Wage-Tenure Contracts in a Frictional Labour Market:

Firms' Strategies for Recruitment and Retention

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Abstract

A common assumption in equilibrium search and matching models of the labour market is that each firm posts a wage, to be paid to any worker hired. This paper considers the implications of firms posting *contracts*, in a random matching model with on-the-job search. More complex contracts enable firms to address both recruitment and retention problems by, for example, increasing the wage with tenure. The effect on the labour market is to reduce turnover, below the level required for efficient matching of workers to firms.

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1. INTRODUCTION

When agents in the labour market are not continuously aware of employment opportunities, they must search for suitable matches. Search frictions reduce the intensity of competition, creating match rents. In this environment there may be a distinction between the problems of recruitment and retention, which would not exist in a competitive labour market. In the equilibrium sequential search literature, where the dynamic process by which workers find jobs is modelled explicitly, there are two approaches to modelling wage determination (Mortensen and Pissarides, 1999), neither of which addresses any distinction between recruitment and retention. In the first approach, the wage is bargained ex-post, after the firm and worker have met, as in Pissarides (2000). In the second, the firm posts the wage in advance of matching and is committed to paying that wage to any worker it hires, as in Diamond (1971) and Burdett and Mortensen (1998). The present paper extends the second approach, by allowing firms to post *contracts*, rather than just wages, and hence to optimise privately with respect to both recruitment and retention of workers. In contrast to Burdett and Mortensen, identical firms post the same contract. But the effect on the labour market is to reduce turnover and the efficiency of matching.

Wage-posting, together with sequential search, gives employers dynamic monopsony power. From Diamond (1971) we know that in the equilibrium of a wage-posting game with identical agents, firms set the monopsony wage. If search is costly we have the Diamond paradox: the only equilibrium is for workers not to participate - since the monopsony wage does not compensate them for sunk search costs. The pure monopsony outcome can be traced to the joint effect of two assumptions: first, searching workers cannot hold offers, so firms do not compete directly with each other when making offers; second, employed workers do not search for alternative jobs, so firms do not face any future competition either.

Burdett and Mortensen's (1998) influential paper demonstrates the effect of on-the-job search. This imposes more competitive pressure on firms, raising the relative welfare of workers. The wage-posting equilibrium for homogeneous workers and firms is characterised by a distribution of wages, and a high aggregate level of labour turnover, since workers change jobs whenever they find a firm offering a higher wage. Firms are indifferent between low wages with high turnover, and high wages with low turnover. In the equilibrium with two types of firm (low and high

productivity), there are two non-degenerate, but non-overlapping, wage distributions.

Burdett and Mortensen's model is interesting for a number of reasons. At a theoretical level, it responds to the challenge of Rothschild (1973), by obtaining a non-degenerate equilibrium wage distribution with homogeneous agents. Furthermore, the equilibrium is consistent with empirical findings, such as wage differentials that cannot be explained by worker characteristics, and the association between wages and firm-size. However, like efficiency wage models, it is vulnerable to the criticism (made by Shimer, 1996) that firms, in posting a single wage, are not optimising – they could do better with a more complex contract.

In this paper, firms are allowed to post general (but anonymous) contracts, in the same sequential search environment with on-the-job search. They chose contracts that wage-discriminate between new and existing workers. In equilibrium, all firms of the same type offer essentially the same contract, and this contract has the effect of reducing turnover. If workers and firms are homogeneous, there is no job-to-job turnover. If firms differ in productivity, equilibrium turnover is lower than the level required for efficient matching of workers to firms.

When firms have a free choice of contracts, we have the pure monopsony outcome again. They can insulate themselves from competition for existing employees, and hence fully exploit their monopsony power in the market for new workers. This result may appear to have little practical relevance, since it depends on the firm's ability to set lump-sum entry or quitting fees; hence I also explore the effect of restricting the set of available contracts. I show that the character of the equilibrium outcome is the same – in particular the low-turnover results still hold – provided that firms are able to vary the wage with tenure. However, contract restrictions modify monopsony power, and hence may enhance matching efficiency.

The paper is organised as follows. Section 2 describes the labour market model, and the contracts that could be used in this environment. In section 3, I assume that firms face no restrictions on the form of contracts, and derive the labour market equilibria first for homogeneous firms, and then for two types of firm differing in productivity. In section 4, I establish the form of the optimal contract when firms are restricted to wage-contracts, and again derive the equilibria for the homogeneous and heterogeneous cases. The results are interpreted and discussed in Section 5.

2. THE MODEL

The structure of the model is the same as that of Burdett and Mortensen (1998). There is a mass of workers and a mass of firms each of measure one. Over time, workers and firms meet each other at Poisson rate λ, which is independent of whether they are currently matched[1]. A firm can match with several workers simultaneously, but a worker must terminate any existing match in order to enter a new one. Workers exit from the labour market (retire) at Poisson rate μ, again independent of employment status, and are replaced by new workers who are initially unemployed. All agents are risk neutral and have zero discount rate[2].

Workers are identical, and have flow utility of unemployment equal to b. Firms differ in productivity: in a firm of type p, every match has constant flow productivity p. The distribution function of firm-types is $P(p)$, which has bounded support, and the lowest firm-type has productivity $\underline{p}>b$. Note that for this labour market to be efficient in matching workers to firms, workers must change jobs if and only if they encounter a more productive match; then, total output is maximised, subject to the constraints imposed by the matching frictions.

The model differs substantively from that of Burdett and Mortensen only in that each firm posts an employment contract, rather than simply a wage. When a worker meets a firm, he observes the contract, and either accepts it and enters the match, or remains in his current state. Workers maximise the expected present value of income, and move between matches if made better off. It is convenient to assume that they accept offers to which they are indifferent.

Contracts are anonymous: a firm offers the same contract to all workers[3]. In particular, it cannot vary the offer according to the current state of a worker (for example, by offering a less desirable contract to the unemployed), or make anything contingent on subsequent offers received by the worker. We will assume that a

[1] Since the matching rate is exogenous, there are no search externalities.

[2] The retirement rate μ plays the role of a discount rate in ensuring that the expected value of each match is finite.

[3] Anonymity is a crucial assumption, common to all wage-posting models. We depart from those models by allowing the form of the contract to be more general, but do not go as far as allowing individualised contracts. The usual justification for anonymity is an information asymmetry: firms do not observe the changing circumstances of workers. There may also be issues of fairness, since workers are identical. Note, however, that if firms were able to observe the alternative opportunities of workers at all times, and could specify contracts individually, the outcome would be the same as under bilateral bargaining with renegotiation, in which the firm had all the bargaining power. Firms would obtain all the rent, and the outcome would be efficient. The form of the contract would be of little interest.

contract can commit the two parties to future payments to be made while the match continues, and that either party may terminate the match at any time. Payments are of two possible types: first, a wage profile $w(t)$, contingent on the worker's tenure, t; and second, instantaneous lump-sum payments, contingent either on tenure, or on the termination of the match by either party.

Workers and firms decide whether to enter or leave matches by comparing the asset value of being in the match with the value of alternative opportunities. An employed worker compares the continuation value (to himself) of the current match with the initial value of any alternative offer. Hence, each match value is affected by the measure $F(v)$ of firms offering contracts of initial asset value to the worker less than or equal to v. The firm's expected payoff from a contract offer depends also on its probability of acceptance – that is, on the distribution of workers' reservation values $G(v)$. F and G will be endogenously determined by the equilibrium contract-offers, but must have the standard properties of a distribution function: they are non-decreasing, right-continuous, and continuous almost everywhere.

The joint continuation value to a worker and firm of a match with productivity p has maximum value J_p^*, constant over time, defined by the Bellman equation:

$$\mu J_p^* = p + \lambda\left(E[\max(J_p^*, Z)] - J_p^*\right) \tag{1}$$

where the random variable Z represents the initial asset value to the worker of an offer from an alternative firm, so has distribution function F. The interpretation of (1) is straightforward: as long as the worker does not retire (which he does at rate μ) the worker and firm jointly obtain the flow p; when an offer arrives (which happens at rate λ) the worker should accept it if and only if it is worth more than the joint continuation value of the match, in which case they jointly obtain the additional value of that offer (since the worker obtains it, and the firm gets zero).

J_p^* is continuous and strictly increasing in p. Since unemployed workers have the same job opportunities as the employed, but flow utility b, the value to a worker of unemployment is:

$$V_u \equiv J_b^* < J_p^* \tag{2}$$

When a relationship is governed by a contract, the worker and firm optimise individually within the constraints of the contract, so the maximum joint value J_p^* is not necessarily achieved. Furthermore, depending on the contract, the value of the

match may vary with its duration. Let $V(t)$ and $Y(t)$ be the continuation values at tenure t for worker and firm respectively. Then $r(t) \equiv F(V(t)-)$ is the retention probability – the probability that the worker will stay in the match if he receives an alternative offer when his tenure is t. The joint value of the match is $J(t) \equiv V(t)+Y(t)$, and $J(t) \leq J_p^*$.

2.1 Equilibria, and the Equilibrium Outcome

The model is analysed as a one-shot game, in which each firm chooses a contract offer to maximise its payoff in the steady-state. Since there are many possible forms for contracts, there will be many different Nash equilibria that have the same effect on the allocation of labour market resources. To capture the essential characteristics of an equilibrium, we define its *outcome*:

DEFINITION: The *outcome* of a Nash equilibrium in contract offers is the distribution of initial values (to workers) of offers made by each firm-type, $F(v \mid p)$.

Note that market distribution of initial offers is $F(v) \equiv \int F(v \mid p) dP(p)$. It is clear that the outcome, thus defined, determines the expected payoff V_u of each worker entering the labour market. We will show below (Proposition 3.2) that it is sufficient to determine all the interesting properties of any equilibrium: the expected payoff of each firm, the distribution of employment across firms, and the turnover between them.

3. UNRESTRICTED CONTRACTS

Assume that there are no restrictions on the form of contracts, and consider the choice of contract for a firm of type p, for given distributions F and G. Since the meeting rate is exogenous, the optimal contract maximises the expected payoff from each worker encountered, which is the probability of acceptance multiplied by the firm's expected return from the consummated match. There is an immediately obvious optimal strategy: maximise the joint value of the match, by giving the whole surplus to the worker ex-post so that he makes efficient quit decisions, and set an entry fee to extract part of the surplus ex-ante:

PROPOSITION 3.1: For a firm of type p, the contract with a constant wage $w = p$ and a fee $\varphi^*(p)=\text{argmax}_\varphi \varphi G(J_p^*-\varphi)$ paid by the worker on acceptance, is optimal.

PROOF: The proposed contract has initial values $J_p^*-\varphi^*$ to the worker and φ^* to

the firm. So the probability of acceptance is $G(J_p^* - \varphi^*)$, and the expected payoff for the firm is $\Pi^* = \varphi^* G(J_p^* - \varphi^*)$. With an arbitrary contract:

$$\begin{aligned}\Pi &= \big(J(0) - V(0) + \varphi\big)G(V(0) - \varphi) \\ &\le \big(J_p^* - V(0) + \varphi\big)G(V(0) - \varphi) \\ &\le \max_v \big(J_p^* - v\big)G(v) = \max_\varphi \varphi G(J_p^* - \varphi) = \Pi^*\end{aligned}$$ □

We will refer to a contract with an entry fee and a wage equal to productivity as a *fee-contract*. Effectively, the firm uses the fee to separate its recruitment strategy from its retention strategy.

It is clear that there are other optimal contracts. For example, a constant wage $p - \mu\varphi^*$ and a fee φ^* paid by the worker on quitting gives the same expected payoffs to both parties. Or if other firms were making low offers, a contract with a low wage and no fee could be optimal. However it is also clear that any optimal contract must have the same recruitment and retention properties as the optimal fee-contract: it must be equally attractive to workers, and induce the same efficient quitting behaviour. Thus, if we are interested in how the labour market allocates resources, we can restrict attention to equilibria in which firms use fee-contracts:

PROPOSITION 3.2: For any Nash equilibrium in contract offers, there exists a Nash equilibrium in offers of fee-contracts with the same outcome, in which every firm (outside a set of measure zero) has the same probabilities of recruiting and retaining workers, and the same expected payoff.

To prove Proposition 3.2 it can be shown that if, from any equilibrium in contract offers, every firm switches to using a fee-contract which has the same initial value to the worker as its original contract (thus leaving the distribution of offers unchanged), then the probability of acceptance of the offer, and the retention probability $r(t)$, are unchanged. So firms have the same expected payoffs as in the original equilibrium. Since employees' reservation values have increased, no other offer can do better, so this is an equilibrium. The details of the proof are provided in the Supplementary Appendix.

This result means that if we can find the conditional offer distribution $F(v \mid p)$ of a fee-contract equilibrium, and determine the corresponding payoffs and labour turnover, we know the characteristics of any equilibrium with the same offer distribution. Thus $F(v \mid p)$ does indeed capture the *outcome* of an equilibrium.

As a first step towards finding fee-contract equilibria, note that whatever the offer distribution F, a firm of type p can make strictly positive profits if and only if it offers a contract of initial value $V_0 \in [J_b^*, J_p^*)$, acceptable to unemployed workers. Hence in any equilibrium all firms make such offers, and all offers are accepted by the unemployed. Then, if the unemployment rate is u, workers enter unemployment at rate $\mu(1-u)$ and leave at a rate λu, so in the steady-state, $u = \mu / (\mu + \lambda)$.

3.1 Equilibrium with Homogeneous Firms

Suppose that all firms have identical productivity $p>b$. Burdett and Mortensen show that if they are restricted to offering a constant wage, there is no equilibrium in which they all offer the same wage. If they were to do so, all employed workers would have the same reservation wage, and there would be a sharp discontinuity in the firms' payoff functions, giving them an incentive to deviate slightly. Hence the only equilibrium is effectively a mixed strategy one, in which firms randomise over an interval of wages, leading to turnover between firms in the steady-state. But if they are not restricted in contract choice, we have instead:

PROPOSITION 3.3: When all firms have the same productivity, p, and an unrestricted choice of contracts, there is a unique equilibrium outcome. All firms offer contracts of initial value $V_0 = b/\mu$; and there is no job-to-job turnover.

PROOF: Suppose all firms use fee-contracts. Then a firm cannot recruit employed workers with an offer worth less than J_p^*, so it maximises its payoff with an offer of initial value J_b^*, which is accepted by the unemployed. Then from (1), since all firms offer J_b^*, we obtain $J_p^* = p/\mu$ and $V_u = J_b^* = b/\mu$, so the unique fee-contract equilibrium has $w=p$, and fee $\varphi = J_p^* - V_u = (p-b)/\mu$, and there is no job-to-job turnover. Applying Proposition 3.2, we have the unique outcome. □

So when contracts are unrestricted, the sequential search model with on-the-job search delivers the monopsony outcome - just as in the model without on-the-job search. In equilibrium, firms price-discriminate between the internal and external markets, separating the problems of recruitment and retention and treating employed and unemployed workers differently. They can fully exploit their monopsony power in the market for new workers and hence extract the whole match rent[4].

[4] There are contracts without fees that support the equilibrium outcome, in which the wage rises sufficiently steeply to deter other firms from bidding away employed workers (Stevens, 2000). But although fees are not *necessary* for the monopsony outcome, firms must be able to set a wage

3.2 Equilibrium and Inefficiency with Heterogeneous Firms

In the homogeneous case the contract-posting equilibrium is efficient: all workers make socially optimal job-acceptance decisions and hence output is maximised subject to the matching frictions. But when jobs differ in productivity, this will no longer, in general, be true. If firms use fee-contracts, an employee receiving an alternative offer compares the joint value of his existing match with the initial value of the alternative offer, which is lower than the joint value of the alternative match. Then some matches will not be consummated when social efficiency would require them to be so, and labour turnover will be inefficiently low.

To illustrate this problem, we determine the fee-contract equilibrium when there are two types of firm, half with low productivity, $\underline{p}$, and half with high productivity $\overline{p}$. In equilibrium there can be no turnover between firms of the same type (since they could attract each other's workers only with an offer bringing zero profit, whereas a lower offer can attract unemployed workers). Low-firms therefore recruit only from unemployment, and offer contracts of initial value V_u. The only job-to-job turnover that is possible in equilibrium is from low-matches to high-matches, and if such turnover occurs it must be achieved by high-firms offering contracts of initial value $J_{\underline{p}}^*$: workers in low-firms earn a wage $\underline{p}$ and hence will not accept less, and high-firms cannot both attract more workers and make positive profit with a higher offer. Alternatively, high-firms may find it better to offer a contract that attracts only unemployed workers.

Thus, there are just two possibilities: a separating equilibrium, in which low-firms offer V_u and high-firms offer $J_{\underline{p}}^*$, and a pooling equilibrium in which both types offer V_u. Clearly, the first of these has socially-optimal matching of workers to firms, whereas in the second, turnover is too low. Proposition 3.4 identifies the condition required for efficient turnover:

PROPOSITION 3. 4: When there two firm-types of equal measure, with $\underline{p} < \overline{p}$, and contract choice is unrestricted, there is a unique equilibrium outcome. If

$$\frac{\overline{p}-b}{\underline{p}-b} \geq 1+\frac{2\mu}{\lambda} \tag{3}$$

it is a separating outcome with efficient turnover. Otherwise it is a pooling outcome

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sufficiently low initially (possibly negative) to have the same effect. From Proposition 4.2, below, we can determine how low a wage is required if the monopsony outcome is to prevail with wage contracts.

with no job-to-job turnover.

PROOF: See Appendix.

So, job-to-job turnover occurs only if the productivity in high-firms exceeds that in low-firms by a positive margin depending on the matching frictions. The outcome with no turnover occurs when the meeting rate λ is low (relative to the retirement rate μ). In this case it is less worthwhile for high-firms to try to attract workers from low-firms. They find it more profitable to offer contracts of low initial value, attracting unemployed workers only, and keeping them until retirement.

In Burdett and Mortensen's equilibrium with two firm-types and constant wages, each type randomises over a wage interval, but the intervals do not overlap so a worker in a low-firm always accepts an offer from a high-firm. Hence labour turnover is socially optimal. But here, when firms are unrestricted, monopsony power is uninhibited, and turnover may be too low. Inefficiency arises because, with heterogeneity, firms can no longer price-discriminate perfectly in equilibrium. They achieve only third-degree price discrimination: effectively they set different wages in the internal and external markets, but they cannot discriminate between workers with different reservation values in the external market. It is well-known that allowing a monopolist to price-discriminate imperfectly can reduce welfare.

Note that the equilibrium contract has the same form as that of Salop and Salop (1976). In their model firms are similarly concerned about turnover, since they must pay a specific training cost at the start of employment, and workers differ in (exogenous and unobserved) quit propensity. But in equilibrium the entrance fee is equal to the training cost – they obtain efficiency with a payment profile that is identical to the productivity profile. Here, in contrast, firms' concern about turnover arises from labour market frictions, and leads them to use a contract that does not follow the productivity profile. It is this divergence that causes inefficiency.

4. WAGE-TENURE CONTRACTS

In practice we rarely observe contracts with entrance fees, although they are often predicted in labour market models. One reason may be that workers are credit-constrained; also firms may have an incentive to renege once the fee has been paid - a guarantee of future employment is not normally possible under employment law. We have seen that quitting fees are a perfect substitute for entry fees, but these may be

illegal, as a form of bonding. Do such arguments justify the assumptions of wage-posting models, and suggest that we should disregard the results in section 3? To investigate this, we determine the contract-posting equilibrium when firms are restricted to wage contracts – but not so restricted that the wage must be constant. We find that even when their freedom to vary the wage is severely limited, the equilibrium outcome is similar in character to that in the unrestricted case.

So, suppose that firms are restricted to *wage-tenure* contracts, $w(t)$. They cannot set lump-sum fees, or lower the wage below some threshold[5] c: $w(t) \geq c$ for all t, where c is strictly below the productivity $\underline{p}$ of the lowest firm-type. The lower bound c may be above or below the flow utility of unemployment, b: if $c \geq b$, it could represent a minimum wage; if $c < b$ it might be interpreted as a subsistence wage for a credit-constrained worker.

4.1 Value Functions for Wage-Tenure Contracts

Consider the value functions for worker and firm, $V(t)$ and $Y(t)$, and the joint value $J(t) \equiv V(t)+Y(t)$, for a match with productivity p and wage-profile $w(t)$. Assume that $w(t)$ is continuous, except possibly at a finite number of points at which it is right-continuous. Provided that $J(t) \geq V(t) \geq V_u$ for all t, the firm will never choose to terminate the match; the worker may do so, but only if he receives a superior offer from another firm. It cannot be optimal for the firm to choose a contract that violates this condition.

Van den Berg (1990) establishes the existence and properties of the value function for a similar non-stationary problem. His analysis extends easily to show that V and Y are continuous and right-differentiable functions of tenure, satisfying:

$$\mu V(t) = V'(t) + w(t) + \lambda\big(E[\max(V(t), Z)] - V(t)\big) \tag{4}$$

$$\mu Y(t) = Y'(t) + p - w(t) - \lambda(1 - r(t))Y(t) \tag{5}$$

where $r(t) \equiv F(V(t)-)$ is the retention probability[6]. Equation (4) is the standard asset-pricing equation for a non-stationary job-search problem: the flow return on the asset V is the appreciation in its value, plus the wage, plus the expected gain from finding a better match. Equation (5) for the firm's payoff Y can be interpreted

[5] Note that to rule out instantaneous fees, we must set a bound on the wage: if firms could set an arbitrarily low wage for a short time, they could effectively impose a fee.

similarly. To ensure existence of the solutions we make the innocuous assumption that the wage is eventually constant: that is, $w(t) = \overline{w}$ for all $t \geq \bar{t}$, for some $\bar{t} \geq 0$. Then *V* and *Y* are constant for $t \geq \bar{t}$; solving the stationary problem for $V(\bar{t})$ and $Y(\bar{t})$ gives boundary conditions for (4) and (5).

4.2 The Recruitment and Retention Problem

The firm's optimisation problem is:

$$\max_{w(t)\geq c} \Pi \equiv \big(J(0) - V(0)\big) G(V(0)) \tag{6}$$

In the unrestricted case in section 3, an optimal strategy sets the wage to maximise the joint value of the match and optimise retention, and an entry fee to optimise recruitment. An analogous approach to problem (6) is to separate it into two steps:

Retention: $$\bar{J}(v) = \max_{w(t)\geq c} J(0) \text{ subject to } V(0) = v \tag{7}$$

Recruitment: $$\max_{v} \big(\bar{J}(v) - v\big) G(v) \tag{8}$$

Problem (7) is to find a wage profile to maximise the joint match value, given that the contract must have an initial value *v* to the worker. Problem (8) is to choose this initial value optimally. The separation is legitimate provided that (7) has a solution for any feasible *v*.

In the unrestricted case a fee-contract solves the recruitment-and-retention problem. An analogous wage-contract would be a *step-contract*: a wage equal to *c* (the lowest acceptable wage) for some initial period $\tilde{t}$, and subsequently equal to productivity *p* (Figure 1a). The lower is *c*, the more closely this contract can approximate a fee-contract. We will show that a step-contract solves the retention problem (7), and hence that step-contracts are optimal wage-tenure contracts.

4.3 Step-Contracts

A step-contract is determined by two parameters $(p, \tilde{t})$[7]. Let $V(t; p, \tilde{t})$ denote the worker's valuation of a step-contract at tenure *t*. It is clear that *V* increases as his tenure approaches $\tilde{t}$, and then remains constant (Figure 1b).

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[6] More precisely, we should define $r(t)$ as the right-hand limit of $F(V(t)-)$, to ensure that it is right-continuous. This makes no difference to the match values since $F(V(t)-)$ and its right-hand limit are equal almost everywhere.

[7] The lower bound *c* is the same for all workers and firms. We allow $\tilde{t} = \infty$, in which case the wage remains at *c* forever, and $V(t) = J_c^*$ for all *t*.

The value function $V(t; p, \tilde{t})$ has a useful property: for $t < \tilde{t}$, it satisfies the differential equation (4) with *w=c*, and this equation does not depend on either of the parameters *p* and $\tilde{t}$. So, we define the *standard value profile W*(.) as a solution of the same differential equation:

$$\mu W(s) = W'(s) + c + \lambda\big(E[\max(W(s), Z)] - W(s)\big) \quad \text{and} \quad W(0) = J_{\underline{p}}^* \qquad (9)$$

W(.) is independent of *p* and $\tilde{t}$, and the boundary condition is chosen arbitrarily to depend on the lowest productivity level $\underline{p}$ (any value above *c* would do). Then, we can show that the worker's valuation of *any* step-contract is some segment of this standard profile *W*. As illustrated in Figures 1b and 1c, we can find a starting point $s_0(p, \tilde{t})$ so that the value function $V(t; p, \tilde{t})$ between 0 and $\tilde{t}$ is identical to the segment of *W*(*s*) between s_0 and $s_0 + \tilde{t}$. Thus, a worker with tenure *t* in a step-contract can be regarded equivalently as having *position s* in the standard value profile. His initial position s_0 is determined by the parameters of his contract; his position then increases one-for-one with his tenure until it reaches $s_0 + \tilde{t}$, after which it remains constant. Formally, we have the following key result:

LEMMA 4.1: (i) The *standard value profile W*(*s*) is continuous and strictly increasing for all *s*, with range (J_c^*, ∞). (ii) For any step-contract $(p, \tilde{t})$, there is a unique $s_0(p, \tilde{t})$ so that a worker with tenure *t* has *position* $s(t) \equiv s_0 + \min(t, \tilde{t})$ in the standard profile: $V(t; p, \tilde{t}) \equiv W(s)$.

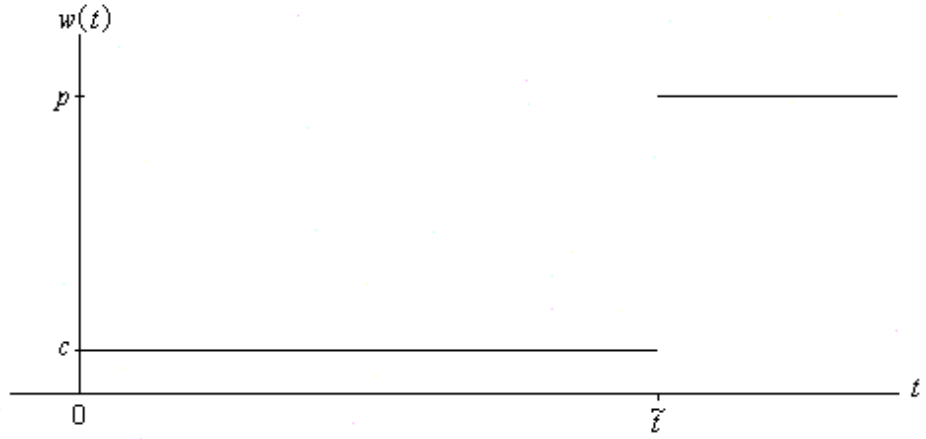
PROOF: (i) See Appendix.

(ii) Let $s_0(p, \tilde{t}) = W^{-1}(J_p^*) - \tilde{t}$ (well-defined by (i)). Then if $s(t) = s_0 + \min(t, \tilde{t})$, *V*(*t*)≡*W*(*s*) satisfies (4) with *w=c* for $t < \tilde{t}$, and $V(t) = J_p^*$ for $t \geq \tilde{t}$, so it is the value function for the step-contract $(p, \tilde{t})$. □

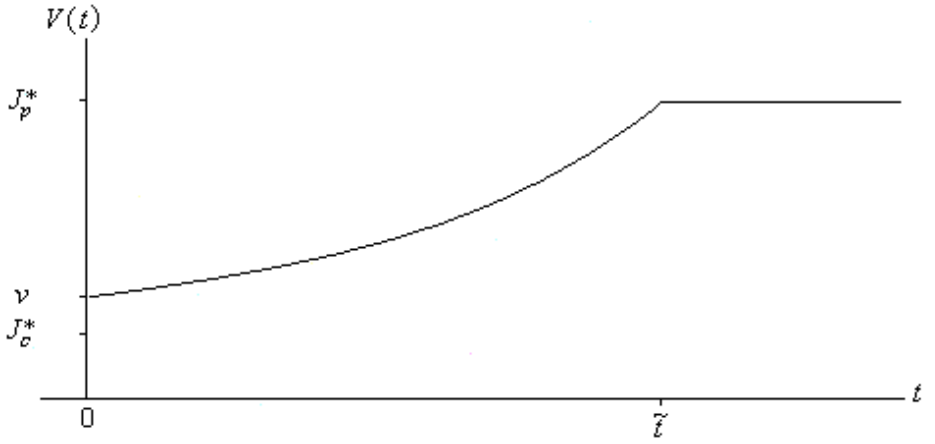
The concept of position gives us a convenient way of comparing step-contracts, which will be exploited to obtain the equilibria in sections 4.5 and 4.6. It captures all the information about a contract that matters to the worker. If he receives an offer from an alternative firm he will accept the offer if and only if it has initial position greater than his current position – that is, if it moves him further along the standard value profile.

FIGURE 1: The Step-Contract $(p, \tilde{t})$

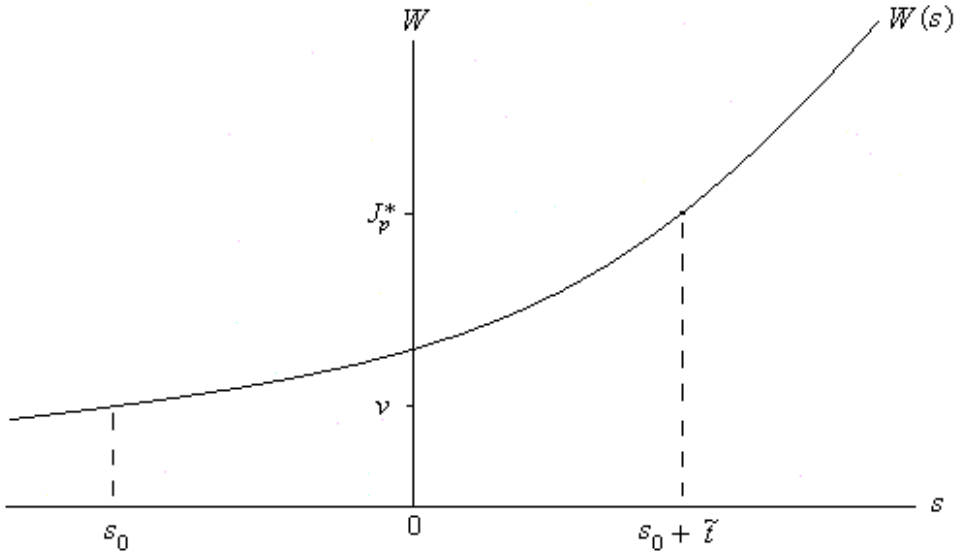
a. Wage Profile



b. Value Profile



c. The Standard Value Profile: $V(t; p, \tilde{t}) = W(s)$ for $t \in [0, \tilde{t}]$



Now consider the retention and recruitment problems (7) and (8), for a firm of type *p*. First, we can see from Figure 1c that the firm can find a step-contract with any desired initial value $v \in (J_c^*, J_p^*]$, by choosing $\tilde{t} = W^{-1}(J_p^*) - W^{-1}(v)$. Next we can show that the worker's value $V(t)$ rises more quickly in the step-contract than in any other contract with initial value *v* (because it reaches J_p^* as soon as possible). Then, since the worker's valuation is higher at every *t* than in any other contract, the probability of his leaving is minimised – the step-contract therefore solves the retention problem. Finally, by choosing the initial value optimally, the firm can solve the recruitment problem (8). Thus we have:

PROPOSITION 4.1: When the firm is restricted to wage contracts, there exists a step-contract that is optimal for the firm.

The full proof is given in the Appendix.

4.4 Equilibria in Wage-Tenure Contracts

In the unrestricted case there is always an optimal fee-contract; similarly in the restricted case there is always an optimal step-contract. The analogy extends further: when looking for equilibria in wage-tenure contracts we can confine attention to those in which all firms use step-contracts. It is shown in the Supplementary Appendix that the direct analogue of Proposition 3.2 holds: every equilibrium has an equivalent equilibrium in step-contracts, with the same outcome and allocation of resources.

To find step-contract equilibria, we can apply Lemma 4.1. An employee's reservation value depends only on his current position *s* in the standard value profile. He accepts an alternative offer if and only if it has initial position $s_0 \geq s$. We can thus transform the problem to work entirely in position *s*, and look for an equilibrium in offers of initial position.

In the following sections, we determine the step-contract equilibrium for the homogeneous and heterogenous cases. Here we make two preliminary points. First, the assumptions $\underline{p} > b$ and $\underline{p} > c$ guarantee that all firms make strictly positive profits in any equilibrium. A step-contract with initial value to the worker of $\max(J_b^*, J_c^*)$ attracts unemployed workers and brings strictly positive profits for the firm, since it employs them (at least for a short time) at a wage less than productivity. Hence, as before, all firms make offers that are accepted by the unemployed, and the equilibrium unemployment rate is $u = \mu / (\mu + \lambda)$. Second, we can rule out the

possibility that in equilibrium a strictly positive mass of firms offers the contract with $w=c$ forever ($\tilde{t} = \infty$). If a mass of firms were offering $w=c$, then a firm using this contract would lose its workers with positive probability to an offer of the same contract. But a step-contract with any finite $\tilde{t}$ would have strictly higher joint value because workers do not leave when they encounter offers of $w=c$. By choosing $\tilde{t}$ sufficiently high the firm could set the initial value to the worker arbitrarily close to J_c*, and hence obtain a strictly higher payoff for itself.

4.5 The Equilibrium with Homogeneous Firms

Suppose that all firms have identical productivity p. We know (Proposition 3.3) that with unrestricted contracts they can extract all the rent to the matching frictions and there is no job-to-job turnover in equilibrium. We will now show (Proposition 4.2 below) that the no-turnover result holds even when firms are restricted to wage contracts, although monopsony power is reduced.

We need only to find the equilibrium in step-contracts, or equivalently, in offers of initial position. In such an equilibrium, employed workers with position s have valuation $W(s)$, and since the lowest productivity $\underline{p}$ is equal to p, their wage steps up when s reaches zero, after which position remains constant. The reservation position of unemployed workers is[8] $s_u \equiv W^{-1}(J_b^*)$; all employed workers, and all offers, have $s \in [s_u, 0]$. Define $a(s)$ as the probability that a worker will accept an offer s (which is the stock of workers with current position less than or equal to s), $\tilde{Y}(s)$ as the firm's continuation value at position s, and $\tilde{r}(s)$ as the retention probability (which is also the distribution of offers of position). Then $\tilde{r}(0) = a(0) = 1$, $\tilde{Y}(0) = 0$, $\tilde{r}(s_u) = 0$, and $a(s_u) = u$. From (5) $\tilde{Y}(s)$ satisfies:

$$\tilde{Y}'(s) = \left(\mu + \lambda(1-\tilde{r}(s))\right)\tilde{Y}(s) - (p-c) \quad \forall s < 0 \tag{10}$$

By equating the flows of workers entering and leaving the stock $a(s)$ we obtain:

LEMMA 4.2: In an equilibrium with offer distribution $\tilde{r}(s)$, the probability of acceptance $a(s)$ of an offer of position s is continuous on $[s_u, 0]$ with right-derivative:

$$a'(s) = \mu - (\mu + \lambda(1-\tilde{r}(s)))a(s) \tag{11}$$

PROOF: See Appendix.

The firm's payoff from an offer s is $\Pi = \tilde{Y}(s)a(s)$. (10) and (11) determine $\tilde{Y}(s)$ and $a(s)$ for a given offer distribution $\tilde{r}(s)$, so we can now look for an equilibrium $\tilde{r}(s)$. Proposition 4.2 states that the equilibrium is unique, with all firms offering the same contract. To prove it, we show first that any equilibrium distribution $\tilde{r}(s)$ has a single mass point. We can then solve (10) and (11) with a distribution of this form, to obtain Π as an explicit function of s, and show that there is a unique optimal offer.

PROPOSITION 4.2: If all firms have the same productivity, p, and are restricted to wage contracts with $w \geq c$ for some $c<p$, there is a unique equilibrium outcome. All firms offer contracts of initial value V_0 to the worker, where $\mu V_0 = \max\left(b, \frac{\lambda p + \mu c}{\lambda + \mu}\right)$, and there is no job-to-job turnover.

PROOF: See Appendix.

Proposition 4.2 means that if firms have *any* freedom to vary wages with tenure (that is, if $c<p$) there is no turnover in equilibrium. The stark contrast with Burdett and Mortensen's result for constant wages occurs because although all workers have the same contract, they have different reservation values depending on tenure – the discontinuity in firms' payoffs disappears.

However, the more firms are constrained in their ability to lower the wage, the less rent they obtain. When c is low enough, firms can extract all the rent ($V_0=b/\mu$) with a contract in which the wage remains at c for a short time. Because the time before the contract steps up is short, there are not enough employed workers with a low enough reservation value to make it worthwhile for other firms to try to bid them away; they prefer to offer the same contract of initial low value, and obtain unemployed workers only. When c is higher, firms would have to leave the wage at c for a long time to obtain all the match rent. But this would give other firms an incentive to try to bid away employed workers. They have to reduce the time before the wage steps up, to discourage this - so they offer contracts of higher initial value to the worker, who thus obtains part of the match rent.

Finally, note that the worker's share of the rent increases with the meeting rate λ, and for any finite c, $V_0 \to p/\mu$ as $\lambda \to \infty$. This is the competitive labour market

[8] For ease of exposition we focus on the case $c<b$, for which s_u is finite. If $b \leq c$, $s_u = -\infty$ and all results hold, with only slight (and obvious) modifications.

outcome, in which workers capture the whole match surplus. In contrast, in the unrestricted case, the equilibrium fee-contract delivers the whole surplus to the firm for all values of λ [9]. Thus, the competitive outcome is the frictionless limit of the model with restricted contracts, whereas the monopsony outcome is the limit of the model with unrestricted contracts.

4.6 Equilibrium and Inefficiency with Heterogeneous Firms

Now suppose there are two types of firm, half with low productivity, $\underline{p}$, and half with high productivity $\overline{p}$, as in 3.2. In the unrestricted case, equilibria had either efficient turnover, or zero-turnover. With wage-tenure contracts it is also possible to have equilibria with partial-turnover, in which high-firms offer contracts of value high enough to attract workers of low tenure in low-firms, but not those of high tenure.

The equilibrium will be efficient, with *full-turnover*, if all firms of the same type offer the same contract, and the contract offered by a high-firm is sufficient to attract any worker in a low-firm. Conversely, *zero-turnover* occurs if all firms offer the same contract. To find these equilibria, and intermediate ones, we look for an equilibrium in which three step-contracts are offered: all low-firms offer initial position s_L, a fraction $h \in [0,1]$ of high-firms offer $s_H \geq 0$, high enough to attract all low-firm workers[10], and the remaining high-firms offer intermediate position s_M. The distribution of offers is then:

$$\tilde{r}(s) = \begin{cases} 0 & s < s_L \\ \frac{1}{2} & s_L \leq s < s_M < 0 \\ 1 - \frac{1}{2}h & s_M \leq s < s_H \\ 1 & 0 \leq s_H \leq s \end{cases} \qquad (12)$$

It is straightforward (although algebraically tedious) to obtain the equilibrium, using the same methods as for homogeneous firms. To illustrate, for the case $c=b$ we have the following result (for which the proof is available in the Supplementary Appendix):

[9] With unrestricted contracts and an infinite meeting rate there is a continuum of additional equilibria. A fee-contract with any fee that shares the rent between worker and firm can be an equilibrium contract. This indeterminacy happens because there are fixed numbers of homogeneous firms and workers and a positive rent even when frictions disappear. When there is perfect competition between workers for jobs, and between firms for workers, the "competitive equilibrium" does not determine the sharing of the rent.

[10] Note that contracts in low-firms step up at $s=0$.

PROPOSITION 4.3: When there two firm-types of equal measure, with $\underline{p} < \bar{p}$, and firms are restricted to wage contracts with $w \geq b$:

(i) there is no zero-turnover equilibrium;

(ii) there is an efficient full-turnover equilibrium if and only if:

$$\frac{\bar{p}-b}{\underline{p}-b} \geq 1+K \quad \text{where } 0 < K < 2\mu/\lambda;$$

(iii) otherwise there is a partial-turnover equilibrium in which some, or all, high-firms offer contracts that are not accepted by workers of high tenure in low-firms.

Compare this result with Proposition 3.4 for the unrestricted case. Condition (ii) shows that with restricted contracts, an efficient outcome occurs for a wider range of relative productivity. Furthermore, when the condition for efficient turnover is not satisfied, the outcome with restricted contracts has partial turnover, whereas there is no turnover at all with unrestricted contracts.

Thus, the restrictions on contracts facilitate efficient matching. If there are no restrictions, firms' attempts to price-discriminate between the internal and external labour markets lock workers into jobs, reducing equilibrium turnover below the efficient level. Restricting their ability to price-discriminate means that wages better reflect productivity, so that workers make more appropriate job-changing decisions. Turnover is higher, and matching more efficient.

5. INTERPRETATION

We have modelled a frictional labour market with on-the-job search, in which firms can post general contracts. More general contracts give firms greater opportunity to exercise monopsony power than when they are restricted to constant wages as in Burdett and Mortensen's model. Firms use contracts in which the value to the worker increases with tenure – either instantaneously, by means of an entry fee, or slowly if entry fees are not possible, by means of a rising wage. Such contracts increase the retention of existing workers, but allow the firms to extract rent at the recruitment stage. Contracts with rising value have the effect of "locking-in" workers and hence reducing labour market turnover and, if firms differ in productivity, reducing the efficiency of matching. They are a form of monopsonistic price discrimination, which inhibits workers from making efficient job-changing decisions. In Burdett and Mortensen's model, firms are constrained to use contracts that are privately

inefficient, but the equilibrium is socially optimal. Here, we have obtained exactly the reverse outcome.

The contract-posting equilibrium has a unique allocation of resources and expected payoffs, but the contracts themselves are not unique. Stevens (2000) characterises a continuum of potential equilibrium contracts for the case of homogeneous firms. Any wage contract can support a symmetric equilibrium provided that its value to the worker rises sufficiently steeply to deter competing firms from bidding away employed workers. The step-contract is the most steeply rising wage contract, weakly dominating the alternatives. The indeterminacy arises from the assumption of risk-neutrality of workers: they are indifferent between wage profiles of the same expected value. Burdett and Coles (2002) have relaxed this assumption, for the homogenous case. It is more difficult to characterise the equilibrium, but interestingly, Burdett and Mortensen's result of turnover between identical firms re-emerges. Nevertheless, our result of rising-wage tenure profiles is robust, and it is this feature that leads to sub-optimal turnover with heterogeneous firms.

Labour contracts of the form predicted by the recruitment-and-retention model, specifying a low wage for an initial period, then subsequently the standard wage for the job, are a common device in Britain, particularly for unskilled jobs (for example, those studied by Manning, 2000). For higher level jobs, large firms often use an incremental pay scale, with new employees entering at a low point, and receiving wage increments at regular predetermined intervals until they reach the top of the scale. Andrews, Bradley and Upward (1999), studying the youth labour market in Lancashire, found that in jobs that provided no training, advertised wage growth was nevertheless 17% during the first year.

More generally, the model is consistent with the mass of accumulated evidence that wages rise, and turnover falls, with tenure (Farber, 1999). There are several well-known competing explanations, but a distinctive feature of this model is that the wage-tenure link leads to a socially sub-optimal level of turnover. In the context of the old debate on internal labour markets (ILMs) the model suggests that ILM strategies may be *both* privately optimal (as argued by Wachter and Wright, 1990) and impediments to efficient employment (as claimed by Ulman, 1992).

The recruitment-and-retention model predicts several determinants of the slope of the relationship between wages and tenure that might be investigated empirically. Ceteris paribus, the relationship is steeper when market frictions are

higher, when the difference between productivity and the utility of unemployment is greater, and when workers are less credit-constrained. This may help to explain differences between tenure coefficients, and mobility rates, for different groups of workers, and for different markets and countries. For example, returns to tenure are higher for skilled than for unskilled workers in Germany, and mobility in Germany is lower than in the US (Dustmann and Meghir, 2001). As noted by Farber (1999), heterogeneity in returns to tenure is an important topic for future research.

Our results suggest that conditions or institutions that restrict firms' ability to wage-discriminate (imperfectly) can be beneficial, in that they promote efficient matching. When workers are unable to accept contracts involving very low wages or entry fees, perhaps because of credit-constraints or legal restrictions on bonding, firms cannot use steeply-rising contracts so they are less able to exercise monopsony power. In particular, a minimum wage can raise efficiency. In Burdett and Mortensen's model a minimum wage raises employment if workers are heterogeneous in their utility of unemployment, *b*. Brown (1999), criticising the applicability of the monopsony argument for a minimum wage, dismisses this example of an apparently positive effect because the equilibrium wage distribution does not have the positive spike at the level of the minimum wage that is observed in practice. However in the model here, when firms respond to labour market frictions by using step-contracts, a minimum wage produces both a wage spike and a more efficient labour market. If we allowed for heterogeneity in the utility of unemployment, a minimum wage would also raise employment.

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APPENDIX

PROOF of **PROPOSITION 3. 4**:

(i) In a *separating equilibrium*, high–firms offer $J_{\underline{p}}*$ and attract both unemployed workers and those already in low–matches; and low–firms offer V_u.

So from (1) $J_p* = p/\mu$ for $p = \underline{p}, \overline{p}$ and $V_u = J_b* = \dfrac{\mu b + \frac{1}{2}\lambda \underline{p}}{\mu(\mu + \frac{1}{2}\lambda)}$.

Let e_L be the measure of employed workers in low-matches. The steady-state condition for e_L is: $e_L(\mu + \frac{1}{2}\lambda) = \frac{1}{2}\lambda u$.

This equilibrium exists if high–firms prefer to offer $J_{\underline{p}}*$ rather than V_u.

Contracts worth $J_{\underline{p}}*$ give payoff: $(J_{\overline{p}}* - J_{\underline{p}}*)(u + e_L) = \dfrac{\overline{p} - \underline{p}}{\mu + \frac{1}{2}\lambda}$

whereas contracts worth V_u give: $(J_{\overline{p}}* - V_u)u = \dfrac{\mu(\overline{p} - b) + \frac{1}{2}\lambda(\overline{p} - \underline{p})}{(\lambda + \mu)(\mu + \frac{1}{2}\lambda)}$

Comparing these gives condition (3).

(ii) In a *pooling equilibrium*, both types offer V_u and attract unemployed workers only. As before $J_p* = p/\mu$ for $p = \underline{p}, \overline{p}$ but now $V_u = b/\mu$. The steady-condition for e_L is $\mu e_L = \frac{1}{2}\lambda u$, and this equilibrium exists if high–firms prefer to offer V_u rather than $J_{\underline{p}}*$. The payoffs are, respectively:

$$(J_{\overline{p}}* - V_u)u = \frac{\overline{p} - b}{\lambda + \mu} \quad \text{and} \quad (J_{\overline{p}}* - J_{\underline{p}}*)(u + e_L) = \frac{(\overline{p} - \underline{p})(\mu + \frac{1}{2}\lambda)}{\mu(\mu + \lambda)}.$$

Comparing these gives the reverse of inequality (3). □

LEMMA A1: If Z is a random variable with distribution function F, and $x_1 > x_2$, then:

(i) $E[\max(x_1, Z)] - E[\max(x_2, Z)] \equiv$

$$x_1F(x_1) - x_2F(x_2) - E\left[Z \middle| x_2 \le Z < x_1\right](F(x_1) - F(x_2));$$

(ii) the function $\Gamma(x) \equiv (\mu + \lambda)x - \lambda E\left[\max(x, Z)\right]$ is continuous and strictly increasing in x.

PROOF: Write $E_i \equiv E[\max(x_i, Z)]$ and $F_i \equiv F(x_i)$. Then:

(i) $E_2 = x_2F_2 + E\left[Z \middle| Z > x_1\right](1 - F_1) + E\left[Z \middle| x_2 < Z \le x_1\right](F_1 - F_2)$

$= x_2F_2 - x_1F_1 + E_1 + E\left[Z \middle| x_2 < Z \le x_1\right](F_1 - F_2)$. Rearranging gives result (i).

(ii) From (i), $x_2 \le E\left[Z \middle| x_2 \le Z < x_1\right] < x_1 \Rightarrow (x_1 - x_2)F_2 < E_1 - E_2 \le (x_1 - x_2)F_1$.

Then $\Gamma(x_1) - \Gamma(x_2) = (\mu + \lambda)(x_1 - x_2) - \lambda\left(E_1 - E_2\right)$

$\Rightarrow 0 < (\mu + \lambda(1 - F_1))(x_1 - x_2) \le \Gamma(x_1) - \Gamma(x_2) < (\mu + \lambda(1 - F_2))(x_1 - x_2)$ □

PROOF of **LEMMA 4.1(i):**

Consider the function W satisfying (9). Using (1) we can write:

$$(W - J_c^*)' = \Gamma(W(t)) - \Gamma(J_c^*) \qquad \text{(A1)}$$

where Γ is defined in Lemma A1. Using the properties of Γ, $W(0) > J_c^* \Rightarrow W(t) > J_c^*$ and $W' > 0$ for all t. Continuity of W follows from integration . Thus W is bounded below, continuous, and strictly increasing. Differentiating (A1), W is strictly concave and hence not bounded above. As $t \to -\infty$ $W' \to 0$ so $W(t) \to J_c^*$. □

PROOF of **PROPOSITION 4.1:** (i) Take any $v \in (J_c^*, J_p^*]$. Using Lemma 4.1, there is a unique step-contract $(p, \tilde{t})$ with initial value v, given by $\tilde{t} = W^{-1}(J_p^*) - W^{-1}(v)$.

(ii) Define $V_s(t)$ as the worker's value of the step-contract $(p, \tilde{t})$, and compare it with $V(t)$ for an arbitrary contract with wage $w(t)$ and $V(0) = v$.

For $t \ge \tilde{t}$, $V_s(t) = J_p^* \ge V(t)$. For $0 \le t < \tilde{t}$, (4) $\Rightarrow V_s' = \Gamma(V_s) - c$ and

$V' = \Gamma(V) - w(t)$, where Γ is an increasing function by Lemma A1(ii).

So whenever $V_s \ge V$, $V_s' \ge V'$. But $V_s(0) = V(0)$. Hence $V_s(t) \ge V(t)$ $\forall t$.

(iii) Similarly compare the joint value $J_s(t)$ of $(p, \tilde{t})$, with $J(t)$ for the arbitrary contract. For $t \ge \tilde{t}$, $J_s(t) = J_p^* \ge J(t)$.

For $0 \le t < \tilde{t}$, first note that $r(t) \equiv F(V(t)-) \le F(V(t))$ and since V_s is continuous and strictly increasing, $r_s(t) = F(V_s(t))$. Adding equations (4) and (5):

$$J'(t) = (\mu + \lambda)J(t) - p - \lambda r(t)(J(t) - V(t)) - \lambda E[\max(Z, V(t))]$$

This equation is satisfied by both contracts. Subtracting, then using the results above for the retention probabilities and applying Lemma A1(i), we obtain:

$$J' - J_s' \ge (\mu + \lambda(1 - F(V)))(J - J_s) \quad \forall t \in [0, \tilde{t})$$

Then $J(\tilde{t}) \le J_s(\tilde{t}) \Rightarrow J \le J_s$ $\forall t$, and in particular $J(0) \le J_s(0)$.

(iv) Hence for $v \in (J_c^*, J_p^*]$ there is a step-contract that solves (7). If $v = J_c^*$, then (p, ∞) is the only contract with initial value v, so it solves (7). Let

$v^* = \arg\max_{J_c^* \le v \le J_p^*} \left(\bar{J}(v) - v\right) G(v)$, and let $(p, \tilde{t}^*)$ be the corresponding step-contract.

(v) An optimal contract for the firm must satisfy $J(0) \ge V(0) \ge \max(J_b^*, J_c^*)$. The payoff from any contract satisfying this condition is:

$$\begin{aligned}\Pi &= \left(J(0) - V(0)\right) G(V(0)) \\ &\le \left(\bar{J}(V(0)) - V(0)\right) G(V(0)) \\ &\le \left(\bar{J}(v^*) - v^*\right) G(v^*) = \text{the payoff from } (p, \tilde{t}^*).\end{aligned}$$

Hence no contract has higher payoff than this step-contract. □

PROOF of **LEMMA 4.2:** For $s_u \le s < 0$, $a(s+\Delta)$ is the total stock of unemployed workers and employed workers with current position less than $s+\Delta$. The number entering this stock in a time period of length Δ is $(1- a(s+\Delta))\mu\Delta$, the workers with higher position who are replaced by unemployed workers. In steady state this is equal to the number leaving, which (ignoring terms of order Δ^2) is given by:

$$(a(s+\Delta) - a(s)) + a(s+\Delta)\lambda\Delta(1-\tilde{r}(s+\Delta))$$

The first term is the employed workers whose position increases beyond $s+\Delta$, and the second is those who accept offers of higher position. Equating the two flows gives:

$$a(s+\Delta) - a(s) = \mu\Delta - a(s+\Delta)\Delta(\mu + \lambda(1-\tilde{r}(s+\Delta))$$

Dividing and letting $\Delta \to 0$ gives the result. □

PROOF of **PROPOSITION 4.2:** Differentiating the payoff function $\Pi = \tilde{Y}(s)a(s)$, and using (10) and (11):

$$\Pi'(s) = \tilde{Y}'a + \tilde{Y}a' = \mu\tilde{Y}(s) - (p-c)a(s) \qquad \text{(A2)}$$

$$\begin{aligned}\Pi''(s) &= 2\mu\tilde{Y}'(s) - \left(\mu + \lambda(1-\tilde{r}(s))\right)\Pi'(s) \\ &= \left(\mu + \lambda(1-\tilde{r}(s))\right)\Pi'(s) - 2(p-c)a'(s)\end{aligned} \qquad \text{(A3)}$$

We first show that any equilibrium $\tilde{r}(s)$ has a single mass point:

- If $\tilde{r}(s)$ is continuous and increasing over an interval, then firms make offers throughout the interval, so profit is constant there: $\Pi' = 0$, and $\Pi'' = 0$. Hence $a'(s) = 0$ in the interval, but from (11) this is impossible if $\tilde{r}(s)$ is not constant.
- Suppose s_1 is a mass point. If $s_1 = s_u$, (11) $\Rightarrow a'(s_1) > 0$. If $s_1 > s_u$ and $a'(s_1) = 0$, then since a is continuous at s_1 but $\tilde{r}(s)$ is not, (11) $\Rightarrow a'(s_1-) < 0$ which is impossible. Hence $a'(s_1) > 0$. For s_1 to be optimal $\Pi'(s_1) \le 0$ so (A2) $\Rightarrow$

$\tilde{Y}'(s_1) < 0$.

- Now suppose $s_2 > s_1$ is also a mass point, and no firm offers $s \in (s_1, s_2)$. Over $[s_1, s_2)$, $\tilde{r}(s)$ is constant, so from (10) $\tilde{Y}'$ is differentiable:

  $\tilde{Y}''(s) = (\mu + \lambda(1 - \tilde{r}(s_1)))\tilde{Y}'(s)$

  and hence $\tilde{Y}'(s) < 0$ throughout the interval. But then from (A2) $\Pi'(s) < 0$ throughout the interval, so we have a contradiction: s_2 cannot be optimal.

So, look for an equilibrium distribution of the form: $\tilde{r}(s) = 0 \ \forall s < s^*$ and $\tilde{r}(s) = 1$ $\forall s \geq s^*$. With this $\tilde{r}(s)$ we can solve (11) for a, (10) for $\tilde{Y}$, and (9) for W:

$a = \mu/(\mu+\lambda) \ \forall s \in [s_u, s^*]$; $(\lambda + \mu)a = \mu + \lambda(1 - \exp(-\mu(s - s^*))) \ \forall s \in [s^*, 0)$;

$\tilde{Y}(0) = 0$; $\mu\tilde{Y} = (p - c)\left(1 - \exp(\mu s)\right) \ \forall s \in [s^*, 0]$;

$W(0) = J^* = p/\mu$; $\mu W(s) = (p - c)\exp(\mu s) + c$.

The value of the offer is $V_0 = W(s^*)$. The first-order (necessary) condition for an equilibrium s^* is that $\Pi'(s^*) \leq 0$ and $s^* \geq s_u$ are complementarily slack.

$$\Pi'(s^*) \leq 0 \quad \Leftrightarrow \ \exp(\mu s^*) \geq \frac{\lambda}{\mu + \lambda} \ \Leftrightarrow \ \mu V_0 \geq \frac{\lambda p + \mu c}{\lambda + \mu} \qquad \text{(using (A2))}$$

$$s^* \geq s_u \quad \Leftrightarrow V_0 \geq V_u \quad \Leftrightarrow \ \mu V_0 \geq b \qquad \text{(using (1) and (2)).}$$

So the first-order condition is satisfied $\Leftrightarrow \mu V_0 = \max\left(b, \frac{\lambda p + \mu c}{\lambda + \mu}\right)$.

From (A3) it is immediately clear that the second-order condition is satisfied, and this condition is also sufficient for an equilibrium. Clearly there is no job-to-job turnover. Since all other equilibria are equivalent to step-contract equilibria, this is the unique outcome. □

SUPPLEMENTARY APPENDIX

Available at: www.restud.com