MSC MACRO HANDOUT

This note is for reference only. It shows how to generalise the two period consumption model to many periods. It involves some mathematics – Hamiltonians – that you are NOT expected to be able to do.

Solving the Infinite Horizon Consumption Problem

The population is made up of a fixed number of identical families, each of which grows at a rate **n**. Each member of the family is treated equally. As a result, each new family member (birth) must be allocated an amount of resources equal to those of existing members. Current family members must therefore save to provide for the new generation. The single period budget constraint in discrete time is given by

$$c_t + \Delta a_t + a_t n = w_t + r a_t \qquad (1)$$

where c is consumption per head, a is financial wealth per head, w is labour income per head and r is the rate of interest, which we assume is constant over time. (See the Solow growth model handout for a derivation of this equation.)

Each individual is free to borrow or lend in every period at the interest rate r. The single period budget constraint is therefore less important than the intertemporal budget constraint. Suppose each individual inherits a_0 in period zero. The period 1 budget constraint can then be written as

$$c_1 - w_1 + a_1\delta - a_0 = 0 \qquad (2)$$

where $\delta = (1-r+n)$. Write out the period 2 budget constraint in the same way, and substitute this into (2) to eliminate a_1. Repeat for period 3 and a_2 and so on. After n periods we obtain

$$\sum_{t=1}^{N}(c_t - w_t)\delta^{t-1} + \delta^N a_N - a_0 = 0 \qquad (3)$$

The No-Ponzi-Game condition (Blanchard & Fischer p49) plus the need to avoid wasted resources implies $\delta^N a_N$ tends to zero as N tends to infinity. (3)

can then be rewritten as

$$\sum_{t=1}^{\infty} c_t \delta^{t-1} = a_0 + \sum_{t=1}^{\infty} w_t \delta^{t-1} \qquad (4)$$

This says that discounted consumption is equal to initial wealth plus discounted labour income, or ‘human wealth’.

The continuous time analogue of (4) is

$$\int_{t=0}^{\infty} c_t e^{-(r-n)t} dt = a_0 + \int_{t=0}^{\infty} w_t e^{-(r-n)t} dt \qquad (5)$$

(Note if δ is close to 1, δ^t is approximately equal to $e^{t(\delta-1)}$ e.g. if t=4, $(0.96)^4 = 0.85$, $e^{4(0.96-1)} = e^{-0.16} = 0.852$)

Equation (5) is a continuous time intertemporal budget constraint. To analyse how financial and human wealth is allocated over time, we need to make some assumptions about utility. Assume each family maximises

$$\int_{t=0}^{\infty} u(c_t) e^{-\theta t} dt \qquad (6)$$

where θ is the 'rate of time preference', and u() is an instantaneous utility function where u'() > 0 , u''() <0.

Consumers maximise (6) subject to the dynamic constraint (1). As the constraint is dynamic, we cannot use the simple Lagrangian technique to solve this problem, but instead we set up a Hamiltonian (see appendix). The Hamiltonian for this problem is

$$H_t = u(c_t)e^{-\theta t} - \lambda_t (c_t - (r-n)a_t - w_t)$$

where c is the *control* variable, a the *state* variable, and lambda is the *costate* variable. The Hamiltonian is made up of the 'value function' being maximised, and lambda times the determinate of the state variable, both at some date t. Pontryagin's maximum principle states that first order conditions for the problem can be obtained by setting the partial derivatives of

$$H_t + a_t d(\lambda_t)/dt \qquad (9)$$

with respect to c_t and a_t to zero.(We ignore the equation associated with the transversality condition for simplicity: see B & F page 40.) Differentiating with respect to c first gives

$$\partial H / \partial c = u'_t e^{-\theta t} - \lambda_t = 0 \qquad (10)$$

Differentiating (10) with respect to time gives

$$\dot{\lambda}_t = e^{-\theta t} (\dot{u}'_t - \theta u'_t) \qquad (11)$$

Now differentiating (9) with respect to ‘a’ gives

$$H_a + \dot{\lambda}_t = (r-n)\lambda_t + \dot{\lambda}_t = 0 \qquad (12)$$

Using (10) and (11) in (12), and dividing through by $e^{-\theta t}$ implies

$$\dot{u}_t' / u_t' = \theta + n - r \qquad (13)$$

which is the Keynes-Ramsey rule for the optimal allocation of consumption over time (because derivatives of u are a function of c). This is a continuous time ‘Euler equation’.

Consider the special case where u(c) = ln(c), so u’ = 1/c. (13) becomes

$$\dot{c}_t / c_t = r - n - \theta \qquad (14)$$

Consumption grows at a rate equal to r-n-θ. Thus if consumption in period zero is c_0, then

$$c_t = c_0 e^{(r-n-\theta)t}$$

Using this expression in the intertemporal budget constraint implies

$$c_0 = \theta[a_0 + \int_{t=0}^{\infty} w_t e^{-(r-n)t} dt] \qquad (15)$$

which states that consumers consume a constant proportion theta of their financial and human wealth.

Another slightly less special case is the CRRA utility function $u(c_t) = c_t^{1-\rho}/(1-\rho)$ where ρ>0. The smaller is ρ, the more willing the consumer is to substitute consumption across time (e.g. as ρ tends to 0, utility tends to linearity in consumption). With CRRA, the Keynes/Ramsey rule becomes

$$\dot{c}_t / c_t = (r - n - \theta) / \rho$$

(see Romer Chapter 2). The log utility function is a special case of CRRA where ρ=1. For a derivation of the Keynes/Ramsey rule with a varying interest rate see Blanchard & Fischer Chapter 2.

Appendix on Hamiltonians

Problem

Maximise $V = \int_0^T F(t, y, u)dt$ (1)

subject to $\dot{y} = f(t, y, u)$ (2)

where t is time, y is the *state* variable (e.g. assets) and u is the control variable (e.g. consumption).

Hamiltonian

The Hamiltonian is defined as

$H(t, y, u, \lambda) = F(t, y, u) + \lambda(t) f(t, y, u)$ (3)

where λ is called the *costate* variable.

Solution

The maximum principle states that the solution to the problem is given by

$\partial H / \partial u = 0$ (4)

$\dot{y} = \partial H / \partial \lambda$ (5)

$\dot{\lambda} = -\partial H / \partial y$ (6)

and the transversality condition $\lambda(T) = 0$. (Strictly the first condition should be stated as 'Max H with respect to u', but this normally reduces to (4) – see Chiang, Elements of Optimisation, page 169.) Equation (5) is just (1) restated. Where do (4) and (6) come from?

Intuition

If (2) holds for all t in the interval 0..T, then the maximisation problem can be restated as

Max $$V + \int_0^T \lambda(t)[f(t,y,u) - \dot{y}] = \int_0^T \{F(t,y,u) + \lambda(t)[f(t,y,u) - \dot{y}]\}dt$$

Given our definition of H, this becomes

Max $$\int_0^T H(t,y,u,\lambda)dt - \int_0^T \lambda(t)\dot{y}dt$$

Integrating both sides of the product rule for differentiation, $d(\lambda y)/dt = yd\lambda/dt + \lambda dy/dt$ implies

$$[\lambda y]_0^T = \int_0^T y d\lambda + \int_0^T \lambda dy$$

where the left hand side is simply λy evaluated at T less its value at 0.

The maximisation can then be rewritten as

Max $$\int [H(t,y,u,\lambda) + y\dot{\lambda}]dt - \lambda(T)y(T) + \lambda(0)y(0)$$

The first and third first order conditions are the result of equating the differential of the expression in the integral with respect to u and y to zero (Chiang, section 7.3).