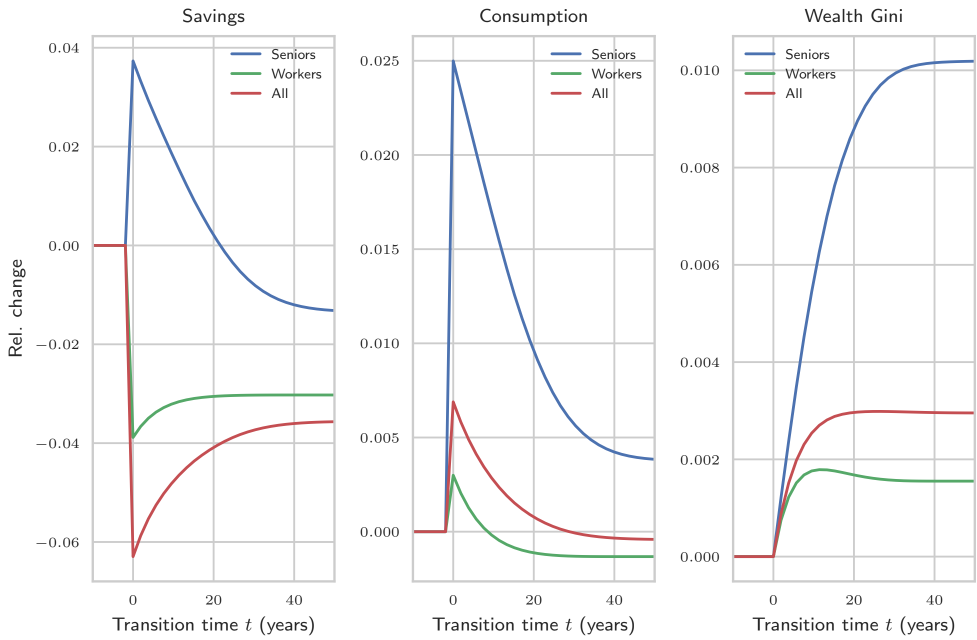

FIGURE 6.2. Dynamic response of savings, consumption, and wealth inequality to a 10% increase in social security. Workers and retirees are defined as agents with age below and above age 65, respectively. Aggregates in thousands of dollars.

shifts down (and consumption shifts up) as they consume the extra transfers, the impact on income dominates.

The consumption–savings decision for retirees is simpler. Social security does not vary with age; the only source of lifecycle variation for seniors is mortality risk. As a result, the transfers are less insurative than they would be during the more volatile period of working life. Retirees find themselves with more permanent income which is immediately available. Consumption and savings both increase; especially so for retirees in lower wealth groups.

Working life is when savings behavior has the most lasting impact on wealth accumulation. Poor workers dissave more than rich workers, due to the differential rates at which the public transfers crowd out private savings. As a result, wealth inequality increases.

Another channel for the increase in wealth inequality is portfolio choice. With more safe pension income, household consumption becomes less sensitive to fluctuations net worth. This serves to reduce the effective level of risk-aversion of agents and thus to encourage investment in the risky asset, as in equation (5.1). Since k carries *idiosyncratic* risk, this will lead to a further increase in wealth inequality.

These channels are in fact two sides of the same coin, highlighting the importance of understanding the role of transfer policies as insurance.

6.3. **Dynamic effects.** Figure 6.2 shows how aggregates and wealth inequality evolve through the transition. The first thing to note is that convergence is slow, on the order of 50 years.

As in the textbook analysis of pay and you go social security in an OLG model, the winners are the current old, who greatly increase both consumption and savings when they receive a windfall which they have not paid for. Further, current workers generally increase consumption too (and reduce savings) as all but the youngest workers will be receiving pension income that future generations will have to pay for.

Over time, workers will start to save more: they expect to spend more time paying the higher income tax and so the transitory bump to their permanent income tapers off. Similarly, after around $\tau = 25$ years, many of the current retirees will be agents who have paid the higher income tax for a significant portion of their working life. As a result, they enter retirement with less financial wealth and thus less capital income than earlier retirees. Their savings gradually fall below those of retirees in the initial steady state. Their consumption remains higher thanks to their higher pension wealth.

Due to these effects on savings, wealth inequality increases gradually across all groups. It reaches an apex around $\tau = 25$, after most of the windfall-receiving generation has died and enough time has passed for the new retirees to have had their savings in working life substantially affected by the reform.

6.4. **Empirical estimates.** There is a large empirical literature estimating how much public pensions crowd out private savings. Attanasio and Brugiavini (2003), Attanasio and Rohwedder (2003), Aguila (2011), Feng et al. (2011), and Lachowska and Myck (2018) use a difference-in-difference design around pension reforms in Italy, the UK, Mexico, rural China, and Poland, respectively. They estimate the degree of substitution between the household savings rate and to pension wealth (expressed in units of current income). They find this coefficient to lie broadly between -0.2 and -0.7. This is a measure of substitution of a flow for a stock. The estimates are surprisingly large: an extra unit of pension wealth leads savings *each year* to be 0.2 to 0.7 units lower. In other words, extra pension wealth will crowd out the stock household wealth more than one to one within five years.

Bottazzi et al. (2006) address this issue by relating the level of household *wealth* to pension wealth, studying three pension reforms in Italy in the 1990s. They find a degree of substitutability around 65% overall, with a higher coefficient (around 80%) for agents well-informed about their pensions.

By contrast, Chetty et al. (2014) uses a regression-discontinuity design to analyze the impact of a pension reform in Denmark in 1998 in which firms became mandated to make automatic contributions to workers' retirement accounts for earnings above a threshold. They find no relationship between private saving and public pension wealth. One suggested explanation for the difference (Lachowska and Myck 2018) is that the reforms studied by Chetty et al. (2014) and Feng et al. (2011) *increase* pension wealth, while the other studies look at reforms that reduced it. This theory is supported by the increasing evidence of a large asymmetry in micro propensities to save and consume. Reactions to negative shocks can be an order of magnitude larger than to positive (Bunn et al. 2017). Further, the discontinuity design relies on the savings behavior of agents around the threshold (earnings of $5,300), while other studies use a broader design.

TABLE 6.2. Effect of different policy reforms, all costing $1,000 per capita per year. Reported as difference relative to baseline, in thousands of dollars.

	Pension		Workers	
	Flat	Wealth test	Income test	Youth flat
Savings	-0.23	-0.28	-0.10	-0.02
Wealth	-4.9	-5.6	-4.2	-0.4
Consumption	-0.05	-0.04	-0.15	0
Wealth Gini	21bp	35bp	52bp	4bp

Finally, Chetty et al. (2014) finds a negligable impact among most agents, but a large reaction conditional on a reaction of savings. It may be that the reform in Denmark, which entailed automatic pension contributions from employers, went unnoticed by many households.

Another useful source of estimates is the literature on the impact on savings of reforms to public health insurance. Maynard and Qiu (2009), De Nardi et al. (2016), and Gruber and Yelowitz (1999) study reforms to Medicaid, the asset-tested medical insurance program. They find a range of elasticities of private savings to changes in the transfer program between 0 and -0.15.

In the model, substitutability is around 0.7, on the higher end of available estimates. This measures is closer to estimates for well-informed agents found in Bottazzi et al. (2006). This makes sense, given the well-known inattention to pension arrangements in the data (Thaler and Benartzi 2004; Chetty 2015).

These studies do not present sufficient cross-sectional evidence to let me validate the model's predictions for savings across wealth groups. However, distributional evidence on marginal propensities to save and consume out of windfalls[24] provide some support for the model, finding large reactions around the bottom of the wealth distribution that decline with net worth and age.

6.5. **Progressive policy and worker subsidies.** In this section I compare the effects described above to a number of different policy specifications. These are summarized in Table 6.2.

6.5.1. *Progressivity of social security.* The experiment above analyzed flat reform to public pensions, affecting all agents equally. In this subsection I look at the effect of a *progressive* reform to social security, by introducing a means-tested subsidy for the poorest retirees. Many countries have such programs in place, such as the Supplemental Security Income (SSI) in the United States, which pays up to $1,100 per month to a couple with net worth below $3,000. In order to retain comparability with the exercise above, I look at a marginal version of this program which adds a small means-tested component to an existing social security setup. I find that progressivity amplifies all the mechanisms set out above.

24. For example, Misra and Surico (2014), Kaplan and Violante (2014), and Kaplan et al. (2014) study fiscal windfalls, Fagereng et al. (2016) look at lottery winnings, and Bunn et al. (2017) use a survey.

I study a reform that introduces an extra transfer schedule $\tilde{\Theta}$ which adds \$950 per year to all agents plus a gradual subsidy of up to an extra \$400 per year for agents with net worth below \$1,000,

$$\tilde{\Theta}(w) = \begin{cases} 950 + 400(1000 - w) & \text{if } w < 1000; \\ 950 & \text{otherwise.} \end{cases}.$$

This way, the reform remains broadly similar to the setup analyzed above, with the addition of a small wealth tested top-up.

The crowding out coefficient of private wealth increases by a further 10 percentage points compared to the flat reform analyzed above. The distributional impact of this progressive reform is also much larger, with the elasticity on the wealth Gini coefficient doubling.

Progressivity amplifies the asymmetry in savings reaction through cross-section. As before, private savings are crowded out by these extra transfers. This time, the transfers are particularly effective insurance for the poor households targeted by the policy. As a result, there is a high degree of substitutability of their savings for transfers. Further, asset-tested transfers carry an implicit wealth tax which strongly impacts the savings of households near or below the eligibility threshold for the program. On the other hand, better-off households with no hope of eligibility will substitute relatively less. As a result, wealth inequality increases even more as a result of this progressive pension reform.

Reactions of aggregates are also amplified, but to a smaller extent. Although the savings of poorer households fall drastically in relative terms, these account for a smaller fraction of aggregate savings and consumption. Indeed, the reaction of savings is only slightly increased, but this feeds through to a large cumulative effect on aggregate wealth.

This example highlights the high sensitivity of distributional variables to fiscal reform, even as aggregates display a more mild reaction. It is crucial therefore to evaluate the (often explicitly distributional) objectives of fiscal policy in a setting that captures the nonlinear and heterogeneous reaction of households.

6.5.2. *Worker subsidies.* To what extent do the effects above carry over to subsidies to *workers* instead of retirees? Rather than providing insurance for lifecycle variation in income, worker subsidies mainly affect the precautionary savings motive. I highlight two scenarios in this section, and consider others in Appendix A.2. The programs are selected to cost the same to the government as the social security increase, so that the accompanying tax increase is identical.

Table 6.2 reports the general equilibrium reactions of chosen moments to the various policies.

Means-tested programs. The first scenario is a productivity-tested transfer program. Since there is predictable lifecycle variation in earnings, income-tested benefits redistribute across the age groups as well as income groups. Looking at a productivity test allows us to isolate the precautionary savings motive. These are not very commonly found in the data, however they can simply be implemented by making a household's benefits depend on income divided by median income for its age.

The results are broadly similar to the pension increase. Savings are crowded out, but effects are weaker at an aggregate level. Households are more likely to be constrained and do not have as long of a time horizon to anticipate the transfers. However, these productivity-tested transfers act as particularly good insurance against productivity shocks. Relieving the precautionary savings motive turns out quantitatively to have a similar effect on savings as social security. However, by contrast, aggregate consumption increases. These transfers are highly welfare improving, helping hand-to-mouth households.

The transfers reduce the incentive to borrow in bonds, so the equilibrium bond rate increases slightly.

Interestingly, even when the means-tested transfers target workers, wealth inequality still increases over the long run. The mechanism is similar. Eligible low-productivity households dissave in response to the subsidies, while high-type households do not. Productivity and wealth are highly correlated, especially for younger households. As a result, the wealth of lower-asset households falls, widening the wealth distribution.

Lump-sum programs to the young. The second scheme I look at is a flat transfer to all agents aged 35 or below. These kinds of programs have been proposed in recent years to address intergenerational inequalities linked to rising house prices and slow wage growth.

In contrast to the other experiments, this policy seems to have little effect. There is no crowding out, there is very little movement on savings, wealth and consumption. There is also not much distributional effect.

In fact, this policy is much closer to being neutral in a Ricardian sense. When young households receive the transfers they anticipate their higher future income taxes which finance the program. The policy pushes young households, who are likely to be hand-to-mouth, away from the financial constraint. Even if the policy is not actuarially fair and changes permanent income, it will have minimal aggregate impact as young households undo its effect.

Indeed, transfers to the young that are paid for later in life are much more likely to be neutral than the converse (social security), as it is easier to save than to borrow.

7. Conclusion

In this paper, I develop a framework for addressing new questions in macroeconomics at the intersection of lifecycle and distributional considerations. To do this, I build a new kind of heterogeneous agent overlapping generations model, with rich characterization of the lifecycle together with a calibration of income shocks on micro data. I discuss the cross-sectional reaction of savings to various fiscal policy reforms, and analyze the dynamic implications for wealth inequality.

I find much heterogeneity in crowding out of private savings by transfer programs. The model explains estimates in the data but suggests a high degree of sensitivity of the savings distribution to progressivity and to the lifecycle timing of transfers. In a world of population aging and increasing inequality, natural targets for fiscal policy, it is important to have a framework for policy analysis that captures the nonlinearity and heterogeneity of household behavior.

Many extensions of this project are worth exploring in future work. Firstly, the supply side is stylized in this paper by design, with a focus on household decisions. In future work I plan to enrich production in order to harness the channels set out in this paper. In Ascari et al. (2018) we add a New Keynesian supply side to these households and look at the cross-sectional impact of a change in the inflation target. Indeed, age patterns in housing and portfolio allocation are very important for monetary transmission.

Second, the debate on the determinants of the rise in wealth inequality in the United States is ongoing. Previous studies have argued about whether this should be ascribed to declining marginal tax rates or to increasing labor income inequality due to skill-biased technological change. This paper argues that government transfer programs can have large impacts on wealth inequality. The model suggests that it takes about 20 years for the full impact on wealth inequality to be felt. There may thus be a connection between the dramatic growth in US government transfer programs from the 1960s and rising wealth inequality in the US from the 1980s.

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Appendix A. Further analysis and robustness results

A.1. **Bequests.** In this section, I discuss how to introduce a bequest motive to the model and show that the model predictions are robust to the addition.

Recall from Section 2 that households choose how much to invest into the annuity in order to decide on the size of their estate were they to die. By giving them utility for this (potential) bequest we can introduce a 'joy of giving' bequest motive (see e.g. De Nardi et al. (2006) and Benhabib et al. (2016)). Agents care only about the total size of their legacy and not how it is distributed between their children. Note that $q_t \geq 0$ at all times – no insurer will sell an annuity worth more than the wealth of the agent.

With bequests, agents maximize

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t-\int_0^t \lambda_s \,\mathrm{d}s}\big(u(c_t)+\lambda_t \tilde{u}(q_t)\big)\,\mathrm{d}t\right], \tag{A.1}$$

over $c_t \geq 0, q_t \geq 0, k_t \geq 0$ subject to the budget constraint below. Joy of giving utility for bequests $\tilde{u}(q_t)$ for is assumed to be:

$$\tilde{u}(q) = \chi\frac{(q-\underline{q})^{1-\tilde{\gamma}}}{1-\tilde{\gamma}}. \tag{A.2}$$

Parameter $\chi > 0$ captures the relative strength of the bequest motive. Here $\underline{q}$ and $\tilde{\gamma}$ are parameters that make the bequest motive non-homothetic when $\underline{q} \neq 0$ or $\tilde{\gamma} \neq \gamma$. In these cases, richer households have relatively stronger bequests motives, encouraging them to save more.

The budget constraint becomes

$$\mathrm{d}w_t = (y_t z_t + \Theta(t,w_t,z_t) + r_t^b w_t + (r^k - r_t^b)k_t - c_t - \pi_t(q_t - w_t))\,\mathrm{d}t + \sigma k_t\,\mathrm{d}B_t, \tag{A.3}$$

implying a Hamilton–Jacobi–Bellman equation

$$\begin{aligned}(\rho+\lambda_t)V(t,w,z) = \max_{c,k,q}\, & u(c)+\lambda_t\tilde{u}(q)+\partial_t V \\ & + \partial_w V\big[y_t z + \Theta(t,w,z) + r_t^b w + (r^k - r_t^b)k - c - \pi_t(q-w)\big] + \partial_{ww}V\frac{\sigma^2}{2}k^2 \\ & + \partial_z V \mu^z(z) + \eta\int_{-\infty}^{\infty}(V(t,w,z') - V(t,w,z))\phi^z(z')\,\mathrm{d}z'.\end{aligned} \tag{HJBb}$$

The following propositions characterize optimal policies.

Proposition 4 (First order conditions)**.** *The agent's optimal policy functions $c(t,w,z)$, $q(t,w,z)$, $k(t,w,z)$ for consumption, wealth at death, and capital holdings at age t, wealth* *level w, and productivity z, are characterized by the first order conditions*

$$c(t,w,z) = (u')^{-1}\big(\partial_w V(t,w,z)\big) \tag{FOC c}$$

$$q(t,w,z) = (\tilde{u}')^{-1}\left(\frac{\pi_a}{\lambda_a}\partial_w V(t,w,z)\right) \tag{FOC z}$$

$$k(t,w,z) = \min\left\{-\frac{\partial_w V(t,w,z)}{\partial_{ww} V(t,w,z)}\frac{r^k - r^b_t}{\sigma^2}, w-\Phi\right\}, \tag{FOC k}$$

for any (t,w,z) such that $t \geq 0, w \geq \Phi$.

Proposition 5 (Euler equations)**.** *Let (w_t, z_t) denote the optimal paths of wealth and productivity given an initial value and the draws of the shock processes. Then the paths of consumption $c_t = c(t,w_t,z_t)$ and bequests $q_t = q(t,w_t,z_t)$ satisfy*

$$\begin{aligned}(\rho - r^b_t + \pi_t - \lambda_t - \partial_w\Theta(t,w_t,z_t))\,dt &= E\frac{du'(c_t)}{u'(c_t)} + \frac{dk_t}{k_t}\frac{du'(c_t)}{u'(c_t)} \\ &= E\frac{d\tilde{u}'(q_t)}{\tilde{u}'(q_t)} + \frac{dk_t}{k_t}\frac{d\tilde{u}'(q_t)}{\tilde{u}'(q_t)}\end{aligned}$$

away from the constraint, i.e. whenever $w_t > \Phi$.

A.1.1. *Initial wealth and bequests.* To close the model with bequests we need to discuss what happens with bequests and link that with the wealth of newborns. Ideally, the distribution of newborn wealth should align with the distribution of after tax bequests.

The model does not keep track of family relationships between agents. Instead, your child is the entering agent who happens by chance to inherit your wealth when you die.

Of course, in real life, bequests are not necessarily received at the time of household formation. Instead, this could be understood as agents borrowing against their future bequest, or as the monetary value of an intangible transmission of human wealth.

Let

$$\Lambda = \int_0^\infty \lambda(t)D(t)\,\mathrm{d}t$$

denote the total mass of agents dying and NB the total mass of newborns.

Recall that agents choose the optimal size q of their total bequest if they were to die in this instant. Let $\tilde{q} = q + \vartheta^q(q)$ denote the size of the bequest after estate taxes and redistribution. This is split into NB/Λ many gifts, each of size $\tilde{q}\Lambda/NB$. A newborn is given a random draw from the distribution of split gifts from agents that actually die. The density of newborn wealth is thus

$$f^w_0(w) = \frac{NB}{\Lambda}\int_0^\infty f^{\tilde{q}}\left(\tfrac{NB}{\Lambda}w|t\right)\lambda(t)D(t)\,\mathrm{d}t, \tag{A.4}$$

Here $f^{\tilde{q}}(\tilde{q}|t)$ is the density of after-tax bequests $\tilde{q}$ for agents of age t, obtained from the joint density through the after-tax bequest policy function $q(t,w,z) + \vartheta^q(q(t,w,z))$.

We can see that the total mass of newborn wealth,

$$NB\int w f_0(w)\,\mathrm{d}w,$$

equals the total mass of after-tax bequests made from dying agents.

Table A.1. Elasticities of aggregates to pension reform size

Reform	Cons.	Saving	Wealth	Wealth Gini
Social security				
+10%	-0.01	-0.31	-0.20	0.03
-10%	-0.01	-0.33	-0.21	0.03
+1%	0.00	-0.31	-0.21	0.03
+100%	-0.01	-0.26	-0.17	0.02
Worker subsidies				
Small income test	-0.02	-0.13	-0.17	0.07
Larger income test	-0.04	-0.21	-0.29	0.13
Flat to sub-35s	0.00	-0.03	-0.01	0.00

The productivity level of newborns is randomly drawn from the long-run (ergodic) distribution of z_t.

In this case, the government budget constraint becomes

$$\frac{\mathrm{d}b^g_\tau}{\mathrm{d}\tau} = r^b_\tau b^g_\tau - c^g_\tau - \int \Theta_\tau \,\mathrm{d}m_\tau - \int \vartheta^q_\tau \,\mathrm{d}m_\tau, \tag{A.5}$$

where the last two terms are the net receipts of the transfer programs and of estate tax/redistribution respectively.

A.2. **Robustness of pension reform analysis.** In this section I run further pension reform experiments, modifying the size, sign and shape of the reform relative to Section 6. Each of these is accompanied by an increase in the level α of the income tax rate in order to keep government budget balance.

First, in the first four rows of Table A.1 I vary the size of the flat pension reform, reporting the elasticity of different variables to change in pension wealth. The reaction of aggregates and of inequality is quite linear in the size and sign of the change.

Next, I analyze different subsidies to workers. Each are these are calculated to cost the same to the government as a 10% increase in social security. Flat lump-sum transfers to households aged below 35 have negligible effect, as agents deduct the benefits from perceived future tax liability. The two income-tested transfer schedules take the form

$$\Theta(t, w, z) = \max\{0, A(B - z)\}1_{t\leq 65}. \tag{A.6}$$

With this continuous formulation (in z), workers with lowest productivity type receive the largest benefit. These taper off at rate A as productivity approaches the eligibility threshold B. The median productivity level is 1.0. The large income test has $(A, B) = (700, 0.5)$, while the smaller income test has $(A, B) = (1500, 1.0)$.

The two income-tested policies have a relatively small impact on aggregates than pension reforms. However, they have a large effect on wealth inequality; especially so for the more progressive policy. These are powerful insurative policies that crowd out the precautionary savings motive, argued by Gourinchas and Parker (2002) to account for the majority of savings of households aged below 40.

Appendix B. Derivations and proofs

In this section, I present further derivations and proofs of the model, omitted from the main text. For full generality, I include the bequest motive of Appendix A.1 in the derivation.

B.1. **HJB equation.** Recall the agents' optimization problem set out in Section 2.1. The agents solve for their value function

$$V(t,w,z) = \max_{(c_s,k_s,z_s)_{s\geq t}} \mathbb{E}_t \int_t^\infty e^{-\rho(s-t)-\int_t^s \lambda_\xi \,\mathrm{d}\xi}\big(u(c_s)+\lambda_s \tilde{u}(q_s)\big)\,\mathrm{d}s \tag{B.1}$$

subject to the budget constraint and borrowing limit at each age, described in the text.

Proposition 6 (HJB)**.** *The function $V(t,w,z)$ obeys the Hamilton–Jacobi–Bellman equation,*

$$\begin{aligned}(\rho+\lambda_t)V(t,w,z) = \max_{c,k,q}\, & u(c)+\lambda_t\tilde{u}(q)+\partial_t V \\ & + \partial_w V\big[y_t z + \Theta(t,w,z) + r_t^b w + (r^k - r_t^b)k - c - \pi_t(q-w)\big] + \partial_{ww}V\frac{\sigma^2}{2}k^2 \\ & + \partial_z V \mu^z(z) + \eta\int_{-\infty}^{\infty}(V(t,w,z')-V(t,w,z))\phi^z(z')\,\mathrm{d}z'\end{aligned} \tag{HJB}$$

on the interior of its domain, with boundary conditions that will be outlined later in this appendix.

Essentially, the HJB equation is the continuous time equivalent of the dynamic programming principle. By rewriting the optimal control problem recursively, it provides a characterization of the solution. The proof is standard in the literature.[25]

Note that the first order conditions pin down the choice variables in the HJB equation only away from the constraint Φ. From this we understand two facts. Firstly, the HJB equation holds only for $w>\Phi$, and in fact the wealth constraint does not appear at all in the form of equation (HJB). Agents act as if they were unconstrained inside the state space. Secondly, we need a special boundary constraint to describe the behavior of V at the wealth constraint. I discuss this condition further down.

Taking first order conditions of equation (HJB) proves Proposition 1.

Note that the first order condition k is the general solution for a Merton (1969) style portfolio choice problem, with the addition of the borrowing constraint limiting leverage.

Proposition 2 set out the Euler equations for consumption and bequests. A proof is as follows.

25. See, for example Dixit (1993) and Bardi and Capuzzo-Dolcetta (2008) for a discussion. For a proof incorporating state constraints, see Soner (1986) and Capuzzo–Dolcetta and Lions (1990). It is important to note that the right solution concept is the *viscosity solution*, for which the value function is the unique solution of the HJB equation (see Achdou et al. (2017))

Proof. Consider the maximized current-value Hamiltonian associated with the agent's problem, with co-states μ, ν :

$$\begin{aligned} H(w,z,t;\mu,\nu) = \max_{c,k,q} u(c) + \tilde{u}(q) + \nu[k\sigma] \\ + \mu\big[y_t z + \Theta(t,w,z) + r_t^b w + (r^k - r_t^b)k - c - \pi_t(q-w)\big]. \end{aligned} \tag{B.2}$$

The following optimality conditions hold

$$0 = \partial_c H = \partial_q H = \partial_k H, \quad d\mu_t = -\partial_w H \, \mathrm{d}t + \nu_t \, \mathrm{d}B_t.$$

Together with the budget constraint, these yield the result. □

HJB boundary conditions. The value function $V(t,w,z)$ has domain

$$\Omega = \left\{(t,w,z) \in \mathbb{R}^3 \mid t \geq 0, w \geq \Phi\right\}.$$

As discussed above, equation (HJB) is a partial differential equation for V that holds on the interior of Ω, but special boundary conditions hold on the boundary of Ω.

The borrowing constraints do not appear in equation (HJB). Instead, they pin down boundary conditions on $\partial\Omega$. These are *state-constraint boundary conditions,* as in Achdou et al. (2017), Soner (1986), and Capuzzo–Dolcetta and Lions (1990).

In the presence of a general utility function for bequests, it may not be possible to find a closed form for the boundary condition satisfied by V on $\partial\Omega$. However, let us focus on the special non-homothetic case where $u(c) = c^{1-\gamma}/(1-\gamma)$ is CRRA and $\tilde{u}(q) = \chi u(q)$ for some constant $\chi \geq 0$. (This is the case studied by Benhabib et al. (2016), where χ represents the relative strength of the bequest motive.)

Proposition 7. *Suppose $u(c) = c^{1-\gamma}/(1-\gamma)$ is CRRA and $\tilde{u}(q) = \chi u(q)$. Then $V(t,w,z)$ satisfies the following* state-constraint boundary condition *at each lower wealth boundary Φ:*

$$\partial_w V(t,\Phi,z) \leq u'\left(\frac{(r_t + \lambda_t)\Phi + y_t z + \Theta(t,\Phi,z)}{1 + \pi_t^{1-1/\gamma}(\chi\lambda_t)^{1/\gamma}}\right) \quad \textit{for all } z,t. \tag{B.3}$$

Proof. (The following calculations hold even without assuming CRRA.) Recall the current-value maximized Hamiltonian associated to the agent's problem, with co-state $\mu = u'(c)$. Optimality conditions imply that $\partial_\mu H|_{\mu=\partial_w V}$ equals the optimal drift of wealth. Capuzzo–Dolcetta and Lions (1990) show that the appropriate boundary condition on the HJB equation corresponding to the constraint on the state variable is

$$\partial_p H|_{p=\partial_w V(w,a)} = 0 \text{ on } \partial\Omega.$$

In other words, the optimal drift of wealth on the boundary must be equal to zero, which makes sense for a reflecting boundary.[26] Further, we know that since $k \geq 0$ and $k \leq w_t - \Phi$, we must have that $k = 0$ on the lower boundary $w = \Phi$. Together, these imply

$$y_t z + r_t^b \Phi + \Theta(t,\Phi,z) - c - \pi_t(q - \Phi) = 0 \tag{B.4}$$

where c and z are optimally chosen from their first order conditions set out in Proposition 1. In other words, optimal savings are zero at the boundary.

26. See, for example, Dixit (1993).

Substituting in the first order conditions, we can rewrite equation (B.4) in terms of the value function:

$$y_t z + (r_t^b + \pi_t)\Phi + \Theta(t, \Phi, z) = (u')^{-1}\big(\partial_w V(t, w, z)\big) + \pi_t (\tilde{u}')^{-1}\big(\frac{\pi_t}{\lambda_t}\partial_w V(t, w, z)\big). \quad \text{(B.5)}$$

This is the general form of the state-constraint boundary condition for our problem, and there is generally no way to rewrite equation (B.5) as a simple condition with V on the left-hand side.

If we now assume the CRRA functional forms of the proposition, the right hand side simplifies to equation (B.3) as required. □

The intuition is simple: the state-constraint boundary condition ensures that at the lower wealth boundary, the optimal drift on wealth is nonnegative, pushing the agents away from the boundary. It does this by ensuring that consumption and the spending on life insurance are both low enough for the agent to grow his wealth back away from the borrowing constraint. The wealth boundary is therefore a reflecting boundary.

Upper wealth constraint. When we solve the problem numerically, we must do so on a bounded grid. This implies that we need to set an upper boundary for wealth, $w \leq \bar{w}$. I choose $\bar{w}$ high enough for the wealth distribution to have negligible mass above $\bar{w}$. This way, the upper bound essentially never binds in equilibrium.

This bound is another state constraint, so we need the relevant boundary condition on the HJB equation. This condition guarantees that optimal drift of wealth at the upper boundary is negative. The optimal drift (i.e. savings policy function) must satisfy

$$y_t z + \Theta(t, \bar{w}, z) + r_t^b \bar{w} + (r^k - r_t^b)k(t, \bar{w}, z) - c(t, \bar{w}, z) - \pi_t(q(t, \bar{w}, z) - w) \leq 0. \quad \text{(B.6)}$$

We need to translate this into a boundary condition for $V(t, w, z)$. To do so, we need to have an idea of the behavior of k for large w. In the CRRA case, I show below that the value function is asymptotically a power function with curvature $1 - \gamma$ in the wealth tail.[27] Then for large enough $\bar{w}$, risky capital holdings satisfy

$$\begin{aligned} k(t, \bar{w}, z) &\approx -\frac{\partial_w V(t, \bar{w}, z)}{\partial_{ww} V(t, \bar{w}, z)} \frac{r^k - r_t^b}{\sigma^2} \\ &= \frac{r^k - r_t^b}{\gamma \sigma^2} \bar{w}. \end{aligned}$$

Substituting this into equation (B.6), we obtain the condition

$$c(t, \bar{w}, z) + \pi_t q(t, \bar{w}, z) \leq y_t z + \Theta(t, \bar{w}, z) + \bar{w}\left(r_t^b + \lambda_t + \frac{(r^k - r_t^b)^2}{\gamma \sigma^2}\right).$$

When $\tilde{u} = \chi u$, as we assumed when discussing the lower boundary condition, we can use the first order conditions for c and z to get the boundary condition at the upper boundary $\bar{w}$:

$$\partial_w V(t, \bar{w}, z) \geq u'\left(\frac{y_t z + \Theta(t, \bar{w}, z) + \bar{w}\left(r_t^b + \lambda_t + \frac{(r^k - r_t^b)^2}{\gamma\sigma^2}\right)}{1 + \pi_t^{1-1/\gamma}(\chi\lambda_t)^{1/\gamma}}\right) \quad \text{for all } t, z. \quad \text{(B.7)}$$

27. I follow Achdou et al. (2017, Online appendix, pp15–6) here.

B.1.1. *Asymptotics and proof of Proposition 3.* In Section 3.1, I argued that household policy functions are asymptotically linear in wealth, and that as a result it is possible to back out an expression for the shape of the tail of the wealth distribution. I provide a sketch proof below. More details – in the case of a non-lifecycle model – can be found in Achdou et al. (2017) and Benhabib et al. (2016).

Lemma 1. *When the utility function is CRRA as in Proposition 7, the agent's policy functions are asymptotically linear in wealth as $w \to \infty$. Namely, for any t,*

$$\frac{c(t,w,z)}{w} \to \zeta_t^c := \frac{\rho + \lambda_t - \frac{1}{2}\frac{(r^k - r_t^b)^2}{\sigma^2}\frac{1-\gamma}{\gamma} - (1-\gamma)(r_t^b + \lambda_t)}{\gamma\left(1 + \lambda_t \chi^{1/\gamma}\right)}, \tag{B.8}$$

$$\frac{q(t,w,z)}{w} \to \zeta_t^q := \frac{\rho + \lambda_t - \frac{1}{2}\frac{(r^k - r_t^b)^2}{\sigma^2}\frac{1-\gamma}{\gamma} - (1-\gamma)(r_t^b + \lambda_t)}{\gamma\left(\chi^{-1/\gamma} + \lambda_t\right)}, \tag{B.9}$$

$$\frac{k(t,w,z)}{w} \to \zeta_t^k := \frac{r^k - r_t^b}{\gamma\sigma^2}, \tag{B.10}$$

$$\frac{s(t,w,z)}{w} \to \zeta_t^s := \frac{r_t^b - \rho}{\gamma} + \frac{(r^k - r_t^b)^2}{\gamma\sigma^2}\frac{1+\gamma}{2\gamma}. \tag{B.11}$$

Proof sketch. Consider an auxiliary version of the household problem, with no borrowing constraint, income, or mortality risk. Then guessing and checking a value function proportional to $w^{1-\gamma}$ yields the solution to the corresponding HJB equation. In this problem, Equations (B.8) to (B.11) can be shown to hold with equality.

Using a limiting argument, in which borrowing constraint and wealth are gradually introduced, we can see that the solution to the full model converges to that of the auxiliary problem as wealth gets large.

The intuition is that when wealth is large enough, the distortions to the consumption–savings problem are relatively so small as to be negligible compared to return on wealth. Hence the solution to the real HJB equation is very close to that of the auxiliary problem outlined above. □

If policy functions are linear for large w, this means that the distribution of wealth behaves like a reflected geometric Brownian motion in the tail. Standard results (Champernowne 1953; Gabaix 2009) show that the stationary distribution of such a process is asymptotically Pareto in the tail, and let us back out the tail exponent ζ.

B.2. **Wealth distribution.** The evolution of the joint distribution of wealth and productivity for a given cohort is pinned down by the *Kolmogorov Forward* (also known as *Fokker–Planck*) equation.

Theorem 1 (Kolmogorov, Fokker, Planck)**.** *Let w_t be a random process obeying the SDE*

$$\mathrm{d}w_t = \mu(w_t, t)\,\mathrm{d}t + \sigma(w_t, t)\,\mathrm{d}B_t$$

for a Brownian motion B_t. If $f(w,t)$ denotes the density function of w_t then f obeys the Kolmogorov forward equation

$$\partial_t f(w,t) = -\partial_w(f(w,t)\mu(w,t)) + \partial_{ww}\left(\frac{\sigma^2(w,t)}{2}f(w,t)\right). \tag{B.12}$$

See Karatzas and Shreve (1991) for a proof.

We can define a differential operator $\mathcal{L}$ as the right-hand side of equation (B.12),

$$\mathcal{L}f(w,t) = -\partial_w(f(w,t)\mu(w,t)) + \partial_{ww}\left(\frac{\sigma^2(w,t)}{2}f(w,t)\right).$$

It turns out that $\mathcal{L}$ is the adjoint operator[28] to the *infinitesimal generator* $\mathcal{A}$ of the process w. The infinitesimal generator is defined by

$$\mathcal{A}f(w,t) = \mu(w,t)\partial_w f + \frac{\sigma^2(w,t)}{2}\partial_{ww}f, \tag{B.13}$$

and is an important differential operator in stochastic analysis which shows up often. Indeed, a more general formulation of the Kolmogorov Forward equation states that for any stochastic process w_t with generator $\mathcal{A}$, the density function $g(w,t)$ satisfies

$$\partial_t g = \mathcal{A}^* g.$$

The generator $\mathcal{A}$ appears on the right hand side of equation (HJB). This deep symmetry linking the HJB and KF equations, where their underlying operators are adjoint, turns out to have wide-reaching consequences. Following Achdou et al. (2017), we can exploit this relationship to obtain a numerical solution to the KF equation for free after solving the HJB equation. See footnote 32 of Appendix C.1 for details.

Applying Theorem 1 to the dynamics of wealth given in equation (2.11) gives a KF equation for the wealth of an individual conditional on survival. By subtracting off the mass $\lambda(a)g(w;a)$ from the right hand side (corresponding to the probability mass of dying agents), we obtain equation (KF) for the density of wealth held at each age for a given cohort.

Existence and uniqueness of equilibrium. Existence and uniqueness results for mean field games, such as the one in this paper, have not yet been fully established, but promising results are forthcoming. Existence and uniqueness has been demonstrated for stationary solutions. For time-dependent solutions, however, we only have existence results so far. Please see Achdou et al. (2017) and Lasry and Lions (2007, 2006) for further discussion.

Anecdotally, the algorithm always converges to the same thing from different initial guesses, which is encouraging. Indeed, numerical experiments indicate that excess savings are a monotonic function of the interest rate.

If we consider the subproblem of solving the HJB and KF equations given interest rates, there exist many existence and uniqueness results for the solution of these PDEs. See, for example Bardi and Capuzzo-Dolcetta (2008). Note that the correct solution concept, for which existence and uniqueness is guaranteed, is the *viscosity solution* of the HJB equation and the *measure-valued solution* of the KF equation.

B.3. **Computation of human wealth and pension wealth.** The appropriate measure of human wealth with idiosyncratic income dynamics is,

$$h(t,w,z) = \mathbb{E}_t\left[\int_t^\infty (\Theta(t',w_{t'},z_{t'}) + y_{t'}z_{t'})e^{-\int_{t'}^{\xi}(r_s^b+\lambda_s)\,\mathrm{d}s}\,\mathrm{d}t' \mid w_t = w, z_t = z\right]. \tag{B.14}$$

28. These are adjoints in the L^2 sense, meaning that $\int(\mathcal{L}f)\cdot g = \int f\cdot(\mathcal{A}g)$ for any functions f and g.

Given the dynamics of z_t, we can work out h_t as follows. Define

$$H(t,w,z;\bar{t}) = \mathbb{E}_t\left[\int_t^{\bar{t}} (\Theta(t',w_{t'},z_{t'}) + y_{t'}z_{t'})e^{-\int_{t'}^{\xi}(r_s^b+\lambda_s)\,\mathrm{d}s}\,\mathrm{d}t' \mid w_t = w, z_t = z\right],$$

the expected discounted income from age t until some horizon $\bar{t}$, so that h_t is the limiting value. Then H satisfies the Feynman–Kac equation[29]

$$\begin{aligned} 0 &= \partial_t H(t,w,z;\bar{t}) + \mathcal{A}H(t,w,z;\bar{t}) + (\Theta(t,w,z) + y_t z) - (r_t + \lambda_t)H(t,w,z;\bar{t}), \\ 0 &= H(\bar{t},w,z;\bar{t}), \end{aligned} \tag{B.15}$$

where $\mathcal{A}$ is the infinitesimal generator for the joint process (w_t, z_t).

We can solve this equation numerically, backwards in time from the terminal condition. In practice, split H into a component of human wealth at infinity (i.e. after the maximal age in the grid, after which the problem is stationary), and a lifecycle component. Solving the stationary analog of equation (B.15) yields the component at infinity. Paste this function as the terminal boundary condition at the maximal age, and solve backwards to get the lifecycle component.

B.3.1. *Pension wealth.* Denote pension wealth PW as $\tilde{h}$ here. It can be computed in the same way, by varying the payoff in equation (B.14) to include only social security transfers,

$$\tilde{h}(t,w,z) = \mathbb{E}_t\left[\int_t^{\infty} (\Theta(t',w_{t'},z_{t'})1_{t'\geq 65})e^{-\int_{t'}^{\xi}(r_s^b+\lambda_s)\,\mathrm{d}s}\,\mathrm{d}t' \mid w_t = w, z_t = z\right]. \tag{B.16}$$

Defining $\tilde{H}$ analogously, the appropriate Feynman–Kac condition becomes

$$0 = \partial_t \tilde{H}(t,w,z;\bar{t}) + \mathcal{A}\tilde{H}(t,w,z;\bar{t}) + (\Theta(t,w,z)1_{t\geq 65}) - (r_t + \lambda_t)\tilde{H}(t,w,z;\bar{t}),$$

and the computational algorithm goes through as before.

Appendix C. Numerical methods

C.1. **Agent's problem.** Recall the agent's problem in Section 2.1 and the associated discussion of the PDEs and boundary conditions in Appendix B.1. In this section I outline the numerical method used for solving these equations.

HJB equation. Due to lifecycle, we need to solve the full time-dependent version of the HJB equation. Although the age domain is $[0,\infty)$, for the numerical method I need to impose an upper bound T on agents' ages. The best way to implement this is to choose a large value of T and assume that from then on, agents solve a fully stationary problem where none of the parameters depend on age. In other words, after age T the agent's policy function no longer change and there are no further lifecycle effects.

I refer to this problem as the agent's problem at infinity. The idea is to begin by solving forward for the value function at infinity. I then impose this as the value function of the agent at age T, and solve the lifecycle problem backwards from there.

Following Achdou et al. (2017), I solve the HJB equation numerically on a grid using an upwind finite-difference method. This is a scheme in which derivatives are replaced with numerical approximations at each grid point, and the algorithm determines implicitly

29. See, for example, Øksendal (2003)

when to use forward or backwards differences for the derivatives. See Candler (1998) and the appendices to Achdou et al. (2017) for more details. However, while these papers use a semi-implicit numerical method, I differ from them by using a fully implicit scheme, as I shall explain below. This family of finite difference methods (whether explicit, semi-implicit or fully implicit) was shown by Barles and Souganidis (1991) to converge to the unique viscosity solution of the HJB equation.[30]

I follow the approach from the numerical appendix to Achdou et al. (2017). Suppose that $i = 1, \ldots, I$ indexes the wealth grid $\underline{w} = w_1 < w_2 < \cdots < w_I = \bar{w}$, and that $n = 1, \ldots, N$ indexes age on the grid $0 = t_1 < t_2 < \cdots < t_N = T$.

Recall the HJB equation equation (HJB):

$$\begin{aligned}(\rho + \lambda_a)V(w, a) = \max_{c,k,z} u(c) &+ \lambda_a \tilde{u}(z) \\ &+ \partial_w V(w, a)\big[y_a + r_a w + (\alpha - r_a)k - c - \lambda_a(z - w_a)\big] \\ &+ \partial_{ww} V(w, a)\frac{\sigma^2}{2}k^2 + \partial_a V(w, a),\end{aligned}$$

By discretizing it onto the grids, we get the following equation for the discretized value V_i^n of wealth w_i at age t_i:

$$\begin{aligned}(\rho + \lambda^n)V_i^n = &\, u(c_i^n) + \lambda^n \tilde{u}(z_i^n) \\ &+ \partial_w V_i^n\big[y^n + r^n w_i + (\alpha - r^n)k_i^n - c_i^n - \lambda^n(z_i^n - w_i)\big] \\ &+ \partial_{ww} V_i^n \frac{\sigma^2}{2}(k_i^n)^2 + \frac{V_i^{n+1} - V_i^n}{\Delta t}\end{aligned} \tag{C.1}$$

30. The viscosity solution is a generalized kind of solution for partial differential equations. HJB equations do not always have a smooth solution – instead it is necessary to consider functions with kinks; a feature which viscosity solutions allow for. HJB equations have been proven under rather general conditions to have a unique viscosity solution, making this the appropriate solution concept for these problems. See for example Bardi and Capuzzo-Dolcetta (2008) for details.

where $\partial_w V_i^n$ is the upwind difference approximation[31] in w and $\partial_{ww} V_i^n$ is a second forward difference. The choice variables c_i^n, z_i^n, k_i^n are determined from the numerical derivatives of V using the first order conditions (Proposition 1). For example, we have

$$c_i^n = (u')^{-1}(\partial_w V_i^n).$$

We will be solving the HJB equation recursively, backwards in time (age). This means that given the vector V^{n+1} we want to solve equation (C.1) for V^n. Note, however, that this is a highly nonlinear system in V^n, as all the choice variables c_i^n, z_i^n, k_i^n all depend on V^n itself. For this reason, equation (C.1) is called a *fully implicit* finite difference scheme.

In practice, if policy functions do not change too much from one period to the next, it is simpler to approximate equation (C.1) using the equation

$$\begin{aligned}(\rho + \lambda^n) V_i^n = &\, u(c_i^{n+1}) + \lambda^n \tilde{u}(z_i^{n+1}) \\ &+ \partial_w V_i^n \big[y^n + r^n w_i + (\alpha - r^n) k_i^{n+1} - c_i^{n+1} - \lambda^n (z_i^{n+1} - w_i) \big] \\ &+ \partial_{ww} V_i^n \frac{\sigma^2}{2} (k_i^{n+1})^2 + \frac{V_i^{n+1} - V_i^n}{\Delta t}.\end{aligned} \tag{C.2}$$

Here we use policy functions derived from *tomorrow's* (known) value function instead of today's. This kind of scheme is referred to as *semi-implicit*, and this is the method outlined by Achdou et al. (2017). It has the advantage that, given V^{n+1}, today's value V_i^n is the solution of a linear problem. Further, by substituting in the definition of the upwind derivatives into the equation, and the problem reduces to a sparse matrix inversion problem, which can be solved very efficiently. This comes from the fact that difference terms depend only neighboring points on the wealth grid, $V_{i\pm1}^n$.

The limitation of this approach however is that it uses c^{n+1} when determining the value function at time t_n. In a relatively stationary problem this is a mild approximation, however in the lifecycle model there may be a big difference between consumption policies today and tomorrow, especially near the borrowing constraint, and especially if the constraint is moving over time.

31. The upwind approximation equals the forward difference $\partial_w^F V_i^n = (V_{i+1}^n - V_{i+1}^n)/\Delta w$ when the drift on wealth (i.e. optimal savings) is positive and the backward difference $\partial_w^B V_i^n = (V_i^n - V_{i-1}^n)/\Delta w$ when savings are negative.

More precisely, the savings policy function is computed from the approximated policies according to

$$s_i^n = y^n + r^n w_i + (\alpha - r^n) k_i^n - c_i^n - \lambda^n (z_i^n - w_i).$$

We can define the forward approximation for the consumption policy as $(c_i^n)^F = (u')^{-1}(\partial_w^F V_i^n)$ as well as the backwards approximation $(c_i^n)^B$, using the same difference approximation of the value function. By doing the same thing for the other choice variables, we can compute the forward and backwards savings rates $(s_i^n)^F$ and $(s_i^n)^B$. The upwind approximation to V is defined as

$$\partial_w V_i^n = \begin{cases} \partial_w^F V_i^n & \text{when } (s_i^n)^F > 0; \\ \partial_w^B V_i^n & \text{when } (s_i^n)^B < 0; \\ \overline{\partial_w} V_i^n & \text{otherwise,} \end{cases}$$

where $\overline{\partial_w} V_i^n$ is set to force savings equal to zero. (If there were only one choice variable, consumption, then it would be set to the marginal utility of income, thus forcing the agent to consume all income.)

Candler (1998) explains how the upwind method is necessary for the finite difference scheme to converge.

For example, if income is zero after a certain fixed retirement age t_r then the natural wealth constraint from that age onward is zero, and the value of negative wealth in retirement is negative infinity. Hence the consumption policy function for negative wealth at age t_r is not defined.

However, one time step before t_r, the agent will receive an income and is therefore able to borrow against it and obtain slightly negative wealth while remaining solvent. Hence an agent at time $t_r - \Delta t$ with slightly negative wealth will be able to consume positively. Consumption at time t_r is therefore very different from the policy function at time $t_r - \Delta t$.

It turns out that for this lifecycle model, the semi-implicit method generally does not work. Instead, we need to solve the fully implicit nonlinear system, equation (C.1), for which the sparse matrix methods do not directly apply.

We solve this nonlinear problem numerically using a Newton methodto find V^n. It turns out that the Newton method can be solved by starting from some initial guess at each time-step, and using a semi-implicit method to solve forward.

Because this involves nested loops, in which we iterate at each time-step, notation quickly becomes unwieldy so I proceed with a simplified example. Suppose the fully-implicit discretized HJB equation we wish to solve is

$$\rho V_i^n = u(c_i^n) + \partial_w V_i^n s_i^n + \frac{V_i^{n+1} - V_i^n}{\Delta t}.$$

As discussed earlier, this is a nonlinear equation in V^n, where the savings, consumptions and the upwind derivative all depend nonlinearly on forward and backward differences of V^n.

Let us change notation, and denote by V^* value tomorrow (previously V^{n+1}), which is known since we are solving backwards, and simply denote by V the value for today which we are trying to find. Then

$$\rho V_i = u(c_i) + \partial_w V_i^n s_i^n + \frac{V_i^* - V_i}{\Delta t}. \tag{C.3}$$

The Newton method solves this equation for V by iterating forward over m from some initial guess V^0, using the update step

$$\rho V_i^{m+1} = u(c_i^m) + (V^{m+1})'_u s_i^m + \frac{V_i^* - V_i^{m+1}}{\Delta t}. \tag{C.4}$$

It is important to note that the last term is an approximation of the time derivative using tomorrow's value V^* and *not* the next guess of today's value V^{m+1}. When this scheme converges and $V^m = V^{m+1}$, we will have obtained a function that satisfies the fully implicit equation (C.3).

Equation (C.4) can be identified as the update step in a semi-implicit upwind finite difference method, and these can be solved efficiently, following Achdou et al. (2017). If we denote by $V^m = (V_i^m)_i$ the I-dimensional vectorization of the value function, we can rewrite equation (C.4) as the I-dimensional equation

$$\rho V^{m+1} = u(c^m) + \mathbf{A}_m V^{m+1} + \frac{V_i^* - V_i^{m+1}}{\Delta t} \tag{C.5}$$

for an appropriate matrix $\mathbf{A}_m$. The problem has a recursive structure in which V_i^{n+1} depends only on $V_{i-1}^n, V_i^n, V_{i+1}^n$. This comes from the continuous formulation of the problem, which ensures that in the discretization of the problem we need only look one grid point in any direction. This recursive structure ensures that $\mathbf{A}_m$ is a sparse matrix, so equation (C.5) can be solved efficiently using sparse matrix inversion methods, such as `spsolve` in MATLAB or `scipy.sparse.linalg.spsolve` in Python.

When the scheme converges at iteration M, the matrix $\mathbf{A} = \mathbf{A}_M$ can be interpreted as the transition matrix for the value function, from t^n to t^{n+1}. This provides an interpretation of the solution scheme as reducing the HJB equation to the form of a standard discrete time Bellman equation associated to a discrete optimal control problem.[32]

To sum up, in the fully implicit method, we iterate the HJB equation backwards over time. At each time-step, we solve for today's value given tomorrow's using a Newton method that amounts to running a semi-implicit method *forward* in equation (C.4). In practice, this Newton method is relatively fast, only requiring three or so iterations.

Boundary conditions. At the boundary of the wealth grid, certain numerical differences do not exist. For example, the backward difference $\partial_w^B V_i^n = (V_i^n - V_{i-1}^n)/\Delta w$ is not defined when $i = 0$. Instead, for those points, we define the missing derivatives by applying the state-constraint boundary conditions from Proposition 7 with equality. It turns out that the upwind derivative always picks the correct difference for this to work (see footnote 31).

Special care needs to be taken when the state-constraints are moving over time, which can happen with the natural borrowing constraint for example. Normally, the state-constraint ensures that consumption is low enough for savings to be zero (or barely positive) at the boundary. However, if the boundary condition at the next time step is tighter, this may not be enough: by saving zero the agent may suddenly find himself in violation of the borrowing constraint. Instead, the borrowing constraint needs to force the agent to save a strictly positive amount, enough to take him from the current borrowing constraint to that of next period.

Hence, if $\underline{w}^n$ denotes today's wealth constraint and $\underline{w}^{n+1} > \underline{w}^n$ denotes the next one, a special boundary condition needs to be applied at each grid point between $\underline{w}^n$ and $\underline{w}^{n+1}$ (inclusive). The boundary condition ensures that savings are high enough for next period's expected wealth to equal $\underline{w}^{n+1}$.

Upper wealth constraint. Implicit in the idea of solving this system by discretizing it onto a grid is the requirement to bound the state variables. In particular, this means we need an upper limit $\bar{w}$ on wealth. In order to do so in a model-consistent way, we impose $\bar{w}$ as a reflecting boundary on the wealth process w_a, just like the lower boundary, which forces savings to be nonpositive at that point. Note that this boundary condition does

32. In fact, standard results from control theory write the HJB equation in the form

$$\rho V = u(c) + \mathcal{A}V,$$

where $\mathcal{A}$ is a special differential operator associated with the control problem called the *infinitesimal generator*. The operator $\mathcal{A}$ is the continuous time analog of the transition matrix for a discrete system, mapping a value function to its continuation value. The matrices $\mathbf{A}_m$ approximate $\mathcal{A}$. There is a deep connection between the HJB and KF equations through $\mathcal{A}$ which I discuss in the appendix.

not bind and will not be selected by the upwind method, and so should not affect the model in an economically significant way.

Finally, a second order boundary condition needs to be imposed at $\bar{w}$, in order to pin down capital holdings at those points. (Note that we did not need to do this at $\underline{w}$ because the conditions $k \geq 0$ and $k \leq w - \underline{w}$ ensure that $k = 0$ at the lower wealth boundary.) We follow Achdou et al. (2017) in imposing

$$\partial_{ww} V(t, \bar{w}) = -\gamma \frac{\partial_w V(t, \bar{w})}{\bar{w}}, \tag{C.6}$$

obtained from the asymptotics of the value function implied by the asymptotic policies in Lemma 1.

KF equation. As outlined by Achdou et al. (2017), we solve equation (KF) using a finite difference method. It turns out that if one chooses the approximations to the derivatives judiciously, we can exploit the adjointness of the HJB and KF equations and solve the KF equation for free (see Footnote 32 – for more details, please see Achdou et al. (2017, online appendix)):

$$\frac{g^{n+1} - g^n}{\Delta t} = (A_n^T - \lambda_n I) g^{n+1}. \tag{C.7}$$

Here $\mathbf{A}_n^T$ is the transpose of the transition matrix obtained on the convergence of the Newton method for the HJB equation at time point t_n, and $\lambda_n I$ takes into account the death rate at t_n.

This implicit method pins down g^{n+1} from g^n, and we can solve it efficiently using sparse matrix methods.

This defines a forward-looking solution scheme for the density of wealth at each age. It is solved starting from a density g^0 which is the initial distribution of wealth (i.e. inheritance).

Stationary equilibrium. To sum up, the algorithm to solve for a stationary equilibrium is as follows.

(1) Make a guess of the stationary interest rate r
(2) Solve the HJB equation backwards given r using the fully implicit method to obtain agents policies
(3) Solve the KF equation forward to obtain the wealth distribution
(4) Using the two steps above work out the aggregate excess demand for bonds given r
(5) Increase guess of r if there is excess supply and decrease it if there is excess demand
(6) Repeat until the bond market clears.

C.2. **Transition equilibrium.** The algorithm to solve for a transition equilibrium in this model requires an extra layer compared to transition in the Achdou et al. (2017) model: every point in time contains agents all ages. Due to lifecycle effects, at time τ we need to solve the agent's problem, a time-dependent system, for each cohort with date of birth $s \leq \tau$. By solving for the decision functions at each age for each cohort, we solve the corresponding KF equations to get the density functions of wealth at each age for each cohort. By selection the density functions for agents of cohort s with age $\tau - s$, we can piece together the wealth distribution at each time τ.

When agents learn of an unexpected shock, they change their policy functions from the initial steady state. This is captured by a change in transition matrix $\mathbf{A}(s, \tau)$. This way, the KF equations change, capturing the shock.

Transition algorithm. Denote transition calendar time by τ. We consider transition experiments from time $\tau = 0$ to some maximal time $\tau = \bar{\tau}$.

For each $\tau \in [0, \bar{\tau}]$ let

$$X_\tau = (D_\tau, \rho_\tau, \sigma_\tau, \alpha_\tau, vol_\tau, \ldots, (\phi^s_\tau)_{s\leq\tau}, (\lambda^s_\tau)_{s\leq\tau}, (\chi^s_\tau)_{s\leq\tau}, (y^s_\tau)_{s\leq\tau}, (BC^s_\tau)_{s\leq\tau})$$

be the tuple consisting of all model parameters (demographic and economic) at time τ, for every cohort born before time τ. We assume stationarity before 0 and after $\bar{\tau}$, so that $X_{-\tau} = X_0$ and $X_{\bar{\tau}+\tau} = X_{\bar{\tau}}$ for all $\tau > 0$.

The following sketches the algorithm used to solve for transition equilibria.

(1) Discretize the transition time interval $[0, \bar{\tau}]$ onto a grid $\boldsymbol{\tau} = \tau_0, \ldots, \tau_N$.

(2) Guess a time path for the interest rates $(r_\tau)_{\tau\in\boldsymbol{\tau}}$

(3) At any time $\tau \in \boldsymbol{\tau}$ there will be agents of every age. We need to know how they make decisions. Hence at each time $\tau \in \boldsymbol{\tau}$, we need to solve the HJB equation for *each cohort* $s \leq \tau$. Assuming an upper bound on lifetimes T, this means that we need to solve it for finitely many cohorts: each date of birth $s \in [\tau - T, \tau] \cap \boldsymbol{\tau}$.

    For each cohort s, work out the age-dependent paths (i.e. lifecycle profiles) of each parameter and endogenous variable for every cohort of agents born at s. These are encoded into $(\tilde{X}^s_a)_{a\geq 0} = (X_{s+a})_{a\geq 0}$. Use the lifecycle paths $(\tilde{X}^s_a)_{a\geq 0}$ for cohort s to solve the HJB equation for each cohort. These are solved using the fully implicit Newton method to solve the lifecycle problem as described earlier.

(4) The next step is to calculate wealth distributions, but we need to be careful when doing this as agents' policy functions change when the shock is announced. At first, each cohort's age-dependent wealth distribution will evolve using the transition matrix for the initial steady state. When the shock occurs, agents' decisions change, so the densities now evolve with a new transition matrix.

    Recall the numerical scheme for obtaining the age- and cohort-specific wealth distributions outlined in Appendix C.1:

$$\frac{g^{n+1} - g^n}{\Delta t} = (A^T_n - \lambda_n I)g^{n+1},$$

    where $\mathbf{A}^T_n$ is the transpose of the transition matrix for the value function obtained in solving the HJB equation with the Newton method. Update the age-dependent wealth distribution using the appropriate transition matrix for each cohort.

(5) With the age-dependent wealth distribution for each cohort $g(-; s, a)$, we can use the age structure $D(a, \tau)$ to obtain the unconditional distribution as usual.

    At each time step τ, we use the policy functions for each cohort together with the age-dependent wealth distribution to work out aggregate wealth and capital holdings for each cohort. By aggregating up using the demographic distribution, we obtain unconditional aggregate wealth W_τ and capital holdings K_τ. Excess demand for bonds at each point in the transition is given by

$$S_\tau = W_\tau - K_\tau.$$

(6) Given the excess demand for bonds S_τ at each time, update the guess of $(r_\tau)_\tau$ to

$$r - \xi \cdot S$$

for some vector of weights ξ with positive entries, a form of Walrasian tâtonnement. These weights can be determined through trial and error, or endogenously using a quasi-Newton method (e.g. the secant method.)

(7) Go back to step 2 and repeat until $(r_\tau)_\tau$ converges and $S_\tau \approx 0$ for all t.

Although this algorithm contains 4 large nested loops, at the inside of which one solves a big matrix inversion problem, the process usually converges in 5-10 minutes on a good computer for reasonable grid sizes.

C.3. **Stability and convergence of numerical scheme.** As discussed in Appendix B.2, the theory has not fully caught up with techniques in this field yet: although there exists a proof of existence of transition solution of the system, uniqueness results are still being sought.

When it comes to the numerical methods, I rely on Barles and Souganidis (1991). They show that the finite difference schemes of the kind we employ to solve the HJB equation are guaranteed to converge to the unique (viscosity) solutions of these equations under mild conditions.