Selling Multiple Complements with Packaging Costs*

Simon Finster†

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Abstract

I consider a package assignment problem where multiple units of indivisible objects are allocated to individuals. The seller can specify additional costs or cost savings on certain packages of objects: e.g., a manufacturer may incur cost savings if they obtain a range of products or services from a single supplier. The objective is to find a socially efficient allocation among buyers. I propose a sealed-bid auction with a novel cost function graph to express the seller's preferences. The graph structure facilitates the use of linear programming to find anonymous, competitive, and package-linear prices. If agents act as price takers, these prices support a Walrasian equilibrium, and I provide additional conditions under which an equilibrium always exists. The auction design guarantees fairness and transparency in pricing, and it admits preferences of the seller or auctioneer over the type and degree of concentration in the market.

Keywords: package assignment, non-linear pricing, Walrasian equilibrium, linear programming, non-additive costs
JEL codes: D44, D47

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†Department of Economics and Nuffield College, University of Oxford, simon.finster@nuffield.ox.ac.uk.

1 Introduction

In many multi-object auctions, it is standard that buyers can submit bids on packages of objects. Submitting preferences over allocations in the market, on the other hand, is typically not allowed, even though cost savings from specific allocations may arise naturally. In procurement auctions, e.g., a manufacturer may want to favour package allocations: by obtaining multiple products or services from a single supplier they can save transaction costs. In the literature on Walrasian equilibrium to date, preferences cannot accommodate such costs either.

I consider a competitive market setting for multiple, indivisible goods, in which the seller's preferences may depend on the *partitioning of supply* among buyers. The objective is to find a socially efficient allocation. I model the seller's preferences over allocations as *packaging costs*, where the key ingredient is a novel graph structure to express marginal cost functions. This *cost function graph* allows me to obtain anonymous, competitive equilibrium prices, which are linear in packages[1] and reflect the structure of marginal costs. The anonymity, package-linearity, and the structure itself guarantee fair and transparent pricing. The seller's marginal costs can reflect cost savings (or additional costs) arising from bundling certain objects together; at the same time, there is flexibility to sell individual objects separately if demand requires it. On the buyers' side, I also allow for a rich set of preferences admitting complementarities as well as some substitutabilities between objects. I characterise competitive equilibria and derive necessary and sufficient conditions for the existence of a competitive (Walrasian) equilibrium.

In my model, a seller ("she") supplies multiple indivisible units of multiple varieties. Several bidders ("he") want to buy multiple units of each variety, and each bidder has a preference over combinations of these varieties. Similar settings, with the goal of characterising competitive equilibria, have been studied by [5] and [6]. However, they consider an exchange economy and a seller whose reservation utility is normalised to zero, respectively. Their model can be straightforwardly extended to reservation values which are additive between objects, but these do not admit preferences over the partitioning of supply among buyers. For this partitioning problem I develop a tractable version of non-additive reservation values, allowing me to provide insights into Walrasian equilibrium.

In the real world, non-additive reservation values or marginal costs are present whenever the bundling of objects results in cost savings or additional expenses for the seller; I call these *packaging costs*. They can take the form of transaction costs (delivery, drawing up a contract), cost savings from realised synergies in the allocation, or even subsidies.

Many examples for realised synergies can be found in procurement. Consider a company holding a procurement auction for two input factors, which could be two different legal services for the same department. One legal team providing both services is likely more effective than two different legal teams, so the company would prefer to obtain the two services from the same law firm. At the same time, it wants to maintain flexibility to be supplied by two different firms if this is significantly cheaper. The two products may also be machines with a servicing contract. One supplier could provide maintenance and employee training for the machines more efficiently

[1] Under package-linear pricing, the same price applies to identical packages, and the price of a collection of several packages equals the sum of prices of the packages contained in the collection. A package-linear pricing function is non-linear in items, i.e. the price of a bundle may be different to the sum of prices of the items contained.

than two different suppliers. The literature in operations research has already taken an interest in related procurement issues. E.g., [4] propose a bidding language that accommodates various types of discounts a procurement manager may want to offer, although not in the context of Walrasian equilibrium. Some decision support systems used in practice[2] also allow for different types of discounts and sophisticated bids ([4], [17], [16]).

Subsidies are especially relevant in auctions held by the government. Consider the allocation of land plots for farming: a government with expert knowledge of land productivity may want to subsidise the allocation of complementary plots in order to achieve higher land productivity.[3] In auctions for biodiversity conservation contracts[4] certain allocations may also be favoured because different measures are more effective if implemented on the same piece of land. The applications extend to transportation and telecommunication licensing, and many more.

While my main results characterise Walrasian equilibria and their existence, my work also contributes to the literature on bidding languages, specifically the OR-of-XOR language (e.g., [24] and [22]).[5] A version of this language was also independently developed in the Product-Mix auction (PMA) by [19, 20, 21], which has been in use by the Bank of England following the financial crisis in 2007 until today. In my model, each buyer has a valuation that is a relaxation of "OR-of-XOR" valuations. Such valuations can be represented in a finite list of "XOR-bids". I identify conditions under which a buyer can use this representation of preferences in the auction, under the relaxation that they may *repackage* their allocated collection of packages in any way desired. The possibility of repackaging a collection of bundles of items in the most profitable way seems natural in many settings.

Work by [2] implies that OR-of-XOR preferences without allowing repackaging satisfy the strong substitutes (between packages) property,[6,7] and it is well known that Walrasian equilibrium exists if all buyers have strong substitutes valuations and the supply bundle is fixed ([12], [2]). Consequently, a package-linear pricing Walrasian equilibrium exists if all buyers have strong substitutes valuations between packages and the supply bundle *of packages* is fixed. However, if the seller partitions a given supply bundle of different *objects* into packages *and* package-linear pricing functions are allowed, her preference cannot satisfy the "strong substitutes between packages" property (see Appendix B). The partitioning problem is significantly more difficult precisely because of this subtlety. Nevertheless, I derive a condition for the existence of Walrasian equilibrium, using linear programming techniques in the spirit of [5]. This approach has the additional advantage of providing a computationally tractable framework. Linear programming relaxations can be solved in polynomial time, and even for associated integer programmes duality theory provides avenues for practical computation (see, e.g., [32]).

Moreover, I establish a second result on equilibrium existence that relies only on the agents'

[2][17] interviewed the companies Ariba/Procuri, CombineNet, Emptoris, and Iasta.

[3]Market design approaches to land allocation have been studied, e.g., in [8].

[4]See, e.g., the well-known BushTender auctions in Victoria (Australia) studied by [27] and others.

[5]XOR bids are standard in combinatorial auctions: e.g., the early package auction iBundle [25] (and many others) uses such bids. However, allowing high-dimensional XOR bids increases the computational complexity of the allocation problem. Using OR-of-XOR language when possible (see also Section 3.4) aims to reduce the computational complexity.

[6]See e.g. [23] or [2] for a definition of strong substitutes.

[7]The attribute "between packages" signifies that packages, not objects, are priced individually. Thus, a buyer may substitute one package for another.

value and cost functions. It states that a non-linear pricing Walrasian equilibrium always exists if buyers' values are superadditive, the seller's costs are subadditive and only one unit per variety is for sale. Although this restricted environment is closely related to [29] (henceforth SY), it is again more general in that my seller's preferences distinguish between different partitionings of objects among buyers. In contrast, SY's seller has a reserve price for each bundle; once the reserve price is met, the bundle can be sold in any partition. I show that these reserve prices can be expressed as cost functions. Furthermore, I derive a new connection between superadditive (subadditive) set functions and their dual set functions. This allows me to characterise properties of those cost functions that are equivalent to superadditive (subadditive) reserve prices.

I describe the general model in Section 2, and discuss the sealed-bid auction and the main result in Section 3. In Section 4, I present the second result on equilibrium existence and my results on the relationship between cost functions and reserve prices. Section 5 is a conclusion. All proofs are deferred to the appendix.

1.1 Walrasian equilibrium and packaging cost: an example

Consider a seller (e.g., the government) who has two distinct and indivisible land plots for sale, and there are two buyers in the market, Kate and Leon. Plot A is particularly suited to growing crop A(lfalfa), and plot B is suited to crop B(eans). Kate wants to buy either plot, but not both, and values plot A at 40 and plot B at 50. Leon, on the other hand, wants to buy both plots, but is not interested in a single plot. He values the combination of both plots AB at 70. All agents have transferable utility. The seller is denoted by index 0, Kate by K, and Leon by L. We are interested in a Walrasian equilibrium, i.e. an allocation $x_S^i \in \{0,1\}, i \in \{K, L, 0\}, S \in \{\emptyset, A, B, AB\}$ and a pricing function $p : \{\emptyset, A, B, AB\} \to \mathbb{R}$ such that buyers demand their allocation (obtain their preferred bundle S) and the seller maximises her profit, at the given prices; and demand equals supply. A pricing function p satisfies by definition $p(\emptyset) = 0$ and it satisfies the standard notion of Walrasian equilibrium if competitive prices $p(A)$, $p(B)$, and $p(AB)$ are such that $p(AB) := p(A) + p(B)$. I call this a linear pricing Walrasian equilibrium. This paper is concerned with a more general version of Walrasian equilibrium, in which $p(AB) \neq p(A) + p(B)$. In the case where a single unit of each of multiple distinct objects is sold, this is called a non-linear pricing Walrasian equilibrium. I will extend this notion to a package-linear pricing Walrasian equilibrium in the main section.

First, assume the seller's marginal cost is zero for the two plots. It is not hard to verify that no linear pricing Walrasian equilibrium exists.[8] Surprisingly, even the more general non-linear pricing Walrasian equilibrium fails: there exists no solution in $p(A), p(B)$, and $p(AB)$.[9] Now I will modify the example. Suppose that Kate does want to buy both plots, and she values the combination AB at 90 (and plot A at 40 and plot B at 50 as before). Suppose Leon also wants to buy plot A alone and values it at 50 (and the bundle AB at 70 as before). The

[8] Suppose $p(A) + p(B) \leq 70$, then Leon demands AB. However, Kate also demands either A or B, unless $p(A) + p(B) > 90$, so markets cannot clear. Suppose $p(A) + p(B) > 70$, then Leon demands nothing. Kate demands either plot A or plot B, but at the given prices, the seller wants to supply both.

[9] Because no linear pricing equilibrium exists, it must be either (i) $p(A) + p(B) < p(AB)$ or (ii) $p(A) + p(B) > p(AB)$. If (i), the seller wants to sell the bundle AB. For Leon to demand AB, it must be that $p(AB) \leq 70$. For Kate to demand neither A nor B, it must be that $p(A) > 40$ and $p(B) > 50$, a contraction. If (ii), the seller prefers to sell A and B separately, but in this case markets do not clear.

agents' valuations v^i for this example are summarised in Table 1. [29] show that a non-linear pricing Walrasian equilibrium always exists, with any number of distinct land plots for sale, if all agents have superadditive values. Superadditivity of values in this example only requires $v^i(AB) \geq v^i(A) + v^i(B)$ for all $i \in K, L, 0$, which is now satisfied.[10] The seller also may have superadditive values for the bundles that she does not sell and retains for her own use instead. For example, the seller (seller 1) may have values $v^0(A) = 10$, $v^0(B) = 20$, and $v^0(AB) = 60$. The seller's profit is defined as the revenue she obtains from sold bundles plus the value she derives from unsold bundles.

	$\emptyset$	A	B	AB
Kate	0	40	50	90
Leon	0	50	0	70
Seller 1 (values)	0	10	20	60
Seller 2 (marg. cost)	0	40	50	60

Table 1: Agents' values and marginal costs over objects

However, such values *cannot* accommodate preferences over the partitioning among agents. In a different model, assume the seller (seller 2) prefers the two plots to be allocated to a single farmer, because, growing alfalfa and beans together, the enterprise may be more likely to thrive and yield higher long-term productivity.[11] The seller may not care if Kate or Leon obtains the bundle of AB as long as one farmer obtains both plots, i.e. the seller is indifferent about the buyer's identity; or she may be prohibited from price discrimination by law. Suppose the seller has a marginal cost function where $c^0(A) = 40$, $c^0(B) = 50$, and $c^0(AB) = 60$. Her profits are defined as revenue minus marginal cost of any bundle sold. Notice that this marginal cost function is not defined arbitrarily. It is the set function dual of v^0: if the value of retaining both plots A and B is 60, and the value of retaining only B is 20, the cost of selling A is 40. I prove that there exists a transformation between a seller with values for retained items, and a seller who maximises revenue minus marginal cost (Proposition 4). The properties of value function and marginal cost function also also interlinked (Lemma 5), and this linkage further illustrates that values for retained items can never accommodate preferences over the partitioning of supply. The

In a non-linear pricing Walrasian equilibrium, and a seller with the value function v^0 for retained land plots, plot A is sold to Leon and plot B to Kate. On the contrary, equipping the seller with the marginal cost function c, the plot AB is allocated to Kate. This is precisely because in the presence of a seller with marginal cost function c, cost savings of 30 (40 + 50 - 60) are realised when allocating the bundle to a single buyer instead of allocating plot A and B separately. As is standard, a non-linear pricing Walrasian equilibrium implements an efficient allocation, and I prove this extends to the notion of package-linear pricing Walrasian equilibrium (Lemma 1). My Theorem 2 and Corollary 4 guarantee that a non-linear pricing Walrasian equilibrium always exists, with any number of distinct land plots for sale, if all buyers

[10] Note that a linear pricing Walrasian equilibrium also exists in this example, supported by $p(A) = p(B) = 50$. In general, this is not the case as shown in Example 1 in [29].

[11] A farm growing different crops may be less susceptible to weather extremes or demand fluctuations, and many other synergies may arise.

have superadditive values and the seller has subadditive marginal costs over the partitioning of supply,[12] and it can be determined through an ascending auction. Note also that the seller's preference with values for retained items is similar to buyers' preferences. Indeed, I describe how this type of seller may disguise as a buyer in the market, and interact with another seller whose preferences are given by marginal cost functions over the partitioning of supply (Proposition 5).

The results illustrated so far provide a baseline for Walrasian equilibrium with packaging costs and connect it to the previous literature in which reserve prices were merely additive. In reality, however, a model of pure complements may still be unsatisfactory. Indeed, most of the time multiple units of distinct varieties need to be allocated, e.g. two land plots of type A, two land plots of type B, and three land plots of type C. Distinct or identical land plots may be complements for some buyers but substitutes for others. The seller may also have richer preferences over the partitioning, e.g., she may prefer allocations in which plots of type A and B are allocated to a single buyer, but plots of type B and C are each allocated to different buyers. She may prefer a more equal distributions of land among buyers, or she may prefer allocations where plots of the same type are allocated to a single buyer. Even more, she may not only favour a specific *type* of concentration (or equal distribution), she may also favour a certain *degree* of concentration; the marginal cost of allocating a plot of a given type may be also be increasing.

All these preferences, and many more dependencies between marginal costs of any bundles of land plots, are allowed in my general model; importantly, the preferences are independent of any buyer's identity. Precisely this requirement allows me to characterise a package-linear pricing Walrasian equilibrium with *anonymous* prices (Proposition 3 and Corollary 1), and I provide a condition for the existence of such equilibrium (Theorem 1).

2 A model of a competitive market with packaging costs

2.1 Preliminaries

I use basic multiset theory in my model and formally define multisets, feasible multisets, and basic operations on multisets.[13]

Definition 1 (Multiset)**.** Given an underlying finite set $A = \{a_1, \ldots, a_n\}$, a multiset is defined as a mapping $m : A \to \mathbb{N}_0$, and written as $\mathbf{m} := (m(a_1), \ldots, m(a_n))$.

Each element $a \in A$ is distinguishable, but occurrences of the same element are indistinguishable. $m(a)$ denotes the multiplicity of a, i.e. there are $m(a)$ occurrences of a in the multiset $\mathbf{m}$. For ease of notation, I denote $m(a)$ as m_a. There are n indivisible, distinguishable varieties in the economy, denoted $j \in N := \{1, \ldots, n\}$, with a maximum supply of Ω_j units per variety. I formulate my model such that packages of complementary objects (on the buyers' side) and packages associated with the seller's packaging costs only consist of distinct objects. Packages of identical objects will be treated as multisets of packages, or "collections of packages". Making this assumption simplifies the exposition of my multi-unit environment, and it is without loss

[12] Note that introducing the seller's cost function to the example, a linear pricing Walrasian equilibrium does not exist: to implement the efficient allocation $x_{AB}^{Kate} = 1$ we must have $p(A) > 50$ so that Leon demands nothing, but Kate demands AB over B only if $90 - p(A) \geq 50$, a contradiction.

[13] See for example [7] for an extensive treatment of multisets.

of generality: relabelling appropriately I can translate any setting with complementarities or packaging costs between identical objects into mine, and I demonstrate this in Section 3.4.

Definition 2 (Package)**.** A package is defined as a set $S \subseteq N$.

A package contains at most one unit of each different variety, and there exist $2^n - 1$ distinct packages in the economy, not including the empty package. The powerset of N is denoted 2^N. If $|S| = 1$, the package may also be denoted by j, where j is the only variety contained in S. I write a multiset of varieties (with underlying set N) as $\mathbf{m} \in \mathbb{Z}_+^n$, a vector $(m_1, m_2, \dots, m_n)$, where each m_j denotes the multiplicity of variety j. I write a multiset of packages (with underlying set 2^N) as $\mathbf{k} \in \mathbb{Z}_+^{|2^N|-1}$, a vector $(k_{S_1}, k_{S_2}, \dots, k_{S_{2^n-1}})$, where each k_{S_x} denotes the multiplicity of package S_x. Given a set $A \subseteq 2^N$, $\mathbf{k}_{-A}$ denotes the vector $(k_{S_1}, k_{S_2}, \dots, k_{S_{2^n-1}})$, where $k_{S_x} = 0$ for all $S_x \in A$. The feasibility of multisets in the economy is restricted by the maximum supply per variety. I denote the universe of all feasible multisets by $\mathbb{K}$. Formally,

$$\mathbb{K} := \left\{ \mathbf{k} \in \mathbb{Z}_+^{|2^N|-1} : \sum_{S \in 2^N, S \ni j} k_S \leq \Omega_j \quad \forall j \in N \right\}$$

If a multiset $\mathbf{k}$ contains only a single set S, i.e. $k_S = 1, k_{S'}, = 0$ for all $S' \neq S$, I denote $\mathbf{k}$ simply by S. For a function $f : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{R}$, $f(S)$ is short for $f(\mathbf{k}$ with $k_S = 1, k'_S = 0 \; \forall S' \neq S)$. The maximum supply of all units is given by the multiset of varieties $\mathcal{N} := (\Omega_1, ..., \Omega_n)$. I define the following basic operations for multisets:

Definition 3. Let $\mathbf{k} = (k_S)_{S \in 2^N}, \mathbf{k}' = (k'_S)_{S \in 2^N} \in \mathbb{Z}_+^{|2^N|-1}$.

(i) Sum. $\mathbf{k} + \mathbf{k}' = (k_S + k'_S)_{S \in 2^N}$. The sum operator for multisets is denoted $+$.

(ii) Multiplication with a scalar. $\alpha \mathbf{k} = (\alpha k_S)_{S \in 2^N}$

(iii) Unpacking. $\mathbf{k}^* = (m_j)_{j \in N}$ with $m_j = \sum_{S \in 2^N, S \ni j} k_S \quad \forall j \in N$

(iv) Cardinality. $|\mathbf{k}| = \sum_{S \in 2^N} k_S$

The unpacking-operator * breaks a multiset of packages $\mathbf{k}$ down into a multiset of the individual varieties contained in $\mathbf{k}$.

2.2 Agents and preferences

There is one seller ("she") in the economy, denoted by 0, and a set of L buyers (bidders) denoted by $l \in \mathcal{L} := \{1, \dots, L\}$. The set of all agents is denoted by $\mathcal{L}_0 := \mathcal{L} \cup \{0\}$.

2.2.1 Buyers

Each buyer ("he") has a preference over packages of goods, which can be summarised by a value function $V^l : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{Z}_+$ with $V^l(\emptyset) = 0$. I add some structure on this value function such that preferences bear the characteristic of superadditivity between complementary objects, and, at the same time, the substitute characteristic of weakly decreasing marginal values between identical varieties and packages. In the literature, superadditivity is the most general concept of complementarity comprising supermodularity and gross complements ([26],[29]).

The preference structure is "OR-of-XOR" (see, e.g., [24] or [22]) with some additional requirements. Hence, each bidder's valuation can be mapped into a finite list of XOR-bids (and this will be shown formally later). However, I also introduce a new and natural feature into this preference: buyers may repackage an allocated collection of packages in any way they desire. Therefore, and to simplify the exposition of my results, we use a different definition of buyers' values.[14] Given a multiset of packages $\mathbf{k}$, a buyer's aggregate value of $\mathbf{k}$ is derived from an assignment of packages in the multiset to positions $\{1, \dots, |\boldsymbol{k}|\}$ (where not all positions have to be filled), treating each copy as a separate package. The aggregate value of $\mathbf{k}$ will be defined by the value-maximal assignment of the copies of packages contained in $\mathbf{k}$.

Definition 4 (Marginal values)**.** A marginal value function is defined as $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$. $v^l(q, S)$ describes the marginal value bidder l derives from package $S \in 2^N$ if he assigns it the qth position in $\{1, \dots, |\boldsymbol{k}|\}$. Interdependencies between valuations of different bidders are not allowed.

Definition 5 (Marginal value aggregation)**.** For any multiset $\mathbf{k}$, a bidder's overall value is defined as

$$V^l(\mathbf{k}) = \max_{\{x_{q,S}\}} \sum_{S \in 2^N} \sum_{q=1}^{\overline{M}} v^l(q, S) x_{q,S} \quad \text{s.t.} \sum_{q=1}^{\overline{M}} x_{q,S} \leq k_S \ \forall S, \sum_{S \in 2^N} x_{q,S} \leq 1 \ \forall q, x_{q,S} \in \{0,1\} \ \forall S, q$$

where $\overline{M} := \max_{q,S}\{q : v^l(q, S) > 0\}$. $x_{q,S}$ is one if package S is assigned to the qth position in $\{1, \dots, |\boldsymbol{k}|\}$, and zero otherwise.

If a bidder is given a multiset of packages, the bidder may not be able to repackage the varieties contained in the multiset of packages, as in Definition 5, due to physical or contractual constraints. I also consider bidders who *can* repackage the assigned varieties into those packages that are most valuable to them.

Definition 6 (Marginal value aggregation with repackaging)**.** For any multiset $\mathbf{k}$, a bidder's overall value is defined as

$$\widetilde{V}^l(\mathbf{k}) = \max_{\{x_{q,S}\}} \sum_{S \in 2^N} \sum_{q=1}^{\overline{M}} v^l(q, S) x_{q,S} \ \ \text{s.t.} \sum_{S \ni j} \sum_{q=1}^{\overline{M}} x_{q,S} \leq \mathbf{k}^*_j \ \forall j, \sum_{S \in 2^N} x_{q,S} \leq 1 \ \forall q, x_{q,S} \in \{0,1\} \ \forall S, q$$

where $\overline{M} := \max_{q,S}\{q : v^l(q, S) > 0\}$. Again, $x_{q,S}$ is one if package S is assigned to the qth position in $\{1, \dots, |\boldsymbol{k}|\}$, and zero otherwise.

Example 2 in the appendix illustrates that the marginal value aggregation is useful and parsimonious when dealing with valuation for packages. It also illustrates the difference between the aggregating marginal values with and without repackaging. With this structure, it is now straightforward to define the properties of superadditivity between different varieties, and weakly decreasing marginal values between identical packages and varieties.

[14] If repackaging is *not* allowed, my definition is equivalent to OR-of-XOR preferences.

Definition 7 (Superadditivity)**.** A marginal value function $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$ is superadditive if for any disjoint sets $S_1, S_2 \in 2^N$ and for all $q \geq 1$ it holds that

$$v^l(q, S_1 \cup S_2) \quad \geq \quad v^l(q, S_1) + v^l(q, S_2).$$

Definition 8 (Decreasing marginal values)**.** A marginal value function $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$ is decreasing if for any package $S \in 2^N$, for all $q \geq 1$, it holds that

$$v^l(q, S) \quad \geq \quad v^l(q + 1, S).$$

If a marginal value function $v^l : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}_+$ satisfies Definitions 7 and 8, I call it *multi-unit-decreasing superadditive* (short MU-decreasing superadditive). I call the aggregate value function constructed from $v^l(q, S)$ according to Definition 5 or 6 *multi-unit concave superadditive* (short MU-concave superadditive) if $v^l(q, S)$ is MU-decreasing superadditive. I make the following assumptions on the bidders' values and utility:

Assumption 1 (Multi-unit-concave superadditivity)**.** Every bidders' value function V^l is MU-concave superadditive.

Assumption 2 (Quasi-linear utility)**.** Every bidder's utility is given by $u^l(\mathbf{k}, p) = V^l(\mathbf{k}) - p(\mathbf{k})$ when he receives a multiset of packages $\mathbf{k}$.

2.2.2 Seller

The seller's ("she") preferences are given by (i) an (incremental) marginal cost function for each individual variety sold, (ii) incremental marginal cost functions which may take effect when varieties are sold as a package, and (iii) a graph characterising cost relations, i.e. *which* incremental marginal cost functions take effect when a given package is sold. These three elements define the seller's cost functions I allow in my model. First, I define the graph characterising cost relations. This graph is called a cost function graph (CFG).

Definition 9 (Cost function graph)**.** A CFG is defined as a directed graph $G := (V, E)$, where the set of vertices is given by $V = 2^N$. The set of edges E defines cost relations subject to the conditions below.

Definition 10 (Path existence and length)**.**

(i) Whenever there exists a sequence of vertices $(S_1, S_2, ..., S_t)$ such that $(S_1, S_2), ..., (S_{t-1}, S_t) \in E$, I say there exists a path from S_1 to S_t and write $\exists(S_1...S_t)$. There always exists a path $(S, ..., S)$, i.e. from any package S to itself.

(ii) Given a path $H := (S_1, S_2, ..., S_t)$, $|H| := t - 1$, i.e. $|H|$ denotes the length of path H.

$(S_1, ..., S_t)$ may not be uniquely defined; but this is not important for my purpose. The CFG serves two purposes: First, it describes the cost relations between packages. Whenever $\exists(T...S)$, i.e. there exists a path from T to S, I say that "package T has a cost relation to package S". This means that a cost increment corresponding to S also contributes to the marginal cost of T. Second, the CFG facilitates the counting of varieties contained in packages, so that overall supply is never violated. Each node in the CFG also corresponds to an incremental marginal cost function. The incremental marginal cost function corresponding to a single variety j contributes to the overall cost of a multiset, whenever a copy of j is sold, if separately or as part of a package; hence, counting these contributions allows me to keep track of overall supply. To fulfil the two described purposes, the graph is required to have the following properties.

Assumption 3 (Permissible cost relations)**.**

(i) If $(T, S) \in E$, then $S \subset T$.

(ii) There exists a directed path from every package S to every single variety j that is contained in S, i.e. $\forall S,\ \forall j \in S,\ \exists(S...j)$.

Assumption 3(i) is intuitive in an economic sense: if a bundle has no "physical" relation to another package (in terms of being a superset of it), it also cannot have a cost relation to it. From Assumption 3(i) it follows that node N is always a source; but G may contain other sources as well. It also follows that G contains no cycles. From Assumption 3(i) and 3(ii) it follows that S is a sink if and only if S is a single variety. This is required because the seller's "physical" supply is given by the number of units available of each individual variety Ω_j. A marginal cost function corresponding to a sink will count the number of units of each single variety allocated.[15] Assumption 3(ii) ensures that whenever a unit of a given package S is allocated, the single varieties contained in that package are counted correctly. It also follows that G is weakly connected. If $\exists(T...S)$, I say that "package T has a cost relation to package S".

Each node of the CFG also corresponds to an incremental marginal cost function $\Delta c(\cdot, S), S \in 2^N$. Suppose one copy of package T is sold and nothing else. Then the marginal cost of T is obtained by adding all incremental marginal costs $\Delta c(\cdot, S)$ to which T has a cost relation, i.e. the marginal cost of T is $\sum_{S \in 2^N : \exists(T...S)} \Delta c(\cdot, S)$. Whenever a package with cost relation to S is sold, the incremental marginal cost $\Delta c(\cdot, S)$ is added to the overall cost of the supplied multiset. I allow the incremental cost to depend on q, the number of packages with cost relation to S sold.

Definition 11 (Incremental marginal cost)**.** Incremental marginal cost functions are defined as $\Delta c : \mathbb{Z}_+ \times 2^N \to \mathbb{Z}$, where $\Delta c(q, S)$ is the incremental marginal cost the seller incurs from selling a given copy of a package T due to its cost relation to package S, when she sells $q - 1$ copies of other packages with cost relation to S, not including the original copy of T. For $q = 0$ and for all S, I define $\Delta c(0, S) := 0$. For all $j \in N$, and for all $q > \Omega_j$, I define $\Delta c(q, j) := \infty$.

Note that I permit negative incremental marginal costs.

Definition 12 (Marginal cost)**.** Given a CFG, and an incremental marginal cost function Δc, the marginal cost of a bundle $S \in 2^N$ is defined as a function $c^0 : \mathbb{Z}_+ \times 2^N \times \mathbb{Z}_+^{2^n - 1} \to \mathbb{Z}_+$. Given $B \subseteq 2^N$, $c^0(k, S, \mathbf{k}_{-B})$ denotes the seller's marginal cost from selling the kth "unit" of package

[15] Should the graph be such that S with $|S| > 1$ is a sink, S would have to be defined as a "new" single variety, available in quantity Ω_S.

S, when she also sells the packages $\mathbf{k}_{-B}$. Let $S \in B$. Then

$$c^0(k, S, \mathbf{k}_{-B}) = \sum_{\substack{\gamma \in 2^N: \\ \exists(S \ldots \gamma)}} \Delta c(q_\gamma, \gamma), \qquad \text{where } q_\gamma := \sum_{\substack{A \in 2^N \setminus B: \\ \exists(A \ldots \gamma)}} k_A + k$$

Note that k_A counts the number of packages of type A supplied, whereas q_γ counts the number of packages of any type $A : \exists(A \ldots \gamma)$ supplied, excluding the packages in B. For single varieties $j \in N$ we have $c^0(k, j, \mathbf{k}_{-j}) = \Delta c(k, j)$, so the marginal cost equals the incremental marginal cost of the kth unit of package j.

Definition 13 (Cost)**.** Given a CFG, and an incremental marginal cost function Δc, the cost of selling a number of packages given by $\mathbf{k} \in \mathbb{K}$ is defined as a function $C^0 : \mathbb{K} \to \mathbb{Z}_+$, and given by

$$C^0(\mathbf{k}) = \sum_{S \in 2^N} \sum_{y=1}^{q_S} \Delta c(y, S), \qquad \text{where } q_S := \sum_{\substack{\gamma \in 2^N: \\ \exists(\gamma \ldots S)}} k_\gamma$$

The seller's overall cost of selling a multiset of packages is given by adding all incremental marginal costs that occur due to cost relations of the packages she supplies. This also takes the interaction of packages with overlapping subsets of single varieties into account; the cost of each additionally supplied package may differ from the cost of a previously supplied, identical package. From Definition 11, it follows immediately that for any $\mathbf{k} = (k_S)_{S \in 2^N}$ for which $\exists\ j \in A \subseteq N : \sum_{S \in 2^N, S \ni j} k_S > \Omega_j$, we have $C^0(k, A, \mathbf{k}_{-A}) := \infty$, i.e. package A (and by extension the multiset $\mathbf{k}$) cannot be sold because the seller is constrained in the supply of at least one variety in A. Note that the seller cannot procure a bundle at the cost of the bundle and then split it up into subsets to be sold individually, or vice versa. I also make the following assumptions on the seller's preferences.

Assumption 4 (Increasing incremental marginal cost)**.** The seller's incremental marginal cost functions Δc are increasing, i.e., for any package $S \in 2^N$ and for all $q \geq 1$, it holds that

$$\Delta c(q, S) \quad \leq \quad \Delta c(q + 1, S).$$

The seller incurs at least the cost of each individual package sold; if she sells two packages, the cost of selling both is weakly greater than the sum of the cost of each individual package, due to Assumption 4 and Definition 13.

Assumption 5 (Seller's quasi-linear profit)**.** The seller's profit is given by $u^0(\mathbf{k}, p) = p(\mathbf{k}) - C^0(\mathbf{k})$ when she sells the multiset of packages $\mathbf{k}$.

Definition 14 (Subadditive marginal cost)**.** Let $S_1, S_2 \in 2^N$ be two disjoint packages, and let $B := \{S_1 \cup S_2, S_1, S_2\}$. The seller's marginal cost is subadditive if it holds that, for all $\mathbf{k}_{-B} \in \mathbb{K}$, and for all $k \geq 1$,

$$c^0(k, S_1 \cup S_2, \mathbf{k}_{-B}) \quad \leq \quad c^0(k, S_1, \mathbf{k}_{-B}) + c^0(k, S_2, \mathbf{k}_{-B}).$$

For general cost function graphs, subadditivity of marginal cost is not obvious; one would have to compute the marginal cost for each possible multiset that could be supplied. For a special class of CFG, however, there is a straightforward criterion, which I describe in section 3.3.1.

2.3 Demand, supply, and equilibrium

The aim is to develop a framework in which equilibrium prices are *package-linear*.[16] Given the objects available for sale, the seller chooses a multiset $\mathbf{k}$ of packages to supply. $\mathbf{k}$ must be feasible, i.e. $\mathbf{k} \in \mathbb{K}$, or $+_{S\in 2^N} k_S S^* \subseteq \mathcal{N}$. An allocation of objects in $\mathcal{N}$ is defined as an assignment $\pi = (\pi(l), l \in \mathcal{L}_0)$ of these objects among the bidders and the seller, such that $+_{l\in\mathcal{L}} \pi(l) = \mathbf{k}$ and $\pi(0) = \mathcal{N} - \mathbf{k}^*$. $\pi(l)$ is the multiset of packages assigned to agent l under the allocation π, where $\pi(l)$ may be the empty set, and $\pi(0) \neq \emptyset$ means that the objects in $\pi(0)$ are not sold. Bidder l's demand correspondence and indirect utility are defined as

$$D^l(p) := \arg\max_{\mathbf{k}\in\mathbb{K}} u^l(\mathbf{k}, p) \text{ and } \mathcal{V}^l(p) := \max_{\mathbf{k}\in\mathbb{K}} u^l(\mathbf{k}, p).$$

I define the seller's supply correspondence $S(p)$ and profit functions as

$$S(p) := \arg\max_{\mathbf{k}\in\mathbb{K}} p(\mathbf{k}) - C^0(\mathbf{k}) \text{ and } \Pi(p) := \max_{\mathbf{k}\in\mathbb{K}} p(\mathbf{k}) - C^0(\mathbf{k}).$$

Definition 15. An allocation π is efficient if it holds for every allocation π' that

$$\sum_{l\in\mathcal{L}} V^l(\pi(l)) - C^0(\mathbf{k}) \geq \sum_{l\in\mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}')$$

where $\mathbf{k} = +_{l\in\mathcal{L}} \pi(l)$ and $\mathbf{k}' = +_{l\in\mathcal{L}} \pi'(l)$.

Given an efficient allocation π, the market value is defined as $V(\mathcal{N}) := \sum_{l\in\mathcal{L}} V^l(\pi(l)) - C^0(\mathbf{k})$. Formally, a pricing function is a function $p : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{R}$ with $p(\emptyset) = 0$. This function is *nonlinear* in varieties $j \in N$, i.e. for any package $S \in 2^N$ we may have $p(S) \neq \sum_{j\in S} p(\{j\})$.

Definition 16. A pricing function $p : \mathbb{Z}_+^{|2^N|-1} \to \mathbb{R}$ with $p(\emptyset) = 0$ is *package-linear* if and only if, for all $\mathbf{k} \in \mathbb{Z}_+^{|2^N|-1}$, $p(\mathbf{k}) = \sum_{S\in 2^N} k_S p(S)$.

Thus, a package-linear pricing function can be represented as a mapping $p : 2^N \to \mathbb{R}$.

Definition 17. A *package-linear* pricing Walrasian equilibrium is given by a *package-linear* pricing function $p^* : 2^N \to \mathbb{R}$ and an allocation π^* such that $+_{l\in\mathcal{L}} \pi(l) \in S(p^*)$ and $\pi^*(l) \in D^l(p^*)$ for every bidder $l \in \mathcal{L}$.

If the pricing function is linear in varieties, the package-linear Walrasian equilibrium is simply the standard Walrasian equilibrium. Walrasian equilibrium and and efficient allocations are connected in the standard way.

[16] SY call a pricing function $p : 2^N \to \mathbb{R}$ *non-linear*, and the equilibrium concept *Non-linear* pricing Walrasian equilibrium (NPW). With only one unit per variety for sale, the non-linear equilibrium and the package-linear equilibrium are the same concept.

Lemma 1. *If (p^*, π^*) is a package-linear Walrasian equilibrium, π^* is an efficient allocation. Furthermore, if π' is another efficient allocation, (p^*, π') is a package-linear Walrasian equilibrium as well.*

Note that I defined efficiency and Walrasian equilibrium assuming bidders have aggregate marginal values without repackaging. If bidders have the technology to repackage bundles they have been allocated in the auction, V^l needs to be substituted by $\widetilde{V}^l$ (in the definition of u^l).

3 The sealed-bid package auction and Walrasian equilibrium

In this section I describe how the agents can submit their preferences in the auction. The auction design is based on the assumption of agents submitting (approximately) truthful preferences. It will be shown later that this results in a Walrasian equilibrium allocation supported by competitive prices; but it is helpful to keep the assumption of truthful behaviour in mind in this section.

3.1 Buyers' bidding language

Each bidder l submits a list of bids to the auctioneer in a sealed envelope. The auctioneer treats each bid as independent from other bids, including the ones made by the same bidder, i.e. the auctioneer only cares about the aggregate list of bids. It is therefore convenient to simply enumerate the bids by $i \in \mathcal{I} := \{1, \ldots, m\}$. I also denote the list of bids submitted from bidder l by $\mathcal{I}^l$, i.e. $+_{l \in \mathcal{L}} \mathcal{I}^l = \mathcal{I}$.

Each bid submitted in the auction is a *package XOR-bid* and each bidder can submit any finite number of package XOR-bid.

Definition 18 (Package XOR-bid)**.** A package XOR-bid is a vector of length 2^n. It specifies a maximum price v^i_S for each available package in the auction (except the empty package) which the bidder is willing to pay, and an overall maximum quantity κ^i.

I denote by x^i_S the allocation of bundle S to bid i. I will show that each package XOR-bid is guaranteed to obtain only packages that maximise its surplus.

Package XOR-bids are the straightforward extension of so called "paired bids" used in the Product-Mix auction by [20] to an environment where packages are sold in addition to single varieties. An allocation $x^i_S, S \in 2^N, i \in \mathcal{I}^l$ maps into a multiset $\pi(l) = \mathbf{k}^l$, where $k^l_S = \sum_{i \in \mathcal{I}^l} x^i_S$. Naturally, the reverse mapping is not possible. I say that a value function V^l can be represented by a list of bids, if (i) there exists a mapping from V^l to the list of bids, and (ii) given an allocation to bidder l, the bidder's value V^l from this allocation is identical to the value entering the auctioneer's objective. Package XOR-bids allow each bidder to represent their preferences:

Proposition 1. *If the auctioneer maximises social welfare, every bidder's value function V^l satisfying Definitions 4 and 5 and Assumption 2 can be represented by a finite list of XOR-bids.*

The proposition above tells us that when bidders cannot repackage their received objects, preferences can be easily translated into the XOR-bid language. However, once bidders are allowed to repackage, the bidding language loses some of its power. Consider the example below.

If bidder 1 gets assigned A and B as separate bundles and she can repackage, her value of this assignment is $v^1(1, AB) = 9$, not $v^1(1, B) + v^1(2, A) = 5 + 1$. Thus, the interpretation of the bids is ambiguous to the seller: because she cannot distinguish which bids were made by which bidder, she may assume that A and B are used as separate bundles, when in fact the bidder uses them as a package, and hence should also pay the package price p_{AB}, instead of $p_A + p_B$. If repackaging is possible, in general, an accurate representation of the *seller's* preferences is not possible. Given an allocation, the bidder is always weakly better off compared to the value the auctioneer assigns to his contribution to welfare. However, when the preferences of the seller and bidders align such that all varieties are complementary to one another, repackaging can be allowed.

Proposition 2. *Assume the auctioneer maximises social welfare, and let bidder l's preference satisfy Assumption 1 (MU-concave-superadditivity) and let the seller's preference satisfy subadditivity of marginal costs according to Definition 14. (Assume also that the standard Definitions 4, 5, 9, 11, 12, 13 hold and standard Assumptions 2, 3, 4 of my model are satisfied.) Then the bidder never has to repackage bundles to achieve the highest aggregate of marginal values, i.e. $V^l = \widetilde{V}^l$.*

Given the XOR-bid structure and Proposition 1, bidder l's indirect utility is

$$\mathcal{V}^l(p) := \max_{x_S^i, i \in \mathcal{I}^l} \sum_{S, i \in \mathcal{I}^l} (v_S^i - p_S) x_S^i \quad \text{s.t.} \sum_S x_S^i \leq \kappa^i \; \forall i \in \mathcal{I}^l \tag{1}$$

For three packages $\{A\}, \{B\}, \{AB\}$, a bid $(v_A, v_B, v_{AB}, 1)$ can be illustrated in price space $p_A \times p_B \times p_{AB}$ as shown in Figure 6 in the appendix.

Note that, allowing repackaging of collections of bids, MU-concave superadditivity of bidders' values and subadditivity of the seller's marginal costs is required such that bidders can represent their preferences accurately in the mechanism. Generally, not allowing repackaging, any buyers' preference that can be represented as a list of XOR-bids is admissible in the auction. If a bidder were interested in at most one unit of each variety, a package XOR-bid would be fully expressive, i.e. allow the representation of *any* preference. Although somewhat more restrictive with multiple units per variety, bidders can express a rich set of preferences in this language. Note also that in the most general environment my mechanism may not always produce an indivisible allocation. Instead, some packages may have to be rationed, e.g., a bid may receive only "half" of a package AB.[17]

3.2 Seller's bidding language

The seller can also perfectly represent her preferences in the auction. She submits her preferences through two components: (i) supply functions, and (ii) a graph describing the relations between supply functions. If she reports her preferences truthfully, the supply functions will correspond precisely to the incremental marginal cost functions, and the graph describing rela-

[17] Formally, one would extend the XOR-bid so that it can also be satisfied with a combination of packages in the convex hull of the given bid, i.e. the bid can be satisfied with any combination of fractions of packages, the sum of which adds up to not more than the maximum quantity specified.

tions between supply functions will correspond to her cost function graph (CFG). The submitted CFG is called a supply function graph (SFG).

The seller submits $2^n - 1$ supply functions. Each is required to be a weakly increasing step function. Analogously to incremental marginal cost functions, a supply function f_j for an individual variety j describes the marginal cost of each unit supplied of variety j. A supply function f_S for package S with $|S| > 1$ describes the incremental marginal cost for every package $S' : \exists(S'...S)$, i.e. the additional cost, or the cost savings, inherited by S', which also contains the objects bundled together in S.

f_S is a step function, where the qth step has length l_S^q and height μ_S^q. Without loss of generality the length of each step can be normalised to one. I denote by y_S^q the amount allocated on step q of f_S. Each supply function consists of an infinite number of steps, and there exists a finite $\bar{q}_S$ such that $\mu_S^q = \infty$ for all $q > \bar{q}_S$. Thus, any step $q > \bar{q}_S$ will never be allocated. I define $\bar{q} := \max\{\bar{q}_S | S \in 2^N\}$.

I make the following two requirements on supply function graphs to ensure non-negative costs and weakly increasing incremental marginal costs.

Requirement 1 (Non-negative costs)**.** $\forall\, \mathbf{k} \in \mathbb{K} : \sum_{S\in 2^N} \sum_{y=1}^{q_S} \mu_S^y,$ where $q_S := \sum_{\substack{\gamma\in 2^N:\\ \exists(\gamma...S)}} k_\gamma$

Requirement 2 (Weakly increasing marginal costs)**.** $\forall\; S, q : \mu_S^q \leq \mu_S^{q+1}$

Because supply function f_S describes the incremental marginal cost that is inherited by all packages S' with a cost relation to S, i.e. $\exists(S'...S)$, I define the allocation on supply function as follows.

Definition 19 (Supply step allocation)**.** Whenever a step on supply function f_S is allocated, a step on every supply function $f_{S'}$ is allocated, for all $S' : \exists(S...S')$.

The allocation on supply curve f_S is limited by the minimum number of steps with finite height among all supply curves $f_\gamma : \exists(S...\gamma)$. Note that this reflects precisely my definition of marginal costs: for $\mathbf{k} = (k_S)_{S\in 2^N}$ for which $\exists\; j \in A \subseteq N : \sum_{S\in 2^N, S\ni j} k_S > \Omega_j$, I defined $C^0(k, A, \mathbf{k}_{-A}) := \infty$. An important property of this supply function graph is that an allocation $(y_S^q)_{S\in 2^N, q\leq\bar{q}}$ uniquely identifies a multiset of packages:

Lemma 2. *Given a supply function graph $G = (V, E)$ satisfying Assumption 3, there exists a linear one-to-one mapping between a supply function allocation $(y_S^q)_{S\in 2^N, q\leq\bar{q}}$ and a corresponding multiset of packages $\mathbf{k}$.*

Now I can rewrite the seller's problem as follows.

$$\begin{aligned}
&\max_{\mathbf{k}\in\mathbb{K}} \{p(\mathbf{k}) - C^0(\mathbf{k})\} \\
\Leftrightarrow\quad &\max_{k_S} \left\{ \sum_{S\in 2^N} k_S p(S) - \sum_{S\in 2^N} \sum_{y=1}^{q_S} \Delta c(y, S) \right\} \qquad \text{where } q_S := \sum_{\substack{\gamma\in 2^N:\\ \exists(\gamma...S)}} k_\gamma \\
\Leftrightarrow\quad &\max_{k_S, y_S^q} \left\{ \sum_{S\in 2^N} k_S p(S) - \sum_{S\in 2^N} \sum_{q=0}^{\bar{q}_S} \mu_S^q y_S^q \right\} \qquad \text{s.t. } y_S^q \leq l_S^q \;\forall S, q
\end{aligned}$$

Subtracting the seller's cost simply amounts to subtracting the amounts allocated on each step multiplied with the height of the corresponding step; Lemma 2 allowed me to substitute some k_γ with the y^q_γ, which are now additional choice variables.

3.3 Solving the auction

The auction input is a list of bids $\{(v^i_S)_{S\in 2^N}, \kappa^i\}_{i\in\mathcal{I}}$, a list $(\mu^q_S, l^q_S)_{S\in 2^N, q\leq \bar{q}}$, and a supply function graph given by a graph G_{SFG}. x^i_S denotes the allocation of bundle S to bid i, y^q_S denotes the allocation on step q of the supply function f_S, and Y_S denotes the total amount of bundle S allocated. The auctioneer's objective is to find the efficiency maximising partitioning of supply. That is, the objective is to find an allocation $\{x^i_S, y^q_S, Y_S\}_{S\in 2^N, i\in\mathcal{I}, q\leq\bar{q}}$ that maximises the sum of the buyers' surplus and of the seller's profit. The SFG determines the relations between the supply functions $(f_S)_{S\in 2^N}$. Lemma 2 tells us that given a SFG with the necessary properties, there exists a linear one-to-one mapping between $\{y^q_S\}_{S\in 2^N, q\leq\bar{q}}$ and Y_S, for all $S\in 2^N$. I denote this mapping by $\phi : \mathbb{Z}_+^{2^n\times\bar{q}} \to \mathbb{Z}^{2^n}$, where $\phi_S(\{y^q_S\}) = Y_S$. The auctioneer's problem can be written as an integer programme, called "IP" and is

IP

$$\max_{\{x^i_S, y^q_S, Y_S\}} \left[\sum_{i,S} v^i_S x^i_S - \sum_{S,q} \mu^q_S y^q_S\right] \text{ s.t. } \begin{cases} \sum_S x^i_S & \leq & \kappa^i & \forall i & \text{bid size constraint} \\ y^q_S & \leq & l^q_S & \forall q, \forall S & \text{step size constraint} \\ \sum_i x^i_S & \leq & Y_S & \forall S & \text{supply constraint} \\ Y_S & = & \phi_S(\{y^q_S\}) & \forall S & \text{SFG relations} \\ x^i_S, y^q_S, Y_S & \in & \mathbb{Z}_+ & \forall S, \forall i, \forall q & \end{cases}$$

The supply constraint must be binding at the optimum, so prices cancel from the buyers' and seller's objective. As a next step, I substitute for Y_S, and relax the integer programme to a linear programme, called "LP". The corresponding dual variables are listed to the right of the constraints.

LP

$$\max_{\{x^i_S, y^q_S\}} \left[\sum_{i,S} v^i_S x^i_S - \sum_{S,q} \mu^q_S y^q_S\right]$$

$$\text{s.t. } \begin{cases} \sum_S x^i_S & \leq & \kappa^i & \forall i & b_i & \text{bid size constraint} \\ y^q_S & \leq & l^q_S & \forall q, \forall S & u^q_S & \text{step size constraint} \\ \sum_i x^i_S - \phi_S(\{y^q_S\}) & \leq & 0 & \forall S & z_S & \text{supply constraint} \\ x^i_S, y^q_S & \geq & 0 & \forall S, \forall i, \forall q & & \end{cases}$$

Linear programming techniques allow me to write the corresponding dual problem, called "DLP". I denote by $\psi(\{z_S\})$ the "dual function" of $\phi(\{y^q_S\})$. Formally, let $\phi(\{y^q_S\}) = \Phi\mathbf{y}^\intercal$, where Φ is a $|S|\times|S|$-matrix, which can be determined by Algorithm 1, and $\mathbf{y} = \left(\sum_q y^q_S\right)_{S\in 2^N}$, i.e. a row vector each entry of which contains the total amount allocated on supply curve f_S. $\phi_S(\{y^q_S\}) = \Phi_S\mathbf{y}^\intercal$, i.e. the row corresponding to package S of Φ multiplied by $\mathbf{y}^\intercal$. Then $\psi(\{z_S\}) = \Phi^\intercal\mathbf{z}^\intercal$, where $\mathbf{z} = (z_S)_{S\in 2^N}$.

DLP

$$\min_{\{b^i, z_S, u^q_S\}} \left[\sum_i \kappa^i b^i + \sum_{q,S} l^q_S u^q_S \right]$$

$$\text{s.t.} \begin{cases} b^i + z_S & \geq & v^i_S & \forall i, S & x^i_S & \text{surplus constraint} \\ u^q_S - \psi_S(\{z_S\}) & \geq & -\mu^q_S & \forall q, \forall S & y^q_S & \text{marginal cost constraint} \\ b^i, u^q_S, z_S & \geq & 0 & \forall S, \forall i, \forall q & & \end{cases}$$

In the following, I define the final auction prices as part of the solution of DLP. I characterise the price structure and discuss some additional properties of the auction prices. Finally, I state my main theorem of this section.

The auction price of bundle S is given by the dual variable of the supply constraint on bundle S, z_S. In the interpretation of z_S as a shadow price, it is the value of the last unit the seller supplies on a given bundle. The auction prices z_S are uniform, anonymous, package-linear prices, i.e. a generalisation of the uniform pricing rule to package-goods. b^i is the surplus generated on bid i. The feasible set of LP is a non-empty, convex polytope and therefore an optimal solution always exists; by strong duality, an optimal solution to DLP exists also.

Proposition 3 reveals an important property of the auction prices. The auction price is determined, not only for each package S of which a positive amount is allocated in the auction, but also for each package to which S has a cost relation.

Proposition 3. *For all $S \subseteq N$,*
(i) it holds that, for all $q \leq \bar{q}$,

$$z_S \leq \sum_{\gamma:\exists(S\ldots\gamma)} \left(u^q_\gamma + \mu^q_\gamma\right)$$

(ii) and if $y^{\widetilde{q_\gamma}}_\gamma \neq 0 \ \forall \gamma : \exists(S\ldots\gamma)$, it holds that, for all $q_\gamma \leq \widetilde{q_\gamma}$,

$$z_S = \sum_{\gamma:\exists(S\ldots\gamma)} \left(u^{q_\gamma}_\gamma + \mu^{q_\gamma}_\gamma\right)$$

For single varieties, we have $z_j = u^q_j + \mu^q_j$ if $y^q_j \neq 0$. That is, the auction prices are given by marginal costs to the extent that they do not have to be adjusted to set non-allocated bids at least indifferent (the u^q_j provide the necessary flexibility); for packages, the z_S equal the sum of incremental marginal costs of each variety or package that package S has a cost relation to (plus adjustments through the u^q_j). There may exist a set of equilibrium prices. The complementary slackness conditions corresponding to primal and dual constraints generate a system of equations which the auction prices have to satisfy. This solution is not necessarily unique; and in addition, the u^q_S may leave some ambiguity. This creates more flexibility for the auctioneer because he can choose from a set of equilibrium prices, and she can specify additional rules to do so.[18] For example, she can define a rule to always choose the lowest equilibrium prices on certain packages,[19] or the highest, or a compromise between prices favouring bidders

[18] Modern LP-solvers can return the set of all integer solutions.
[19] A set of lowest equilibrium prices may not always exists (see example 2 in the appendix).

and prices favouring the seller. The following corollary provides an additional characterisation of the auction prices.

Corollary 1. Let q be the last step of supply function f_S that is allocated, i.e. $y_S^q \neq 0$ and $y_S^{q+1} = 0$. Let also $y_\gamma^{\widetilde{q_\gamma}} \neq 0 \ \forall \gamma : \exists (S...\gamma)$. Then, for all $q_\gamma \leq \widetilde{q_\gamma}$,

$$\begin{aligned} &\text{(i)} && u_S^q + \mu_S^q \leq \mu_S^{q+1} \\ &\text{(ii)} && \psi_S(\{z_S\}) = z_S - \sum_{\gamma:\exists(S...\gamma),\gamma\neq S} \left(u_\gamma^{q_\gamma} + \mu_\gamma^{q_\gamma}\right) \end{aligned}$$

The term $\psi_S(\{z_S\})$ describes the price gain from selling all objects in S together in package S, in addition to the price satisfying all other cost relations. If this price gain was strictly less than μ_S^q, the seller would prefer not to sell the package, and if this price gain was strictly greater than μ_S^{q+1}, the seller would prefer to sell an additional package S. Finally, the auction prices I characterised support indeed an equilibrium. Moreover, this equilibrium exists if and only if it can be characterised by an optimal solution of LP.

Theorem 1. A package-linear pricing Walrasian equilibrium exists if and only if any optimal solution to IP is also an optimal solution to LP, i.e. the optimum values of IP and LP coincide.

my theorem generalises the results by [5] and [6] in that my seller's preferences are non-additive in varieties. Note that my auction is a uniform price auction for multiple differentiated packages. Because the allocation rule is such that not only units of the same variety, but also packages composed of different varieties compete against one another for an efficient allocation, my auction can be seen as set of simultaneous uniform-price auctions for packages, with additional competition between packages. Thus, the results from [30] on asymptotic efficiency apply to my setting. Under asymptotic environmental similarity[20] and other standard assumptions, they show that any equilibrium must be asymptotically ex-ante efficient. If a large number of bidders *for each package* participates in my auction, with the assumptions of [30] being satisfied for each package separately, it appears my auction is also asymptotically ex-ante efficient. Related results have been shown by [11] and [13].

The assumptions imposed on the agents' preferences are only to comply with the bidding language of my mechanism. While it is suited for complementary varieties, under a different bidding language, the sealed-bid auction will still yield a Walrasian equilibrium, if it exists, if some varieties are substitutes. In this case one needs to exercise caution for specific configurations of reserve prices, for instance, if some varieties are substitutes for the auctioneer (superadditive reserve prices), but complements for some bidders. If bidders know that reserve prices are superadditive, and are allowed multiple bids, they have an incentive to bid on individual varieties instead of a package.

3.3.1 Complete supply function graphs

When cost savings from packaging two varieties together are independent of other varieties present, a special class of cost functions graphs can be formulated. This class of CFGs may be

[20] Asymptotic environmental similarity requires the probability of some bidder i winning the hth unit with bid b to converge uniformly to the probability of any bidder j winning the h'th unit with bid b, for all i, j, h, h', b.

applicable to the procurement of production factors: if service A and service B are delivered by the same provider, the manufacturer incurs cost savings, and similarly for services A and C. However, services B and C, if delivered by the same provider cause additional costs, e.g. because of a substitutability paired with a high contingency risk. If all three services A, B, and C were delivered by the same provider, this could be modelled precisely: cost savings from A and B, A and C, and additional cost from B and C enter the cost function, plus an additional cost savings term to account for the interaction of A, B, and C together.

More generally, a cost function graph of this class is such that each package has a cost relation to all packages that are subsets of itself; and these cost relations are achieved with a minimal number of edges, i.e. there are directed paths from each package S to its subsets of size $|S|-1$ only. The following introduces some additional notation needed to generate results on these special graphs, including some closed form solutions.

Definition 20 (Levels)**.** Let $x, y \in 2^N$.

(i) $x \subset_r y := \{x \mid x \subseteq y, |y|-|x|= r\}$. Similarly, $y \supset_r x := \{y \mid y \supseteq x, |y|-|x|= r\}$. I say y is r levels above x, or x is r levels below y.

(ii) $x \subset_{\geq r} y := \{x \mid x \subseteq y, |y|-|x|\geq r\}$. Similarly, $y \supset_{\geq r} x := \{y \mid y \supseteq x, |y|-|x|\geq r\}$. I say y is at least r levels above x, or x is at least r levels below y.

(iii) $x \subset_{\leq r} y := \{x \mid x \subseteq y, |y|-|x|\leq r\}$. Similarly, $y \supset_{\leq r} x := \{y \mid y \supseteq x, |y|-|x|\leq r\}$. I say y is at most r levels above x, or x is at most r levels below y.

Note that $x \supset_0 y$ implies $x = y$.

Definition 21 (Complete CFG)**.** A CFG is complete if and only if for each S, for all $\gamma \subset_1 S$, we have $(S, \gamma) \in G$, and for all other $\gamma \subset_{\geq 2} S$, and for all $\gamma \not\subseteq S$, we have $(S, \gamma) \notin G$ (see Fig. 3 in Appendix C.3).

Lemma 3 (Subadditivity in complete CFG)**.** *Suppose the CFG is complete. If the incremental marginal cost satisfies*

$$\Delta c(q, S) \geq 0 \ \forall S : |S| = 1 \quad \text{and} \quad \Delta c(q, S) \leq 0 \ \forall S : |S| > 1$$

then the marginal cost is subadditive.

The following lemma makes the crucial connection between the supply function graph and the overall allocation per variety.

Lemma 4 (ϕ-mapping for complete SFG)**.** *Let Y_S denote the overall amount of bundle S allocated. Given the supply function graph defined above, it holds that*

$$Y_S = \phi_S(\{y_S^q\}) = \sum_{r=0}^{n-|S|} \sum_{q, \gamma \supset_r S} (-1)^r y_\gamma^q$$

Duality then implies

Corollary 2 (ψ-mapping for complete SFG)**.**

$$\psi(\{z_S\}) = \sum_{r=0}^{|S|-1} \sum_{\gamma \subset_r S} (-1)^r z_\gamma$$

Following Corollary 1, the term $\sum_{r=0}^{|S|-1} \sum_{\gamma \subset_r S} (-1)^r z_\gamma$ describes the incremental price gain from selling all objects in S together in package S. In some cases the seller may wish to specify positive price increments. The subsequent corollary provides an additional characterisation result for this case, and is immediate from Proposition 3.

Corollary 3. If $\mu_S^q \geq 0$, and if $y_S^q \neq 0$, then $z_S \geq z_\gamma \; \forall \gamma \subseteq S$.

While a very general class of supply function graphs allow me to solve the partitioning problem, closed form solutions simplify the implementation of supply function graphs.

3.4 Complementarities and packaging costs between identical objects

If the market designer anticipates complementarities between identical objects, they may want to allow bidders to submit specific bids on packages containing those identical objects. To accommodate this, I simply introduce different notation relabelling identical objects. I demonstrate this "preprocessing" with an example. Suppose there are two distinct varieties A and B with supply $\Omega_A = \Omega_B = 2$. The two units of variety A may *either* be complementary for the buyer *or* a packaging cost may be associated with the multiset $\{AA\}$, i.e. the auctioneer wants to allow for the multiset $\{AA\}$ to be treated as a package with a separate price $p(AA) \neq 2p(A)$. An admissible XOR-bid is a vector $\boldsymbol{b} := (v_A^i, v_B^i, v_{AA}^i, v_{AB}^i)$, and, as usual, each bidder may submit a finite list of these XOR-bids. The relabelling is done by the auctioneer for the workings of the mechanism: bids are simply processed differently, but bidders and the seller are not presented with relabelled objects. I reformulate the setup such that *each unit of supply* of variety A obtains a different index, i.e. we have $N := \{A_1, A_2, B\}$, and a package is defined simply as $S \subseteq N$. To process the bid vector correctly, it is augmented as $\tilde{\boldsymbol{b}} = (v_{A_1}^i, v_{A_2}^i, v_B^i, v_{A_1A_2}^i, v_{A_1B}^i, v_{A_2B}^i, v_{A_1A_2B}^i)$, where $v_{A_1}^i = v_{A_2}^i = v_A^i$, $v_{A_1A_2}^i = v_{AA}^i$, $v_{A_1B}^i = v_{A_2B}^i = v_{AB}^i$, and $v_{A_1A_2B}^i = \max\{v_{AA}^i, v_{AB}^i\}$.

The seller submits supply functions f_A, f_B, f_{AB}, and f_{AA}, and a supply function graph defining the cost relations between those packages. Note that here only one admissible supply function graph exists (see Fig. 4 in Appendix C.3). The supply functions and the supply function graph are then augmented as follows. The auctioneer defines supply functions f_{A_1}, f_{A_2}, f_{A_1B}, f_{A_2B}, $f_{A_1A_2}$, $f_{A_1A_2B}$ such that $\mu_{A_1}^1 = \mu_A^1$, $\mu_{A_2}^1 = \mu_A^2$, $\mu_{A_1B}^1 = \mu_{AB}^1$, $\mu_{A_2B}^1 = \mu_{AB}^2$, and $\mu_{A_1A_2}^1 = \mu_{AA}^1$. All other steps heights μ_S^q are set to ∞ and f_B remains the same (step lengths are normalised to one). The supply function graph is augmented as shown in Fig. 5 in Appendix C.3. Note that $f_{A_1A_2B}$ may be omitted, unless the seller wants to define a packaging cost associated with AAB. However, in that case the bidders should be offered to submit a separate bid price for v_{AAB}^i as well.

The linear programming arguments leading to Proposition 1, Corollary 1, and Theorem 1 can then be applied as before. One thing to note are the auction prices: solving the auction, the output is $z_{A_1} = z_{A_2} =: z_A$, z_B, $z_{A_1A_2} =: z_{AA}$, $z_{A_1B} = z_{A_2B} =: z_{AB}$, (and $z_{A_1A_2B}$). The fact that prices for identical objects are the same follows from the surplus constraints of the dual DLP, keeping in mind the preprocessing rules for values (e.g., $v^i_{A_1} = v^i_{A_2} = v^i_A$).

4 Distinct complements and packaging costs

Equilibrium existence in the presence of complementarities is not well understood in the literature, but important contributions have been made in recent years. SY establish existence of a non-linear pricing Walrasian equilibrium when buyers and the seller have superadditive values.[21] I extend their ascending procedure to establish existence of an integer-valued equilibrium in my setting, restricting supply to one unit per variety. If the seller's costs are expressed through incremental marginal cost functions and CFGs, an equilibrium can also be found via linear programming.

For "gross substitutes and complements" preferences, existence of a linear pricing Walrasian equilibrium was shown in [28], and generalised by [31]. Both of these classes of preferences are generalised by the concept of demand types introduced in [2]. [9] study pricing equilibria for economies where buyers have graphical valuations with pairwise complementarities or substitutabilities, and [10] establish existence of a linear pricing Walrasian equilibrium for an environment where one unit per variety is sold and bidders have sign-consistent tree valuations. In all those studies, the seller cannot express preferences over the partitioning of supply.

The environment is restricted now such that $\Omega_j = 1$ for all j, i.e. one unit per variety is for sale. A set of packages is now characterised by $\mathbf{k} \in \{0,1\}^{|2^N|-1}$. Each bidder's aggregate value function is identical to his marginal value function $v^l(1,S), S \in 2^N$. I omit the first argument and define bidder l's value function as $v^l : 2^N \to \mathbb{Z}_+$. Every buyer's value function is superadditive, and his utility is quasi-linear. The seller's marginal cost function is identical to her incremental marginal cost for the first unit, i.e. $c^0(1, S, \mathbf{k}_{-S}) = \Delta c(1, S)$ for all $\mathbf{k}_{-S}$. Again, I simplify and define the seller's marginal cost from selling a package $S \in 2^N$ as $c^0 : 2^N \to \mathbb{Z}_+$. The seller's aggregate cost is defined as before and simplifies to $C^0(\mathbf{k}) = \sum_{S\in 2^N} c^0(S)k_S$. The seller's marginal cost function is subadditive and her utility is quasi-linear.

For comparability with SY, I use a different notation for sets of packages: instead of a vector $\mathbf{k}$, I view a set of packages as a partition δ of N into packages, i.e. $\delta = \{A_1, \dots, A_k\}$ where $A_t \subseteq N, t = 1, \dots, k$, $A_{t_1} \cap A_{t_2} = \emptyset$ for all $A_{t_1} \neq A_{t_2}$, and $\bigcup_{t=1}^k A_t \subseteq N$. As before, $\mathbb{K}$ denotes the universe of all partitions of N. Consequently, the seller's profit is given by $u^0(\delta, p) = \sum_{S\in\delta} \left(p(S) - c^0(S)\right)$ when she sells the set of packages δ.

The definitions of demand and supply correspondences, and indirect utility and profits carry over from the previous section. Note that an allocation $\pi = (\pi(0), \pi(1), \dots, \pi(L))$ can also be interpreted as a partition δ of N defined by $\delta = \{\pi(l) | \pi(l) \neq \emptyset \text{ and } l \in \mathcal{L}_0\}$, where $\pi(l)$ disappears if it is the empty set. The package-linear pricing Walrasian equilibrium coincides with SY's non-linear pricing Walrasian equilibrium.

[21] [29] also propose an incentive-compatible mechanism for their setting. Results in a similar spirit are established in [15] for unimodular demand types. However, as I show in Appendix B, in SY's and my setting the seller's demand type is not unimodular.

4.1 Modelling the seller's preferences

The difference of this restricted environment to [29] are the seller's preferences. SY describe their seller as "revenue-maximising", while I model a profit-maximising seller. Examining how the models relate, I make two observations. First, note that in SY one needs to distinguish between the seller's utility and true reserve prices, whereas my reserve prices simply correspond to the seller's marginal cost function. Second, SY's seller preferences can be reformulated such that the seller acts as a profit-maximiser with a well-defined cost function.

In SY, the seller has a "reserve price" function $v^0 : 2^N \to \mathbb{Z}_+$. If she doesn't sell a package S, $v^0(S)$ is also the "utility" (or "revenue") she derives from keeping S. v^0 is assumed to be superadditive. Example 1 demonstrates the subtlety in defining the seller's value functions and reserve price function when they are strictly super- or subadditive.

Example 1. I revisit the seller from Section 1.1. She possesses the bundle $\{AB\}$ which is worth 60 to her, and she values good A alone at 10, and good B alone at 20. Then, her reserve price for good A should be 40: at this price, she is indifferent between keeping the bundle and selling good A alone. Similarly, the reserve price for good B should be 50. In SY's basic ascending auction, by allowing the seller to choose her supply set *given the current prices*, these reserve prices are implicit. If (p_A, p_B, p_{AB}) are the auction prices, the seller chooses to sell good A individually only if $p_A + 20 \geq 60$ and $p_A + 20 \geq p_{AB}$, i.e., good A's true reserve price is $60 - 20 = 40$.

In my model, the seller's reserve price for a bundle corresponds to the marginal cost she incurs when supplying that bundle. If she is selling a set of packages, the cost of supplying the set is the sum of marginal costs of packages contained in the set. In contrast to SY, the cost of supplying a set of packages depends on the partition of supply into packages, where each package is allocated to a different bidder. I now describe the relationship between SY's seller and mine formally. Let S^c denote the complement for any set $S \in 2^N$, i.e. $S^c = N \setminus S$. First, I state the definition of the *dual* of a set function $f : 2^N \to \mathbb{R}$.[22]

Definition 22. For any bundle $S \in 2^N$, given $f : 2^N \to \mathbb{R}$ with $f(\emptyset) = 0$, define the transformation:

$$g(f, S) = f(N) - f(S^c)$$

$g(f, \cdot)$ is called the *dual* of f. Note that $g(f, N) = f(N)$.

Definition 23 (Set-cover submodularity)**.** Given a finite set N, and a function $f : 2^N \to \mathbb{R}$, f is *set-cover submodular* if $\forall\ S_1, S_2 \in 2^N$ with $S_1 \cup S_2 = N$

$$f(S_1) + f(S_2) \geq f(S_1 \cup S_2) + f(S_1 \cap S_2)$$

If the inequality sign in the above equation is reversed, f is *set-cover supermodular*, and if the equation holds with equality f is *set-cover modular*.

Note that set-cover submodularity is weaker than submodularity, because it is only required for every two subsets of N the union of which fully covers N. In particular, set-cover sub-

[22] This is standard in matroid theory, see e.g. [18] or [14].

modularity does not imply subadditivity. However, the following lemma exhibits a connection between superadditive and set-cover submodular functions.[23]

Lemma 5. *Given a superadditive (subadditive) function* $v^0 : 2^N \to \mathbb{R}$*, its dual* $c^*(v^0, \cdot)$ *is set-cover submodular (set-cover supermodular).*

The following proposition relates SY's and my model.

Proposition 4. *Let the seller's preferences be represented by a reserve price function* $v^0 : 2^N \to \mathbb{Z}_+$*,* $v^0(\emptyset) = 0$*. The seller's objective maximising revenue as defined in SY is equivalent to maximising quasi-linear profits with cost function* $C^0 : \mathbb{K} \to \mathbb{Z}_+$*, where*

(i) marginal costs $c^0 : 2^N \to \mathbb{Z}_+$ *are such that* c^0 *is the dual of* v^0

(ii) $C^0(\delta) = c^0\left(\bigcup_{S\in\delta} S\right)$

From (i) and Lemma 5 it follows that c^0 is set-cover submodular. A set-cover submodular cost function can have strictly subadditive and strictly superadditive elements, as shown in example 3 in the appendix. Property (ii) in Proposition 4 states that the seller in SY only cares about which objects are contained in the set of objects she sells overall, in contrast to my seller whose preferences take into account how objects are partitioned among buyers. Thus, even with identical marginal cost functions, their and my ascending auction may generate different results.

4.2 The ascending auction and equilibrium existence

I use a modified version of the basic ascending procedure by SY to determine an indivisible, package-linear Walrasian equilibrium. The rules of my ascending procedure are identical to SY on the bidders' side, but differ for the seller. Bidders bid straightforwardly in this auction: in each round, every bidder bids as if the current round was the last round of the auction, i.e. he maximises his profits given the current auction prices. Let $p(t, S)$ denote the price of bundle $S \in 2^N$ at time t. Bidder l bids straightforwardly with respect to his valuation function v^l if at every time $t \in \mathbb{Z}_+$ for any price vector (pricing function) $p(t)$, he bids $A_l(t) \in D^l(p(t)) = \arg\max_{S\subseteq N}\left\{v^l(S) - p(t, S)\right\}$. I require $A_l(t) = \emptyset$ when $\emptyset \in D^l(p(t))$.

In the ascending procedure the seller states her supply and bidders state their demand at current auction prices. If a package is overdemanded, the price on this package is increased by one in the next round. The procedure stops as soon as no package is overdemanded. Formally, the procedure is described as follows:

Step 1: The seller states her initial reserve prices c^0, and the auctioneer fixes the initial price vector $p(0) : 2^N \to \mathbb{Z}$ such that $p(0, S) = c^0(S)$ for every $S \in 2^N$. Set $t = 0$ and continue with step 2.

Step 2: In each round of the auction process $t = 0, 1, \ldots$, the auctioneer announces the current price vector $p(t)$ and chooses a supply set $\delta \in S(p(t))$. Subsequently, every bidder l reports his demand $A_l(t)$ given the current price vector $p(t)$. Then, the auctioneer determines if any of the packages are overdemanded. Package S is overdemanded with respect to supply set δ and reported demand bundles $A_l, l \in \mathcal{L}$, if it is demanded by more than one bidder, or demanded by

[23] Set-cover submodular and set-cover supermodular functions can both be subadditive or superadditive.

one bidder but not in the supply set δ. If no package is overdemanded, continue with step 3. If some package is overdemanded, increase the price of every package S that is overdemanded with respect to δ and $A_l, l \in \mathcal{L}$ by 1, i.e. $p(t+1,S) = p(t,S)+1$. Every other price $p(t,S'), S' \neq S$ remains the same. Increase the counter $t = t+1$ and continue with step 2.

Step 3: Every package $S \in A_l(t)$ is allocated to bidder l at price $p(t,S)$. If there is a package $B \in \delta(t)$ that is not demanded by any bidder at $p(t)$, B remains with the seller if $p(t,B) = c^0(B)$. If $p(t,B) > c^0(B)$, B is allocated for price $p(t,B)$ to a bidder who reported B in his demand set in a previous round was among the last to forfeit B. The auction terminates.

Theorem 2. If all bidders bid straightforwardly in this auction setting, the ascending auction terminates in a package-linear Walrasian equilibrium after a finite number of rounds.

Equilibrium existence follows for the model from section 2 when restricting supply to one unit per variety.

Corollary 4. When $\Omega_j = 1, j \in N$, there always exists a package-linear Walrasian equilibrium.

Example 4 in the appendix further illustrates that my ascending auction and SY's can lead to different auction outcomes, even with identical marginal cost values for each package. I also demonstrate that my ascending auction generalises SY's procedure, because their seller is similar to the buyers: she does not care about the partitioning of a set of varieties in her supply, as long the reserve price is met. Thus, their seller can "disguise" as a buyer.

4.3 The ascending buy-back auction

Suppose the seller from SY's model participates in my auction as a bidder. My seller's own supply is set to zero, but the SY-seller gives her supply to my seller, and then participates in the ascending procedure attempting to buy back her objects. If she wins a bundle, this bundle is effectively not sold, and she receives the price of every bundle sold to a different bidder. I call this the "buy-back" auction to distinguish it from the standard SY-ascending auction. The seller attempting to buy back her objects is called the SY-seller. I denote objects relating to the SY-procedure with a superscript SY and show the following:

Lemma 6. *SY's auction and the buy-back auction terminate in the same allocation (up to ties).*

Proposition 5. *If the buy-back auction starts at prices $p(-1,S) = v^0(S) - 1\ \forall S \in 2^N$, there exists a price path that is identical in the buy-back auction and the SY ascending auction, and the final auction prices at the end of this price path support the same allocation in both procedures.*

Hence, the buy-back auction strictly generalises SY by adding my seller with subadditive marginal costs over the partitioning of supply.

5 Conclusion

In many combinatorial auctions and markets, it is natural for the seller to have a preference over the partitioning of objects among buyers. I establish a new graph structure to express such preferences, while maintaining the tractability of the allocation problem through a linear programming formulation. To the best of my knowledge, mine is the first Walrasian-equilibrium auction setting in which the seller can express preferences over allocations.

I characterise the existence of an indivisible Walrasian equilibrium with package-linear pricing. Equilibrium prices have a modular structure: the price of a given package is tied to the marginal costs of varieties or subsets of varieties contained in the given package, according to the cost relations specified by the seller. In an environment with one unit per variety for sale, superadditive utilities for buyers, and subadditive cost for the seller, I show that an integer-valued package-linear pricing equilibrium always exists. I also uncover a relationship between valuations and costs defined on sets of indivisible goods. In order to characterise properties of these value and cost functions, I derive a new relationship between superadditive (subadditive) set functions and their duals.

While my results are theoretical, I aim to inspire practical auction design. My methods allow the seller to express vastly richer preferences than described in the previous literature and in common auction design. At the same time, I guarantee transparency in pricing between different packages that contain identical objects, and I address another important concern in combinatorial auctions with non-linear pricing: if not all bundles receive bids, sensible bundle prices can be easily constructed. Allowing sellers in practice to express richer preferences will likely improve efficiency, and a more accurate representation of marginal costs may be helpful to generate higher revenue.

Finally, the graph structure I developed in this paper may be of independent interest in other allocation problems.

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Appendix

A Proofs

Proof of Lemma 1. This proof proceeds by analogy with [29] and [5] who show the result for an environment with a single unit per variety for sale, and without explicit seller's costs. Let $\mathcal{A}$ denote the universe of all feasible allocations, i.e.

$$\mathcal{A} := \left\{ \pi \in \mathbb{Z}_+^{|2^N| \times |\mathcal{L}_0|} : \mathop{+}_{l \in \mathcal{L}} \pi(l) = \mathbf{k} \text{ for some } \mathbf{k} \in \mathbb{K}, \text{ and } \pi(0) = \mathcal{N} - \mathbf{k}^* \right\}$$

(p^*, π^*) is a package-linear pricing Walrasian equilibrium, so for any bidder $l \in \mathcal{L}$ and any allocation $\pi' \in \mathcal{A}$, we have

$$V^l(\pi^*(l)) - \sum_{S \in \pi^*(l)} p^*(S) \geq V^l(\pi'(l)) - \sum_{S \in \pi'(l)} p^*(S)$$

Let $+_{l \in \mathcal{L}} \pi^*(l) = \mathbf{k}$ and $+_{l \in \mathcal{L}} \pi'(l) = \mathbf{k}'$. I sum over $l \in \mathcal{L}$ and add and subtract the seller's cost

$$\begin{aligned} &\sum_{l \in \mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k}) - \left(\sum_{l \in \mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}') \right) \\ \geq &\sum_{l \in \mathcal{L}} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\mathbf{k}) - \left(\sum_{l \in \mathcal{L}} \sum_{S \in \pi'(l)} p^*(S) - C^0(\mathbf{k}') \right) \end{aligned} \tag{2}$$

Because $\pi^* \in S(p^*)$, we have, for all $\pi' \in \mathcal{A}$,

$$\sum_{l \in \mathcal{L}} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\mathbf{k}) \geq \sum_{l \in \mathcal{L}} \sum_{S \in \pi'(l)} p^*(S) - C^0(\mathbf{k}')$$

From equation (2), it follows that, for all $\pi' \in \mathcal{A}$,

$$\sum_{l \in \mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k}) - \left(\sum_{l \in \mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}') \right) \geq 0$$

and so π^* is efficient.

Now let π' be an efficient allocation. Then $V(\mathcal{N}) = \sum_{l \in \mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}')$. It also holds that $V(\mathcal{N}) = \sum_{l \in \mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k})$ because π^* is efficient as part of the equilibrium. The equilibrium is also bidder-optimal and seller-optimal, given prices. Thus, we obtain the following two inequalities:

$$\begin{gathered} \mathcal{V}^l(p^*) \geq V^l(\pi'(l)) - \sum_{S \in \pi'(l)} p^*(S), \qquad \text{for all } l \in \mathcal{L} \text{ and} \\ \sum_{l \in \mathcal{L}} \sum_{S \in \pi^*(l)} p^*(S) - C^0(\mathbf{k}) = \Pi(p^*) \geq \sum_{l \in \mathcal{L}} \sum_{S \in \pi'(l)} p^*(S) - C^0(\mathbf{k}') \end{gathered}$$

Suppose that one of these two inequalities were strict, then we would obtain

$$\begin{aligned}
V(\mathcal{N}) &= \sum_{l\in\mathcal{L}} V^l(\pi^*(l)) - C^0(\mathbf{k}) \\
&= \sum_{l\in\mathcal{L}} \left[V^l(\pi^*(l)) - \sum_{S\in\pi^*(l)} p^*(S) + \sum_{S\in\pi^*(l)} p^*(S) \right] - C^0(\mathbf{k}) \\
&= \sum_{l\in\mathcal{L}} \mathcal{V}^l(p^*) + \Pi(p^*) \\
&> V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S) + \sum_{l\in\mathcal{L}} \sum_{S\in\pi'(l)} p^*(S) - C^0(\mathbf{k}') \\
&= \sum_{l\in\mathcal{L}} V^l(\pi'(l)) - C^0(\mathbf{k}') \\
&= V(\mathcal{N}),
\end{aligned}$$

This is a contradiction, and consequently it holds that

$$\begin{aligned}
\mathcal{V}^l(p^*) &= V^l(\pi'(l)) - \sum_{S\in\pi'(l)} p^*(S), \qquad \text{for all } l\in\mathcal{L} \text{ and },\\
\Pi(p^*) &= \sum_{l\in\mathcal{L}} \sum_{S\in\pi'(l)} p^*(S) - C^0(\mathbf{k}'), \quad \text{i.e., } \pi' \in S(p^*).
\end{aligned}$$

It follows that (p^*, π') is also a package-linear pricing Walrasian equilibrium. $\square$

Proof of Proposition 1. To prove Proposition 1 I show two things:

(a) There exists a mapping from V^l to a list $\{v_S^i, S\in 2^N, i\le \overline{M}\}$.

(b) Suppose we have an allocation to bidder l given by $x_S^i, S\in 2^N, i\in\mathcal{I}^l$ and mapped into the multiset $\mathbf{k}^l$. Then the bidder's value V^l from this allocation is identical to the value entering the auctioneer's objective function.

(a) is straightforward given the structure on V^l, which is defined by bidder l's marginal values $v^l(q,S), S\in 2^N, q\le \overline{M}$. The bidder can simply make a set of bids $\{(v^l(q,S),1)_{S\in 2^N}, q = 1,...,\overline{M}\}$. (b) follows from the bidder's aggregation of marginal values. The value entering the welfare maximising auctioneer's objective is simply given by $\sum_{S,i\in\mathcal{I}^l} v_S^i x_S^i$. An allocation to a bidder is easily aggregated to the corresponding multiset such that $k_S^l = \sum_{i\in\mathcal{I}^l} x_S^i$. For a given allocation $x_S^i, S\in 2^N, i\in\mathcal{I}^l$, the bidder's overall value is defined as

$$V^l(\{x_S^i\}) = \max_{\{x_{q,S}\}} \sum_{S\in 2^N}\sum_{q=1}^{\overline{M}} v^l(q,S)x_{q,S} \quad \text{s.t.} \ \sum_{q=1}^{\overline{M}} x_{q,S} \le \sum_{i\in\mathcal{I}^l} x_S^i, \ \sum_{S\in 2^N} x_{q,S} \le 1 \ \forall q, x_{q,S}\in\{0,1\}$$

Note that every allocation $x_S^i, S\in 2^N, i\in\mathcal{I}^l$ respects $\sum_S x_S^i \le \kappa^i$ because of Definition 18, and we can normalise $\kappa^i = 1$. Thus, $x_{q,S} = x_S^i$ is feasible in bidder l' value aggregation for some order of bids in $\mathcal{I}^l$. It follows immediately that $V^l(\{x_S^i\}) \ge \sum_{S,i\in\mathcal{I}^l} v_S^i x_S^i$. Suppose that $V^l(\{x_S^i\}) > \sum_{S,i\in\mathcal{I}^l} v_S^i x_S^i$. Then, the order in the marginal value aggregation of at least two packages must have switched compared to the auctioneer's assignment $\mathcal{I}^l$. But all marginal values were available to the auctioneer to choose from, i.e. the bidder's marginal value aggregation was a feasible allocation in the social welfare maximisation problem. Thus $\{x^i_S\}$ was not optimal. $\square$

Proof of Proposition 2. Because the bidder's marginal values are superadditive between varieties and disjoint packages, the bidder could never gain from repackaging a given package into two (or more) disjoint subsets of that package. Consequently, repackaging can only occur if the bidder is allocated two disjoint packages and merges them.

Because the seller's marginal cost is subadditive between varieties and disjoint packages, she would never have a strict incentive to sell two disjoint packages separately if she could sell them bundled in one larger package. For contradiction, suppose that the bidder received two disjoint packages S_1 and S_2 on two different bids. Let $v^l(S_1, q)$ and $v^l(S_2, q')$ denote the corresponding marginal values submitted, with $q \geq q'$. Then, by assumption of MU-concave superadditive preferences, it must be that

$$v^l(S_1 \cup S_2, q) \geq v^l(S_1, q) + v^l(S_2, q') \tag{3}$$

By subadditivity of the seller's marginal cost, we have

$$c^0(q, S_1 \cup S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) \leq c^0(q, S_1, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) + c^0(q, S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) \tag{4}$$

Adding equation 3 and 4 yields the auctioneer's (partial) social welfare objective

$$\begin{aligned} v^l(S_1 \cup S_2, q) - c^0(q, S_1 \cup S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) &\geq v^l(S_1, q) + v^l(S_2, q') \\ &\quad - c^0(q, S_1, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}) - c^0(q, S_2, \mathbf{k}_{-(S_1 \cup S_2, S_1, S_2)}), \end{aligned}$$

hence a contradiction to allocating S_1 and S_2 on two different bids. In case of indifference, the bidder must also be indifferent between repackaging S_1 and S_2 to $S_1 \cup S_2$, and leaving them as separate bundles. $\square$

Proof of Lemma 2. First, I show that given a supply function graph G, any supply function allocation $(y^q_S)_{S \in 2^N, q \leq \bar{q}}$ maps into a unique vector $(k_S)_{S \in 2^N}$. The proof is constructive; I provide an algorithm to determine k_S.

ALGORITHM 1: Construct multiset from allocation on SFG

Input : supply function graph $G = (V, E)$, allocation $(y_S^q)_{S \in G, q \leq \bar{q}}$
Output: $(k_S)_{S \in 2^N}$
Initialise list of successfully visited nodes $\mathscr{V} := \emptyset$.
while $V \neq \mathscr{V}$ **do**
  Select a node $S \in V \setminus \mathscr{V}$
  if $\exists A \in V \setminus \mathscr{V} : \exists(A...S)$ **then**
    skip S ;
  else
    $k_S = \sum_{q=1}^{\bar{q}} y_S^q - \sum_{A \in \mathscr{V}: \exists(A...S)} k_A$;
    mark S as successfully visited: $\mathscr{V} = \mathscr{V} \cup S$
  end
end

To proof that algorithm 1 constructs a unique image from any permissible input, I demonstrate that (a) algorithm 1 stops, (b) each node S is successfully visited at some point, (c) the time at which S is successfully visited is irrelevant. That the mapping is linear is obvious from the definition of k_S in the algorithm.

(a) and (b) follow because Assumption 3(i) implies that there are no cycles in G. Thus, as long as $V \neq \mathscr{V}$, there exists some $S \in V \setminus \mathscr{V}$ for which the if-condition is false. Consequently, all nodes are added to $\mathscr{V}$ at some point. It is without loss of generality to assume that the algorithm does not get stuck in a trivial loop, i.e. it does not select a sequence of nodes that allow no successful visit and are skipped and revisited indefinitely. (c) follows because once the if-condition is false for a given node S, the set of nodes $A \in \mathscr{V} : \exists(A...S)$ remains unaltered. Once S *could* be successfully visited, is does not matter when it is actually selected for the successful visit, i.e. other nodes may be selected first.

The reverse mapping is straightforward. Given a supply function graph G, a multiset $\mathbf{k} = (k_S)_{S \in 2^N}$ maps in the following supply function allocation: for all $S \in 2^N$, $y_S^{q_S} = 1 \; \forall q_S \leq \sum_{A:\exists(A...S)} k_A$ and $y_S^{q_S} = 0 \; \forall q_S > \sum_{A:\exists(A...S)} k_A$. □

Proof of Proposition 3. To prove this proposition I use a slightly different formulation of the auctioneer's partitioning problem. The supply function graph required that whenever a step on supply function f_S is allocated, a step on every supply function f_γ for all $\gamma : \exists(S...\gamma)$ is allocated as well. Therefore, I reformulate objective and constraints in the following way: Summing over the steps q, I introduce the identity of the package due to which a step on a supply curve is allocated. Formally, I call these steps $q'(S)$; they are the steps which are allocated on a given supply curve due to the allocation of package S. I suppress S in the problem; but it is important that the steps q' are different for each S.

Note that this problem formulation is not practical for solving the auction because the q' would have to be linked to the corresponding package S, and additional rules would have to be specified for which packages are allocated on which steps. However, this alternative problem formulation is useful to find out more about the price structure.

With the new definition of q', it holds that $\sum_{q'} y_S^{q'} = Y_S$. I rewrite LP as follows:

LP

$$\max_{\{x^i_S, y^{q'}_S\}} \left[\sum_{i,S} v^i_S x^i_S - \sum_{q'} y^{q'}_S \left(\sum_{\gamma:\exists(S\ldots\gamma)} \mu^{q'}_\gamma \right) \right]$$

$$\text{s.t.} \begin{cases} \sum_S x^i_S & \leq & \kappa^i & \forall i & b_i & \text{constraint on bid size} \\ y^{q'}_S & \leq & l^{q'}_\gamma & \forall q', \forall S, \forall \gamma : \exists(S\ldots\gamma) & u^{q'}_\gamma & \text{constraint on step size} \\ \sum_i x^i_S - \sum_{q'} y^{q'}_S & \leq & 0 & \forall S & z_S & \text{supply constraint} \end{cases}$$

and all variables are non-negative. Then, the corresponding dual problem is

DLP

$$\min_{\{b^i, z_S, u^q_S\}} \left[\sum_i \kappa^i b^i + \sum_{q', S, \gamma:\exists(S\ldots\gamma)} l^{q'}_\gamma u^{q'}_\gamma \right]$$

$$\text{s.t.} \begin{cases} b^i + z_S & \geq & v^i_S & \forall i, S & x^i_S & \text{surplus constraint} \\ \sum_{\gamma:\exists(S\ldots\gamma)} u^{q'}_\gamma - z_S & \geq & -\sum_{\gamma:\exists(S\ldots\gamma)} \mu^{q'}_\gamma & \forall q', \forall S & y^{q'}_S & \text{marginal cost constraint} \end{cases}$$

and all variables b^i, $u^{q'}_S$ and z_S are non-negative. From the marginal cost constraint it follows that $z_S \leq \sum_{\gamma:\exists(S\ldots\gamma)} \left(u^{q'}_\gamma + \mu^{q'}_\gamma \right)$ for all S and for all q'. Complementary slackness implies that, if $y^{q'}_S > 0$, $z_S = \sum_{\gamma:\exists(S\ldots\gamma)} \left(u^{q'}_\gamma + \mu^{q'}_\gamma \right)$. As described above, $q' = q'(S)$ refers to the specific steps on which package S is allocated. However, in the original problem formulation, it does not matter which package is allocated on which step, because only the entirety of allocated steps is relevant for the seller's cost. Because $q'(S)$ could indeed be any of the steps on f_S, on which a positive amount is allocated, the equality holds for all $q_\gamma \leq \tilde{q}_\gamma$, when $y^{\widetilde{q_\gamma}}_\gamma \neq 0$. The inequality holds for any $q \leq \bar{q}$ because of the marginal cost constraint. □

Proof of Corollary 1. From CC 7 and CC 8 follows

$$\mu^q_S \leq \psi(\{z_S\}) \leq \mu^{q+1}_S$$

Together with CSC 4, (i) follows. (ii) is immediate from Proposition 3 and CSC 4. □

Proof of Theorem 1. First, I note the complementary slackness conditions (CSC) from LP and DLP. The CSC are standard in linear programming; they are derived and used by analogy with [1], and the subsequent argument follows [5].

CSC 1 : If $x^i_S \neq 0$, then $b^i = v^i_S - z_S$, for all $i \in \mathcal{I}, S \subseteq N$.

CSC 2 : If $\sum_S x^i_S < \kappa^i$, then $b^i = 0$, for all $i \in \mathcal{I}$.

CSC 3 : If $\sum_i x^i_S - \phi(\{y^q_S\}) < 0$, then $z_S = 0$, for all $S \subseteq N$.

CSC 4 : If $y^q_S \neq 0$, then $u^q_S + \mu^q_S = \psi(\{z_S\})$, for all $q \leq \bar{q}, S \subseteq N$.

CSC 5 : If $y^q_S < l^q_S$, then $u^q_S = 0$, for all $q \leq \bar{q}, S \subseteq N$.

I establish the following lemma to help characterise a solution of LP.

Lemma 7. *If $\{x_S^i, y_S^q\}$ and $\{b^i, u_S^q, z_S\}$, respectively, are solutions to LP and DLP, $\{x_S^i, y_S^q\}$ and $\{b^i, u_S^q, z_S\}$ satisfy the following "characteristic conditions":*

CC 1 : $\sum_S x_S^i \leq \kappa^i$, *for all* $i \in \mathcal{I}$.

CC 2 : $y_S^q \leq l_S^q$, *for all* $q \leq \bar{q}, S \subseteq N$.

CC 3 : $\sum_{i,\gamma:\exists(\gamma\ldots S)} x_\gamma^i \leq \sum_q y_S^q$, *for all* $S \subseteq N$.

CC 4 : If $z_S > 0$, *then* $\sum_{i,\gamma:\exists(\gamma\ldots S)} x_\gamma^i = \sum_q y_S^q$, *for all* $S \subseteq N$.

CC 5 : If $v_S^i - z_S < \max_{S' \neq S}\{v_{S'}^i - z_{S'}, 0\}$, *then* $x_S^i = 0$, *for all* $i \in \mathcal{I}, S \subseteq N$.

CC 6 : If $\max_S\{v_S^i - z_S\} > 0$, *then* $\sum_S x_S^i = \kappa^i$, *for all* $i \in \mathcal{I}, S \subseteq N$.

CC 7 : If $\mu_S^q < \psi(\{z_S\})$, *then* $y_S^q = l_S^q$, *for all* $S \subseteq N$.

CC 8 : If $\mu_S^q > \psi(\{z_S\})$, *then* $y_S^q = 0$, *for all* $S \subseteq N$.

I also note again Corollary 1(ii): If $y_S^q > 0$, then for all $q' \leq q$, it holds that

$$\psi(\{z_S\}) = z_S - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S} \left(u_\gamma^{q'} + \mu_\gamma^{q'}\right) \tag{5}$$

The subsequent lemma makes the well known connection between a primal-dual solution and Walrasian equilibrium.

Lemma 8. *Prices $\{z_S\}$ given by the solution of DLP support the allocation $\{x_S^i, y_S^q\}$ given by the solution of LP as a package-linear pricing Walrasian equilibrium.*

Proof. Assume the allocation $\{x_S^i, y_S^q\}$ and prices $\{z_S\}$ are solutions of LP and DLP as defined above. Then, by Lemma 7, conditions CC 1 - CC 8 hold. I show that CC 1 - CC 8, together with the CSC and the constraints of LP and DLP, imply that the $\{z_S\}$ support $\{x_S^i, y_S^q\}$ as a package-linear pricing Walrasian equilibrium. I prove (a) that there is no surplus improvement possible for any bid i, (b) that for a bidder who received a multiset of packages no surplus improvement is possible from receiving a different partition of the multiset of packages, (c) that, given a partition of the allocated supply and given the final auction prices, the seller cannot improve her profit by allocating more or less of a given package, and (d) that, given a partition of the allocated supply and given the final auction prices, the seller cannot improve her profit by choosing a different partition of supply.

(a): If there is no positive surplus on any of the packages of a bid i, CC 5 ensures that this bid is allocated nothing. CC 6 implies that, if a strictly positive surplus can be made on any of the packages, the bid is allocated the maximum quantity bid for, and from CC 5 follows that the maximum quantity is allocated only on packages that maximise the bid's surplus. CC 1 ensures that the maximum quantity bid for is always respected.

(b): A bidder does not desire to repackage goods that he received in a certain partition (Proposition 2), or is prohibited from doing so, i.e. if he received the multiset of objects $\mathbf{k}$, he does not rearrange single varieties to obtain a different multiset $\mathbf{k}'$ with $\mathbf{k}^* = \mathbf{k}'^*$. However, I still have to show that the bidder would not be strictly better off with a different partitioning of supply at the given auction prices. Let $\mathbf{k}$ denote the multiset of objects received by bidder l through the set of winning bids $\mathcal{W}^l$ (this set contains only those bids of l with a strictly positive allocation), i.e. $\mathbf{k} = \left(\sum_{i\in\mathcal{W}^l} x^i_S\right)_{S\in 2^N}$. First, recall Proposition 1: given an allocation $\mathbf{k}$, the bidder's value derived from $\mathbf{k}$ equals its contribution to social welfare as it appears in the auctioneer's objective, i.e. $V^l(\mathbf{k}) = \sum_{i\in\mathcal{W}^l,S} x^i_S v^i_S$. Then, the bidder's utility is given by $u^l(\mathbf{k}, z) = \sum_{i\in\mathcal{W}^l,S}(v^i_S - z_S)x^i_S$. Suppose there is a different partitioning $\mathbf{k}' \neq \mathbf{k}$ where $\mathbf{k}^* = \mathbf{k}'^*$, with a corresponding set of winning bids $\mathcal{W}'^l$ and allocation $\{x'^i_S\}$, which gives bidder l strictly higher utility, i.e. $u(\mathbf{k}', z) = \sum_{i\in\mathcal{W}'^l,S}(v^i_S - z_S)x'^i_S > u(\mathbf{k}, z)$. There exist at least two bids $i, j \in \mathcal{W}'$, for which $x'^i_S \neq x^i_S$ and $x'^j_S \neq x^j_S$ for some S. Let $\mathcal{D} := \{i \in \mathcal{W}^l \cap \mathcal{W}'^l : \exists S \text{ s.t. } x'^i_S \neq x^i_S\}$, i.e. $\mathcal{D}$ contains the winning bids in $\mathcal{W}$ which are also in $\mathcal{W}'$, but with some allocations shifted to different packages. By the surplus constraint of DLP we have $v^i_S - z_S \leq b^i$ for all $i \in \mathcal{D}, S \in 2^N$; so no additional surplus can be generated through shifting allocations on those bids. Let $\mathcal{C} := \mathcal{W}' \setminus \mathcal{W}$, i.e. the bids in $\mathcal{W}'$ that were not winning in the auction. By the contraposition of CC 6 it must hold that $z_S \geq v^i_S$ for all $i \in \mathcal{C}, S \in 2^N$. It follows that $u^l(\mathbf{k}', z) = \sum_{i\in\mathcal{W}',S}\left(v^i_S - z_S\right)x'^i_S \leq \sum_{i\in\mathcal{W}^l,S} b^i = u^l(\mathbf{k}, z)$, i.e. bidder l maximises his surplus with the multiset received in the partition prescribed by the auction.

(c): If a step on supply curve f_S is allocated, i.e. $y^q_S > 0$, then CC 8 implies $\psi(\{z_S\}) \geq \mu^q_S$. Together with equation (5) we have $z_S \geq \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S}(u^q_\gamma + \mu^q_\gamma) + \mu^q_S$, i.e. the seller always sells package S at a weakly positive surplus. Furthermore, if $z_S > 0$, it follows by CC 4 that $\sum_{i,\gamma:\exists(\gamma\ldots S)} x^i_\gamma = \sum_q y^q_S \ \forall S \subseteq N$, i.e. the amount of all packages with cost relation to S sold equals $\sum_q y^q_S$. Suppose $z_S > \mu^q_S + \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S}(u^q_\gamma + \mu^q_\gamma) \Leftrightarrow z_S - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S}(u^q_\gamma + \mu^q_\gamma) > \mu^q_S$, i.e. a strictly positive surplus is made on package S. Then, Proposition 3(i) implies that

$$\sum_{\gamma:\exists(S\ldots\gamma)}\left(u^q_\gamma + \mu^q_\gamma\right) - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S}\left(u^q_\gamma + \mu^q_\gamma\right) \geq z_S - \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S}\left(u^q_\gamma + \mu^q_\gamma\right) > \mu^q_S$$

and thus $u^q_S + \mu^q_S > \mu^q_S$. With CSC 5, $u^q_S > 0$ implies that the entire supply function step is allocated. Conversely, let $z_S < \mu^q_S + \sum_{\gamma:\exists(S\ldots\gamma),\gamma\neq S}(u^q_\gamma + \mu^q_\gamma)$, i.e. a strict loss would be made on a package. For contradiction, suppose $y^q_S > 0$. Then, by equation (5), $\psi(\{z_S\}) < \mu^q_S$, and therefore, by CC 8, $y^q_S = 0$, a contradiction. Hence, the supply function step cannot be not allocated at all. Finally, by CC 2, a supply step is never allocated more than its maximum step size.

(d) I claim that, given a partition of supply that is a solution to LP, the seller cannot improve her profit by choosing a different partition of supply. To see this, recall Lemma 2, which states that the mapping from the allocation on the supply function graph to a multiset of packages, i.e. $(y^q_S)_{S\in 2^N, q\leq\bar{q}} \to (Y_S)_{S\in 2^N}$, is one-to-one. In (c), I have shown that the seller cannot improve upon the allocation on the supply function graph, given the dual prices. Then, it is immediate that the resulting partition of allocated supply is seller-optimal also. □

Using the lemma above, a standard argument establishes the theorem. O_P denotes the value of an optimal solution to P, O_{LP} the value of an optimal solution to LP, and O_{DLP} the value of an optimal solution to DLP. First, let $O_P = O_{LP}$. Then, there exists $\{x^i_S, y^q_S\}$ as an optimal solution to IP and LP, and $\{x^i_S, y^q_S\}$ is efficient. By Lemma 8, the dual variables $\{z_S\}$ of DLP support this allocation as a package-linear pricing Walrasian equilibrium.

Now suppose there exists an equilibrium, i.e. prices $\{z_S\}$ that support $\{x^i_S, y^q_S\}$ as an equilibrium allocation. By Lemma 1, the allocation is efficient. Let $b_i := v^i_S - z_S$ for all $i \in \mathcal{I} : x^i_S > 0$, and let $u^q_S := \psi_S(\{z_S\}) - \mu^q_S$ for all $q \leq \bar{q}_S : y^q_S > 0, S \in 2^N$. Note that the u^q_S are defined recursively by equation 5: for $j \in N$, we have $\psi(\{z_j\}) = z_j$, so $u^q_j := z_j - \mu^q_j$ for all $q \leq \bar{q}_j, j \in N$. Given the u^q_j, one can go on to define u^q_S for all S, for which it holds that for all $\{\gamma : \exists(S...\gamma), \gamma \neq S\}$, $\gamma \in N$, and so on. The u^q_S are the seller's surplus on each individual supply step, and the b^i are the buyers' surplus on each bid. Because the $\{z_S\}$ are competitive equilibrium prices, it must be that $u^q_S, b^i \geq 0$. Efficiency implies that

$$\textstyle\sum_{i,S} v^i_S x^i_S - \sum_{S,q} \mu^q_S y^q_S \quad \geq \quad \sum_{i,S} v^i_S (x^i_S)' - \sum_{S,q} \mu^q_S (y^q_S)' \quad \forall \quad (x^i_S)', (y^q_S)'$$

i.e. u^q_S and b^i are feasible in DLP *and* $\{x^i_S, y^q_S\}$ are optimal in LP. By strong duality it holds that $O_{LP} = O_{DLP}$. Thus, we have

$$\begin{aligned}
O_{LP} &= O_{DLP} \\
&\overset{(1)}{\leq} \sum_i \kappa^i b^i + \sum_{q,S} l^q_S u^q_S \\
&\overset{(2)}{=} \sum_{S,i:x^i_S=\kappa^i} \kappa^i (v^i_S - z_S) + \sum_{q,S:y^q_S=l^q_S} l^q_S (\psi_S(\{z_S\}) - \mu^q_S) \\
&\overset{(3)}{=} \sum_{S,i:x^i_S=\kappa^i} \kappa^i (v^i_S - z_S) + \sum_{q,S:y^q_S=l^q_S} l^q_S (z_S - \sum_{\gamma:\exists(S...\gamma),\gamma\neq S} \left(u^{q'}_\gamma + \mu^{q'}_\gamma\right) - \mu^q_S) \\
&\overset{(4)}{=} \sum_{S,i} x^i_S (v^i_S - z_S) + \sum_S Y_S z_S - \sum_{q,S} y^q_S \mu^q_S \\
&\quad \underbrace{+ \sum_S \left(\sum_{\gamma:\exists(\gamma...S),\gamma\neq S} Y_\gamma \right) z_S - \sum_S \left(\sum_{\gamma:\exists(\gamma...S)} Y_\gamma \right) \left(\sum_{\gamma:\exists(S...\gamma),\gamma\neq S} \left(u^{q'}_\gamma + \mu^{q'}_\gamma\right) \right)}_{:=F} \\
&\overset{(5)}{=} \sum_{i,S} v^i_S x^i_S - \sum_{S,q} \mu^q_S y^q_S \\
&\overset{(6)}{=} O_{IP}
\end{aligned}$$

(1) follows from DLP's objective function. (2) follows by definition of b_i and u^q_S above. (3) follows by equation (5). (4) follows because the amount on a given supply curve allocated equals the sum of the quantities of all packages sold that have a cost relation to the given supply curve, i.e.

$$\sum_{q,S:y^q_S=l^q_S} l^q_S z_S = \sum_S Y_S z_S + \sum_S \left(\sum_{\gamma:\exists(\gamma...S),\gamma\neq S} Y_\gamma \right) z_S$$

(5) follows because $\sum_i x^i_S = Y_S$ for all S for which $z_S > 0$, and because, using Proposition 3

(substituting for z_S) and simplifying sums, we have

$$F = \sum_S Y_S \left(\sum_{\gamma:\exists(S\ldots\gamma)} \left(u_\gamma^{q'} + \mu_\gamma^{q'} \right) \right) - \sum_S \left(\sum_{\gamma:\exists(\gamma\ldots S)} Y_\gamma \right) \left(\mu_S^{q'} + u_S^{q'} \right) = 0$$

The previous equation is equal to zero simply because of the distributive law in arithmetic. Finally, (6) follows from the definition of the auctioneer's indivisible allocation problem P.

I have shown that $O_{LP} \leq O_{IP}$. It also holds that $O_{LP} \geq O_{IP}$ because any solution of IP is feasible in LP, and the claim follows.

□

Proof of Lemma 7. CC 1 and CC 2 follow from the constraint on the bid size, and the constraint on the step size in LP. CC 3 is obtained by summing up the bundle S supply constraints from LP. In particular, I sum over all $\gamma : \exists(\gamma\ldots S)$. By definition of the supply functions, $\sum_{\gamma:\exists(\gamma\ldots S)} Y_S = \sum_q y_S^q$ for any package $S \subseteq N$. CC 4 follows from CSC 3: the contrapositive of CSC 3 states that, if $z_S > 0$, $\sum_i x_S^i - \phi(\{y_S^q\}) = 0$ (> 0 is ruled out by the supply constraint of LP). If $z_S > 0$, then by Proposition 3, Requirement 1, and because $u_S^q \geq 0 \; \forall S, y$, it holds that $z_\gamma > 0$ for all $\gamma : \exists(\gamma\ldots S)$ *on which some* y_γ^q is allocated. Therefore, I can take the sum of tight supply constraints across $\gamma : \exists(\gamma\ldots S)$ (if $x_\gamma^i > 0$ for some i, then we must also have $y_\gamma^q > 0$ for some q); we obtain CC 4. CC 5 is derived from CSC 1: Assume $x_S^i > 0$ and $x_{S'}^i > 0$ for some $S \neq S'$. Then, by CSC 1, $v_S^i - z_S = v_{S'}^i - z_{S'} = b^i$. So if $v_S^i - z_S < v_{S'}^i - z_{S'}$ for some S', then $x_S^i = 0$. Also note that it always holds that $b^i \geq 0$ and $b^i \geq v_S^i - z_S$, by the surplus constraint of DLP. So if $v_S^i - z_S < 0$, then it cannot be that $v_S^i - z_S = b^i$, and thus $x_S^i = 0$. Together, we obtain CC 5. $b^i \geq v_S^i - z_S$ also implies that, if $\max_S\{v_S^i - z_S\} > 0$, then $b^i > 0$. The contrapositive of CSC 2 then implies $\sum_S x_S^i = \kappa^i$, and thus CC 6. The marginal cost constraint of DLP is $u_S^q \geq \psi(\{z_S\}) - \mu_S^q$. Thus, if $\mu_S^q < \psi(\{z_S\})$, then $u_S^q > 0$, and so CC 7 follows by the contrapositive of CSC 5. Finally, if $\mu_S^q > \psi(\{z_S\})$, then we cannot have $u_S^q = \psi(\{z_S\}) - \mu_S^q$, because u_S^q is positive. Hence, the contrapositive of CSC 4 implies $y_S^q = 0$, and therefore CC 8. □

Proof of Lemma 3. First, note that for complete CFG, the definition of marginal cost simplifies to

$$c^0(k, S, \mathbf{k}_{-(B_1,\ldots,B_t)}) = \sum_{\gamma \subseteq S} \Delta c(q_\gamma, \gamma), \qquad \text{where } q_\gamma := \sum_{A \supset \gamma, A \in 2^N \setminus (B_1,\ldots,B_t)} k_A + k$$

Let $B := (S_1 \cup S_2, S_1, S_2)$ and let $q_\gamma := \sum_{A \supset \gamma, A \in 2^N \setminus B} k_A + k$. Then we have, for any disjoint sets $S_1, S_2 \in 2^N$,

$$\begin{aligned}
& c^0(k, S_1, \mathbf{k}_{-B}) + c^0(k, S_2, \mathbf{k}_{-B}) \\
=& \sum_{\gamma \subseteq S_1} \Delta c\left(q_\gamma, \gamma\right) + \sum_{\gamma \subseteq S_2} \Delta c\left(q_\gamma, \gamma\right) \\
=& \sum_{\gamma \subseteq S_1 \cup S_2} \Delta c\left(q_\gamma, \gamma\right) - \sum_{\gamma \subseteq S_1 \cup S_2, \gamma \not\subseteq S_1, S_2} \Delta c\left(q_\gamma, \gamma\right) \\
\geq & \sum_{\gamma \subseteq S_1 \cup S_2} \Delta c\left(q_\gamma, \gamma\right) \\
=& c^0(k, S_1 \cup S_2, \mathbf{k}_{-S})
\end{aligned}$$

□

Proof of Lemma 4. Let $W \subseteq S \subseteq \delta \subseteq N$, and let $y(S, \delta, t, W)$ denote the amount allocated on supply function f_S that is due to the allocation of a bundle $\delta \supset_t W$, t levels above W. Let $dist(x, y) := ||x| - |y||$ for any $x, y \subseteq N$. I first establish a series of facts.

Fact (1). Given any reference supply set $W \subseteq S$, I can write the amount allocated on supply function f_S as

$$\sum_q y_S^q = \sum_{t=dist(S,W)}^{n-|W|} \sum_{\delta \supset_t W} y(S, \delta, t, W).$$

Lemma 9. *Given are sets $x \subseteq z \subseteq N$, and a number q with $|x| \leq q \leq |z|$. Let $R := \{y \subseteq N | x \subseteq y \subseteq z, |y| = q\}$. Then, $|R| = \binom{|z|-|x|}{q-|x|}$.*

Proof. This is a standard combinatorics problem. First, note that $q - |x|$ objects can be added to x such that y contains q objects. These objects also need to be different from those contained in x, and they need to be contained in z. Hence, there are $|z| - |x|$ different objects, of which $q - |x|$ many can be added to x. This is possible in $\binom{|z|-|x|}{q-|x|}$ different ways. □

Fact (2). Given S, δ, W, let $r := dist(S, W)$ and $t := dist(\delta, W)$. By Lemma 9 above, there exist $\binom{t}{r}$ supply functions f_S relative to f_W, on which the amount $y(S, \delta, t, W)$ is allocated. Of course, t (or δ or W) is redundant; it is only written for clarity of exposition.

Fact (3). $y(S, \delta, t, W)$ does not depend on S. If a step on supply function f_δ is allocated, then a step on each supply function $\gamma \subseteq \delta$ is allocated. Thus, the allocation of bundles on supply functions in the graph "between" W and δ *due to the allocation of bundle* δ has to be the same amount.

Fact (4). For any $t \geq 1$, we have, by the binomial theorem, $\sum_{r=0}^{t} \binom{t}{r} (-1)^r = 0$.

We now have

$$\begin{aligned}
& \sum_{r=0}^{n-|S|} \sum_{q} \sum_{\gamma \supset_r S} (-1)^r y_\gamma^q \\
= \quad & \sum_{r=0}^{n-|S|} \sum_{\gamma \supset_r S} (-1)^r \sum_{q} y_\gamma^q \\
\overset{(1)}{=} \quad & \sum_{r=0}^{n-|S|} \sum_{\gamma \supset_r S} (-1)^r \sum_{t=dist(\gamma,S)}^{n-|S|} \sum_{\delta \supset_t S} y(\gamma, \delta, t, S) \\
= \quad & \sum_{r=0}^{n-|S|} \sum_{\gamma \supset_r S} (-1)^r \sum_{t=r}^{n-|S|} \sum_{\delta \supset_t S} y(\gamma, \delta, t, S) \\
\overset{(2),(3)}{=} \quad & \sum_{r=0}^{n-|S|} (-1)^r \sum_{t=r}^{n-|S|} \binom{t}{r} \sum_{\delta \supset_t S} y(\cdot, \delta, t, S) \\
\overset{(4)}{=} \quad & \sum_{t=0}^{n-|S|} \sum_{\delta \supset_t S} y(\cdot, \delta, t, S) \sum_{r=0}^{t} (-1)^r \binom{t}{r} \\
= \quad & \sum_{\delta \supset_0 S} y(\cdot, \delta, 0, S) \sum_{r=0}^{0} (-1)^r \binom{0}{r} \\
= \quad & y(\cdot, S, 0, S) \\
= \quad & Y_S
\end{aligned}$$

□

Proof of Lemma 5. To simplify notation, I write $c^*(v^0, S)$ as $c^*(S)$ for any $S \in 2^N$. Let $S_1^c, S_2^c \in 2^N$ and $S_1^c \cap S_2^c = \emptyset$. Note that $c^*(N) = v^0(N)$. Also note that $S_1^c \cap S_2^c = \emptyset \Leftrightarrow S_1 \cup S_2 = N$. Because v^0 is superadditive we have

$$\begin{aligned}
& v^0(S_1^c \cup S_2^c) \geq v^0(S_1^c) + v^0(S_2^c) \\
\Leftrightarrow \quad & v^0(N) - c^*((S_1^c \cup S_2^c)^c) \geq 2v^0(N) - c^*(S_1) - c^*(S_2) \\
\Leftrightarrow \quad & c^*(S_1) + c^*(S_2) \geq c^*(S_1 \cup S_2) + c^*(S_1 \cap S_2)
\end{aligned}$$

The proof for subadditive v^0 is analogous to the above. □

Proof of Proposition 4. In SY, the seller's supply correspondence is defined as

$$S^{SY}(p) = \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta} p(A) \right\}$$

In my ascending auction the seller's supply correspondence is defined as

$$S(p) = \arg\max_{\delta \in \mathbb{K}} \left\{ \sum_{A \in \delta} \left(p(A) - c^0(A) \right) \right\}$$

By definition of SY's ascending auction, $p(B) = v^0(B)$ for any bundle B that is assigned to the seller during the procedure; and from the proof of SY's Theorem 2, we know that $p(\pi(0)) = v^0(\pi(0))$. In partition δ, where bundle $B \in \delta$ goes to the seller, we have $p(B) = v^0(B)$. Hence, we have

$$\begin{aligned}
S^{SY}(p) &= \arg\max_{\delta\in\mathbb{K}} \left\{ \sum_{A\in\delta\setminus B} p(A) + v^0(B) \right\} \\
&= \arg\max_{\delta\in\mathbb{K}} \left\{ \sum_{A\in\delta\setminus B} p(A) + v^0(B) - v^0(N) \right\} \\
&= \arg\max_{\delta\in\mathbb{K}} \left\{ \sum_{A\in\delta\setminus B} p(A) - c^*(N\setminus B) \right\} \\
&= \arg\max_{\delta\in\mathbb{K}} \left\{ \sum_{A\in\delta\setminus B} p(A) - c^*\left(\bigcup_{A\in\delta\setminus B} A \right) \right\}
\end{aligned}$$

Part (i) and (ii) of the proposition immediately follow, as c^* is by definition the dual of v^0, and $c^*\left(\bigcup_{A\in\delta\setminus B} A\right)$ may be interpreted as the seller's cost function. $\square$

Proof of Theorem 2. The proof proceeds exactly as the proof of Theorem 2 in [29]. The auction procedure terminates at some time t^*, because demand will cease entirely as soon as the price of a package exceeds the package's maximum value among all bidders. The price of the empty package is always zero.

Let $p^* = p(t^*)$ and let $A_l^* = A_l(t^*)$. Furthermore, let $\delta^* = \delta(t^*) \in S(p^*)$ denote the supply set in $S(p^*)$ that is chosen at time t^* by the seller. First, I establish an allocation π^* such that (p^*, π^*) constitutes a package-linear Walrasian equilibrium. Because at p^* no package is overdemanded, for any bidder $l \in \mathcal{L}$ with $A_l^* \neq \emptyset$, his demand A_l^* must be in δ^*. If $\bigcup_{l\in\mathcal{L}} A_l^* = N$ holds, define the allocation as follows: $\pi^*(l) = A_l^*$ for all $l \in \mathcal{L}$ and $\pi^*(0) = \emptyset$. Then (p^*, π^*) is a package-linear Walrasian equilibrium.

If $\bigcup_{l\in\mathcal{L}} A_l^* \subset N$, there is at least one package B in the chosen supply set δ^* which is not demanded by any bidder at time t^* (in SY such package is called a *squeezed-out* package). Now we have to distinguish multiple cases:

Case 1: $p^*(B) = c^0(B)$

The final auction price of bundle B is still fixed at the initial reserve price. This means, B was never overdemanded. If it was demanded in some earlier round by some bidder, this bidder demands now a different, more profitable package. Let

$$\delta_0^* = \{B \in \delta^* \mid p^*(B) = c^0(B) \text{ and } B \neq A_l^* \text{ for all } l \in \mathcal{L}\}$$

be the set of all squeezed-out packages. Let $\pi^*(0) = \bigcup_{B\in\delta_0^*} B$ and allocate $\pi^*(0)$ to the seller (at zero cost). Let $\mathbb{K}_0^*$ denote the universe of all partitions of the elements contained in δ_0^*. Because $\delta^* \in S(p^*)$, we have

$$\sum_{B\in\gamma} \left[p^*(B) - c^0(B)\right] \leq \sum_{B\in\delta_0^*} \left[p^*(B) - c^0(B)\right] = 0$$

for all $\gamma \in \mathbb{K}_0^*$. Trivially, the partition of objects retained by the seller is irrelevant, and the seller is indifferent between selling $\pi^*(0)$ or not.

Case 2: $p^*(B) > c^0(B)$
Because $p^*(B) > c^0(B)$, package B was demanded by some bidder in an earlier round. I denote by t the last round in which B was demanded by some bidder l. Just as in SY, the auction rule determines that package B may be allocated to bidder l, at the current price $p^*(B)$. Hence, we need to demonstrate that it is still profit-maximising for bidder l to receive package B at the current price. By the auction rule and Assumption 2, we must have

$$\mathcal{V}^l(p(t)) = u^l(B, p(t)) = v^l(B) - p(t, B) \geq 1 \tag{6}$$

and $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. Given equation (6) we have for bidder l, who is allocated the squeezed-out package B,

$$u^l(B, p^*) = v^l(B) - p^*(B) \geq 0 \tag{7}$$

By analogy with SY, I distinguish the following two sub-cases.

Case 2A: If $A_l^* = \emptyset$, simply assign bidder l the squeezed-out bundle, i.e. $\pi^*(l) = B$. $A_l^* \in D^l(p^*)$ and $A_l^* = \emptyset$ imply that $\mathcal{V}^l(p^*) = 0$. By definition of $\mathcal{V}^l$ we have $\mathcal{V}^l(p^*) \geq u^l(B, p^*)$. Together with equation (7) this implies $u^l(B, p^*) = 0$, and hence $\pi^*(l) \in D^l(p^*)$.

Case 2B: If $A_l^* \neq \emptyset$, we assign bidder l what he demanded at time t^* *and* the squeezed-out bundle, i.e. $\pi^*(l) = A_l^* \cup B$. Because the seller chose a supply set $\delta^* \ni \{A_l^*, B\}$, we have

$$p^*(A_l^*) - c^0(A_l^*) + p^*(B) - c^0(B) \geq p^*(\pi^*(l)) - c^0(\pi^*(l)). \tag{8}$$

Superadditivity of bidder l's utility implies that

$$v^l(\pi^*(l)) \geq v^l(A_l^*) + v^l(B). \tag{9}$$

Subadditivity of the seller's cost implies that

$$c^0(\pi^*(l)) \leq c^0(A_l^*) + c^0(B). \tag{10}$$

And because $A_l^* \in D^l(p^*)$, we have

$$v^l(A_l^*) - p^*(A_l^*) \geq v^l(\pi^*(l)) - p^*(\pi^*(l)) \text{ and} \tag{11}$$
$$v^l(A_l^*) - p^*(A_l^*) \geq 0. \tag{12}$$

From (8) and (10) follows

$$p^*(A_l^*) + p^*(B) \geq p^*(\pi^*(l)) \tag{13}$$

Then, using equation (13), (9), and (7) (in this order), we obtain

$$\begin{aligned}
v^l(\pi^*(l)) - p^*(\pi^*(l)) &\geq v^l(\pi^*(l)) - [p^*(A_l^*) + p^*(B)] \\
&\geq \left[v^l(A_l^*) - p^*(A_l^*)\right] + \left[v^l(B) - p^*(B)\right] \\
&\geq v^l(A_l^*) - p^*(A_l^*),
\end{aligned}$$

Using equation (11), it follows that

$$\begin{aligned}
v^l(\pi^*(l)) - p^*(\pi^*(l)) &= v^l(\pi^*(l)) - [p^*(A_l^*) + p^*(B)] \\
&= v^l(A_l^*) - p^*(A_l^*),
\end{aligned}$$

and therefore

$$p^*(\pi^*(l)) = p^*(A_l^*) + p^*(B). \tag{14}$$

Bidder l is therefore happy to receive bundle B in addition to his demanded bundle A_k^*, and pay the price that is set for the bundle $\pi^*(l)$.

Exactly as in SY, this process can be repeated for every squeezed-out bundle B with $p^*(B) > c^0(B)$. Every bidder l who is not allocated any squeezed-out bundle receives his demanded package, i.e. $\pi^*(l) = A_l^*$. δ^* is a partition of N chosen by the seller, and thus $(\pi^*(0), \ldots, \pi^*(L))$ is an allocation of N. The seller's profit is, from equation (14),

$$\sum_{l \in \mathcal{L}} \left[p^*(\pi^*(l)) - c^0(\pi^*(l))\right] = \sum_{A \in \delta^*} \left[p^*(A) - c^0(A)\right] = \Pi(p^*)$$

and consequently, the allocation π^* is in the seller's supply correspondence. It follows that (p^*, π^*) is a package-linear pricing Walrasian equilibrium. $\square$

Proof of Lemma 6. In SY, an allocation π is efficient if it holds for every allocation π' that

$$\sum_{l \in \mathcal{L}_0} \left[v^l(\pi(l))\right] \geq \sum_{l \in \mathcal{L}_0} \left[v^l(\pi'(l))\right] \tag{15}$$

In my model, an allocation π is efficient if for every allocation π' it holds that

$$\sum_{l \in \mathcal{L}} \left[v^l(\pi(l)) - c^0(\pi(l))\right] \geq \sum_{l \in \mathcal{L}} \left[v^l(\pi'(l)) - c^0(\pi'(l))\right] \tag{16}$$

The SY auction terminates in an allocation π that is efficient in the sense of their definition (theorem 2 in SY), i.e. if equation (15) holds for all allocations π'. In my model, the ascending auction procedure also terminates in an efficient allocation π in the sense of my definition, i.e. if equation (16) holds for all allocations π'. The efficient allocation in the buy-back auction procedure is equivalent to the efficient allocation in SY: setting $c^0(S) := 0 \ \forall S \in 2^N$ and adding the seller as a participant in the auction, i.e. running the auction with the set of buyers $\mathcal{L}' = \mathcal{L} + \{0\} = \mathcal{L}_0$, (15) and (16) are equivalent. The auction rules apart from the seller's choosing of the supply set are identical in SY and my auction, so the claim follows. $\square$

Proof of Proposition 5. Note that every conventional bidder bids identically in both procedures, up to ties. I simply split up the SY-seller into 2^n dummy bidders, denoted by $l_S, S \in 2^N$. Define bidder l_S's utility function as follows:

$$v^{l_S}(B) := \begin{cases} v^0(S) & \text{if } B \supseteq S \\ 0 & \text{otherwise} \end{cases}$$

Each bidder l_S has the highest bid on bundle S among all dummy bidders, because v^0 is super-additive. Let bidder l_S demand bundle S whenever he weakly prefers to demand bundle S to another bundle, but let him demand the empty set when she weakly prefers to do so.

Let the ascending auction start at $t = -1$ with starting prices $p(-1, S) = v^0(S) - 1 \; \forall S \in 2^N$. Let two instances of each bidder l_S participate. Bidders $l_S, S \in 2^N$ each demand bundle S. The seller (my seller) offers some supply set. Regardless of the non-dummy bidders' demand, each bundle $S \in 2^N$ is overdemanded in $t = -1$, so prices in $t = 0$ are increased by one. The dummy bidders all demand the empty set for all $t = 0, 1, \dots$, so if at some round $t \geq 0$ the auction ends with squeezed-out bundles, they can be allocated to the dummy bidders if they were the last to demand them. It is without loss of generality to stipulate that such squeezed-out bundles *are* allocated to dummy bidders (if they were the last to demand them), and not to regular bidders who might have demanded them at $t = -1$ as well. Then, in all rounds $t = 0, 1, \dots$, the supply correspondence and the demand correspondences are chosen to maximise identical profit and utility functions in both auction procedures. Thus, the supply correspondence and demand correspondences are identical in every round of both auction procedures, and it immediately follows that an identical price path resulting in the same allocation exists. $\square$

B The seller's demand type

If all buyers have strong substitutes valuations and supply is fixed, it is well known that Walrasian equilibrium exists. If the seller is partitioning a supply of different objects into packages, however, her preferences are not strong-substitutes-between-packages under non-linear (or package-linear) pricing (see also Footnote 7). Consequently, a Walrasian non-linear pricing equilibrium need not exist. Example 2 in [29] (also [3]) illustrates this nicely: There are three objects A, B, C for sale, one unit of each, and there are three bidders 1,2, and 3. The bidders' values are given in Table 2, and the seller's reserve prices are zero for all possible packages. It can be verified that Walrasian equilibrium does not exists under linear pricing [3] or non-linear pricing [29]. All bidders demand only one package, hence their valuation is trivially strong-substitutes-between-packages type. However, the seller's valuation is *not* strong-substitutes-between-packages. The reason is simply that she may be induced by an infinitesimal price change to switch from selling the partition $\{AB\}$ to selling the partition $\{A, B\}$. [2] show that a valuation is strong-substitutes if and only if it corresponds to a demand type defined by a unimodular set of vectors. A set of vectors in n dimensions is unimodular if every linearly independent subset of the set of vectors containing n vectors has determinant 0 or ± 1. For this to hold, an single vector can have at most one non-zero entry of the same sign. A demand type contains all the vectors describing the directions in which demand could change due to an infinitesimal generic price change. The seller's "demand" type characterisation would contain the vectors $\pm(1,1,0,-1,0,0,0)$. The entries of the vector correspond to the change in supply of packages $(A, B, C, AB, AC, BC, ABC)$ induced by an arbitrary, small price change, i.e. there exists an arbitrary, small price change that could make the seller prefer selling the partition $\{AB\}$ to selling the partition $\{A, B\}$. Hence, the seller's demand type does not satisfy unimodularity.

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	10	8	2	13	11	9	14
Bidder 2	0	8	5	10	13	14	13	15
Bidder 3	0	1	1	8	2	9	9	10

Table 2: Bidders' values over objects

C Examples and additional figures

C.1 Example for Section 3

Example 2. Consider the sale of two units of good A and two units of good B, which may be sold separately or in packages. Only $\{AB\}$ is considered a package; the multiset $\{A, A, B\}$, could only be bundled as $\{\{AB\}, \{A\}\}$, but not as $\{\{AA\}, \{B\}\}$.

Bidder values. Suppose there is one bidder, "bidder 1", with the MU-concave superadditive value function given by Table 3. Then, for $\mathbf{k} = (k_A, k_B, k_{AB})$, we have, e.g., $V^1(1,0,0) = 3$, $V^1(1,0,1) = 9+3$, $V^1(0,1,1) = 5+9$, $V^1(1,1,0) = 5+1$, and $V^1(1,1,1) = 5+9+0$. Note that the bidder cannot repackage objects A and B if he receives them separately, so if he receives three distinct packages one of them is worthless to him. Consider now a different MU-concave

Table 3: Bidder 1's marginal values

q	$v^1(q, A)$	$v^1(q, B)$	$v^1(q, AB)$
1	3	5	9
2	1	2	9
> 2	0	0	0

Table 4: Bidder 1's marginal values

q	$v^1(q, A)$	$v^1(q, B)$	$v^1(q, AB)$
1	3	5	9
2	1	2	5
> 2	0	0	0

superadditive value function given in Table 4. In Table 3 we had $V^1(0,1,1) = v^1(1,B) + v^1(2,AB) = 5+9$; in this example, we have $V^1(0,1,1) = v^1(1,AB)+v^1(2,B) = 9+2$. Of course a convention could be introduced according to which $V^1(0,1,1) = v^1(1,B) + v^1(2,AB) = 5+5$ in this example also. However, it would be difficult to argue why to choose such convention over another. The marginal value aggregation by Definition 5 seems least restrictive to us. One may reasonably require, however, that bidders are capable of repackaging the objects they receive, in accordance with Definition 6. Suppose this is the case; then, if the bidder was given A and B in separate packages, we have $V^1(1,1,0) = v^1(1,B) + v^1(2,A) = 5+1$ and $\widetilde{V}^1(1,1,0) = v^1(1,AB) = 9$.

Seller's cost. Suppose there is a seller with two units of A and two units of B available for sale. Her preference is given first as her marginal cost function in Table 5. Note that $k_{-S} = (k_B, k_{AB})$

Table 5: Seller's marginal cost

k_{-S}	$c^0(1, A, k_{-S})$	$c^0(2, A, k_{-S})$	$c^0(1, B, k_{-S})$	$c^0(2, B, k_{-S})$	$c^0(1, AB, k_{-S})$	$c^0(2, AB, k_{-S})$
(0,0)	1	2	1	2	1	4
(1,0)	1	2	1	2	2	∞
(0,1)	2	∞	2	∞	2	∞
(1,1)	2	∞	2	∞	3	∞
(0,2)	∞	∞	∞	∞	∞	∞
(2,0)	1	2	1	2	∞	∞
(2,2)	∞	∞	∞	∞	∞	∞
(1,2)	∞	∞	∞	∞	∞	∞
(2,1)	2	∞	2	∞	∞	∞

if $S = A$, $k_{-S} = (k_A, k_{AB})$ if $S = B$, and $k_{-S} = (k_A, k_B)$ if $S = AB$. The overall cost function is not straightforwardly defined; take for example $C^0(1,1,1)$. We could plausibly have $C^0(1,1,1) = c^0(1, A, (0,0)) + c^0(1, B, (1,0)) + c^0(1, AB, (1,1)) = 1 + 1 + 3$, or $C^0(1,1,1) = c^0(1, AB, (0,0)) + c^0(1, B, (0,1)) + c^0(1, A, (1,1)) = 1 + 2 + 2$, or $C^0(1,1,1) = c^0(1, A, (1,1)) + c^0(1, B, (1,1)) + c^0(1, AB, (1,1)) = 2 + 2 + 3$, etc., depending on how one defines marginal costs of different goods to interact with one another.

The representation of the seller's preferences in terms of incremental costs and cost relations has two advantages: first, it is much more compact, and second, it allows for a simple and *unambiguous* aggregation of incremental costs to overall costs. The incremental cost representation of the same preference is given in Table 5 and the graph in Figure 2. Then, by Definition 13 above, the overall cost is given by, e.g., $C^0(2,1,0) = \Delta c(1, A) + \Delta c(2, A) + \Delta c(1, B)$, and $C^0(1,1,1) = \Delta c(1, A) + \Delta c(1, B) + \Delta c(2, A) + \Delta c(2, B) + \Delta c(1, AB)$.

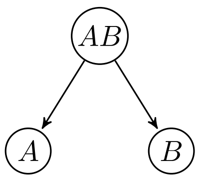

Figure 1: Cost function graph

q	$\Delta c(q, A)$	$\Delta c(q, B)$	$\Delta c(q, AB)$
1	1	1	-1
2	2	2	0
> 2	∞	∞	∞

Table 6: Incremental marginal cost

Value representation. Bidder 1's preferences (Table 3) can be expressed simply by submitting his two sets of marginal values in two XOR-bids, $(v_A, v_B, v_{AB}, \kappa) = (3, 5, 9, 1)$ and $(1, 2, 9, 1)$. These bids can be represented geometrically; e.g., $(v_A, v_B, v_{AB}, \kappa) = (3, 5, 9, 1)$ is shown in Fig. 6 in Appendix C.3.

Cost representation. The preference given in Table 6 and Example 2 can be directly submitted to the auctioneer and serves as input in the auction algorithm. Note again the compactness of

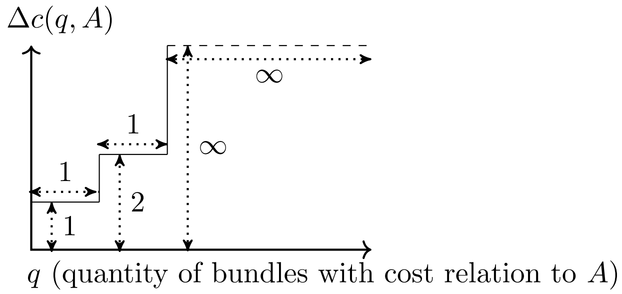

Figure 2: Marginal cost function/supply function f_A submitted by the seller

the bidding language compared to expressing marginal costs directly (Table 5). $\Delta c(q, S)$ is now simply renamed into μ_S^q. The incremental marginal cost function corresponding to package A from Table 6 is given in Fig. 2. *Solving the auctions.* Bidder 1 submits the bids $i = 1, 2$, and there are two additional bidders who submit bids 3 and 4, respectively. The seller submits the supply schedule discussed above. Supply schedule and bids are listed as auction input in Table 7. The efficient allocation is underlined in the bid list. The seller has a cost saving if she sells the first units of A and B as a package, hence bid 4 obtains $\{AB\}$. For the second units, the seller is indifferent between selling separately or as a package, so bid 1 and 3 win. Prices have to satisfy the set of equations corresponding to the constraints of DLP. One can verify that the set of equilibrium prices is given by $(z_A, z_B, z_{AB}) \in \{(4, 5, 9), (5, 4, 9), (5, 5, 9), (5, 5, 10)\}$, which all support the unique equilibrium allocation. Note that a lowest equilibrium price vector does not exist, but a highest equilibrium price vector does. Proposition 1 (ii) tells us, e.g., that $z_{AB} = z_A + z_B + \mu_{AB}^1 + u_{AB}^1$. Corollary 1 (i) tells us that $\mu_{AB}^1 + u_{AB}^1 \leq \mu_{AB}^2 = 0$. Thus, equilibrium prices must satisfy either $z_A + z_B = z_{AB}$ or $z_A + z_B = z_{AB} - 1$. The surplus constraints pin down the set of equilibrium prices precisely.

Supply functions f_A, f_B, f_{AB}

q	μ_A^q	μ_B^q	μ_{AB}^q
1	1	1	-1
2	2	2	0
> 2	∞	∞	∞

List of XOR-bids submitted

i	v_A^i	v_B^i	v_{AB}^i	κ^i
1	3	$\underline{5}$	9	1
2	1	2	9	1
3	$\underline{5}$	3	8	1
4	6	2	$\underline{11}$	1

Table 7: Auction input

C.2 Examples for Section 4

Example 3. In Table 8, c^0 is a subadditive. Its dual $v^*(c^0, \cdot)$ is set-cover supermodular by Lemma 5, but of course it is also subadditive. The strictly subadditive elements A, B, and AB are untouched by the set-cover requirement.

In Table 9, v^0 is superadditive. Its dual is set-cover submodular, and elements A, BC, and ABC are strictly subadditive. However, it also exhibits strictly superadditive elements B,C, and BC. Strictly superadditive and strictly subadditive elements of a cost function correspond to negative and positive synergies between objects that are sold together. Selling B costs the seller 2, and selling C costs the seller 0. Selling $\{BC\}$ or $\{B, C\}$ (each object to a different bidder) costs the seller 3. But adding A to the set of objects sold, i.e. selling any partition of $\{A, B, C\}$, takes away the cost increment of 1 the seller incurred from selling B and C: Any partition of $\{A, B, C\}$ now costs merely 4. Hence, in this example both negative and positive synergies are present.

Table 8: Set-cover supermodular and subadditive function

S	A	B	C	AB	AC	BC	ABC
$c^0(S)$	3	2	0	4	2	1	4
$v^*(c^0, S)$	3	2	0	4	2	1	4

Table 9: Superadditive v^0 and set-cover submodular c^*

S	A	B	C	AB	AC	BC	ABC
$v^0(S)$	1	2	0	4	2	2	4
$c^*(v^0, S)$	2	2	0	4	2	3	4

Example 4. Table 10 states the bidders' and the seller's values for goods A, B, and the package AB. Bidders are numbered from B1 to B6, the seller is denoted by S. The dual of these values yields $c^0(A) = 4$, $c^0(B) = 6$, and $c^0(AB) = 8$. Table 11 illustrates the basic ascending auction from SY (the seller chooses a revenue-maximising supply set wrt. v^0, or a profit-maximising supply set wrt. $C^0(\delta) = c^0\left(\bigcup_{S\in\delta} S\right)$, in each round). Table 12 illustrates my procedure, i.e. the ascending auction where the seller chooses a profit-maximising supply set wrt. $C^0(\delta) = \sum_{S\in\delta} c^0(S)$ in each round. Both procedures terminate in the respective efficient allocation defined in the corresponding environment: in SY's model, the two individual objects A and B are allocated, e.g. to B1 and B3; in my model, the bundle AB is allocated, e.g. to B5.

C.3 Additional figures

Different examples for cost/supply function graphs, referred to in the main text, are shown in Figs. 3 to 5.

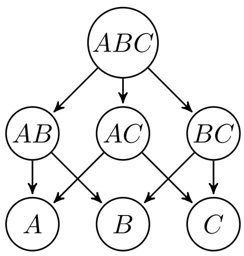

Figure 3: Complete CFG with three different varieties A, B, C

Fig. 6 illustrates the package XOR-bid $(v_A, v_B, v_{AB}, \kappa) = (3, 5, 9, 1)$ geometrically. The black dot at $(3, 5, 9)$ corresponds to the bid. Beyond the light blue faces, the bid demands nothing

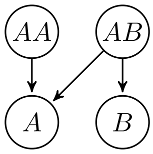

Figure 4: SFG with package AA

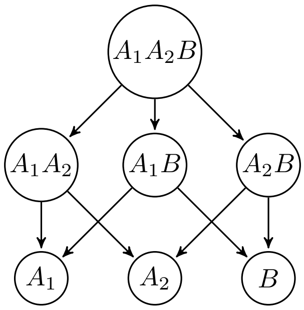

Figure 5: Augmented SFG with three different varieties A_1, A_2, B

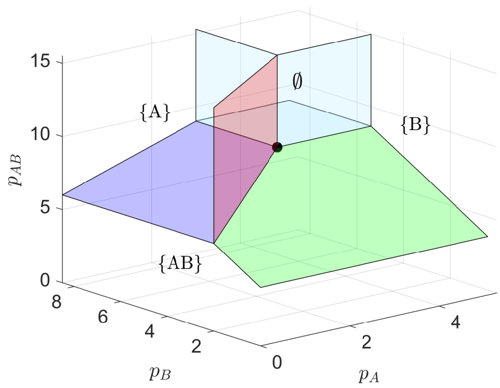

Figure 6: Combination XOR-bid for three packages

Table 10: Values and costs

	A	B	AB
B1	5	0	5
B2	5	0	5
B3	0	7	7
B4	0	7	7
B5	0	0	11
B6	0	0	11
S (values)	2	4	8
S (costs)	4	6	8

Table 11: Revenue-maximising ascending auction

current price	supply set	demand B1	B2	B3	B4	B5	B6	squeezed-out
$p(0) = (2, 4, 8)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(1) = (3, 5, 9)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(2) = (4, 6, 10)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(3) = (5, 7, 11)$	$\{A, B\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	A,B

Table 12: Profit-maximising ascending auction, cost = sum of individual package costs

current price	supply set	demand B1	B2	B3	B4	B5	B6	squeezed-out
$p(0) = (4, 6, 8)$	$\{AB\}$	A	A	B	B	AB	AB	
$p(1) = (5, 7, 9)$	$\{A, B\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB	AB	
$p(2) = (5, 7, 10)$	$\{AB\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB	AB	
$p(3) = (5, 7, 11)$	$\{AB\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	AB

because prices are too high. Consider now the area that is not beyond the light blue faces; it is divided in three areas. For prices above the blue face and left of the red face bundle $\{A\}$ is demanded, for prices above the green face and right of the red face bundle $\{B\}$ is demanded, and for priced below the blue and the green face bundle $\{AB\}$ is demanded. At prices that lie on the red face, the bid is indifferent between receiving $\{A\}$ or $\{B\}$, and similarly for prices on the green or blue face. At prices that lie on the intersection of all three faces, the bid is indifferent between all three bundles. At prices on any of the light blue faces, the bid is indifferent between receiving nothing and the bundle from the neighbouring demand region. A detailed description of the geometric representation of such preferences is given in [2].