The Neoclassical growth model with infinite horizon intertemporal consumption

The set up is the same as the previous handout on the neoclassical (Solow) model, except that we no longer assume a fixed savings propensity. The key equations from that handout were

$y = f(k)$ (1)

$f'(k) = r$ (2)

$w = f(k) - kf'(k)$ (3)

$y = c + dk/dt + nk$ (4)

With a fixed savings propensity we also had

$c = y(1-s)$ (5')

This is now replaced by the Keynes/Ramsey rule

$[d(u'(c))/dt]/u'(c) = \theta + n - r$ (5)

which can also be written as

$u''(c)\, dc/dt / u'(c) = \theta + n - r$

For a derivation of this rule see the handout on the infinite horizon consumption problem. Equation (5) has the same properties as the K/R rule for the two period problem: $(1+\theta)u'(c_1) / u'(c_2) = (1+r)$, except that now population growth acts like impatience.

The model has two dynamic equations, in consumption (5) and capital (4) , rather than just one in capital. In steady state $dk/dt = 0$ and $d(u'(c))/dt = 0$, which implies

$r^* = \theta + n$ (6)

and

$y^* = c^* + nk^*$ (7)

using * for steady state values. Combining (6) and (1) shows that steady state capital per head (and hence output) is determined by technology (f), preferences (θ) and n. This was also true in the Solow model, except now the 'deep' parameter θ replaces the fixed savings propensity s.

To examine the dynamics of the model, we can construct a <u>phase diagram</u> in the space defined by the two dynamic variables, c and k. For simplicity, consider the special case of a log utility function i.e. (5) becomes

$$dc/dt = c[r - \theta - n] = c[f\,'(k) - \theta - n] \tag{8}$$

First draw the steady state lines in the normal way. There is only one value of k at which dc/dt = 0 in (8), so this curve is a vertical line. The curve defined by (4) is more complex. It goes through the origin: if k=0 then there is nothing to consume. As k increases consumption increases at first: maximum consumption is reached at the golden rule point $f\,'(k) = n$. The two lines cross at the equilibrium $f\,'(k) = \theta + n$, which as $f\,''(k) < 0$ is to the left of the golden rule level of k.

Now consider how each variable is moving either side of these steady state lines. Above the dk/dt=0 line, consumption is higher so capital must be decreasing. Below this line, consumption is lower so there is positive net saving. To the left of the dc/dt=0 line, capital is low so $f\,'(k)$ is greater, so c must be increasing. The reverse is also true. This allows us to determine the direction of movement in each quadrant of the diagram. Because the economy always moves away from equilibrium in two quadrants, there is a unique trajectory (the 'stable saddle path') along which the variables will lie in disequilibrium. On all other paths, either the Keynes-Ramsey rule or the transversality condition for consumption is violated.

Suppose the economy starts at some arbitrary point. To get on to the saddle path, consumption must 'jump' immediately. Capital cannot jump in this manner because it can only change gradually through saving - it is a 'backward looking' variable in the sense that its current value is influenced by its past. There is no similar link for consumption: consumers can choose to save or borrow at a level which is completely independent of past saving decisions.