Week 5 - Solutions

PART A

1. $$MRS_{2,1} = \frac{dx_2}{dx_1} = -\frac{u_1}{u_2} = -\frac{\beta x_1^{\beta-1} x_2^{1-\beta}}{(1-\beta) x_1^{\beta} x_2^{1-\beta-1}} = -\frac{\beta}{1-\beta}\frac{x_2}{x_1}$$

2. $MRS_{2,1}$ condition: $-\frac{\beta}{1-\beta}\frac{x_2}{x_1} = -\frac{P_1}{P_2}$ Budget constraint: $P_1x_1 + P_2x_2 = m$

$$MRS_{2,1} \Rightarrow \quad x_2 = \left(\frac{1-\beta}{\beta}\right)\left(\frac{P_1}{P_2}\right)x_1 \quad \text{sub in} \Rightarrow \quad P_1x_1 + P_2\left(\frac{1-\beta}{\beta}\right)\left(\frac{P_1}{P_2}\right)x_1 = m$$

$$\Rightarrow \quad \frac{m}{P_1} = x_1\left(\frac{1-\beta}{\beta}+1\right) = x_1\left(\frac{1-\beta+\beta}{\beta}\right) = \frac{x_1}{\beta} \quad \Rightarrow \quad \boxed{x_1 = \beta\frac{m}{P_1}}$$

Sub into budget constraint: $P_1\left(\frac{\beta m}{P_1}\right) + P_2x_2 = m \Rightarrow P_2x_2 = m - \beta m$

$$\Rightarrow \boxed{x_2 = (1-\beta)\frac{m}{P_2}}$$

3. Take logs: $\ln x_1 = \ln\beta + \ln m - \ln P_1 \Rightarrow$ $\varepsilon_{x_1,P_1} = -1$

$\varepsilon_{x_1,P_2} = 0$

$\varepsilon_{x_1,m} = 1$

$\ln x_2 = \ln(1-\beta) + \ln m - \ln P_2 \Rightarrow$ $\varepsilon_{x_2,P_1} = 0$

$\varepsilon_{x_2,P_2} = -1$

$\varepsilon_{x_2,m} = 1$

4. $$\left.\frac{\partial x_1}{\partial P_1}\right|_u = \frac{\partial x_1}{\partial P_1} + \frac{\partial x_1}{\partial m}x_1$$

$$= \frac{-\beta m}{P_1^2} + \frac{\beta}{P_1}x_1$$

$$= \beta\left(\frac{x_1}{P_1} - \frac{m}{P_1^2}\right)$$

$$= \beta\left(\frac{x_1}{P_1} - \frac{P_1x_1}{\beta}\frac{1}{P_1^2}\right)$$

$$= \beta\left(\frac{x_1}{P_1}\right) - \frac{x_1}{P_1} = \boxed{-(1-\beta)\frac{x_1}{P_1}}$$

$$\left.\frac{\partial x_2}{\partial P_2}\right|_u = \frac{\partial x_2}{\partial P_2} + \frac{\partial x_2}{\partial m}x_2$$

$$= -(1-\beta)\frac{m}{P_2^2} + \frac{(1-\beta)}{P_2}x_2$$

$$= (1-\beta)\left(\frac{x_2}{P_2} - \frac{m}{P_2^2}\right)$$

$$= (1-\beta)\left(\frac{x_2}{P_2} - \frac{P_2x_2}{(1-\beta)}\frac{1}{P_2^2}\right)$$

$$= (1-\beta)\frac{x_2}{P_2} - \frac{x_2}{P_2} = \boxed{-\beta\frac{x_2}{P_2}}$$

5. $V(p_1,p_2,m) = \left(\frac{\beta m}{p_1}\right)^{\beta}\left(\frac{(1-\beta)m}{p_2}\right)^{1-\beta} = m\left(\frac{\beta}{p_1}\right)^{\beta}\left(\frac{1-\beta}{p_2}\right)^{1-\beta}$

PART B

1. MRS$_{2,1}$ condition : $\frac{-\beta}{1-\beta}\frac{x_2}{x_1} = \frac{-p_2}{p_1}$ Utility function : $x_1^{\beta}x_2^{1-\beta} = u$

MRS$_{2,1}$ condition $\Rightarrow$ $x_2 = \left(\frac{1-\beta}{\beta}\right)\left(\frac{p_1}{p_2}\right)x_1$ sub in $\Rightarrow$ $x_1^{\beta}\left(\frac{1-\beta}{\beta}\right)^{1-\beta}\left(\frac{p_1}{p_2}\right)^{1-\beta}x_1^{1-\beta} = u$

$\Rightarrow x_1\left(\frac{p_1}{p_2}\frac{1-\beta}{\beta}\right)^{1-\beta} = u$ $\Rightarrow$ $\boxed{x_1 = u\left(\frac{p_2}{p_1}\right)^{1-\beta}\left(\frac{\beta}{1-\beta}\right)^{1-\beta}}$

Similarly $\boxed{x_2 = u\left(\frac{p_1}{p_2}\right)^{\beta}\left(\frac{1-\beta}{\beta}\right)^{\beta}}$

2. $\left.\frac{\partial x_2}{\partial p_2}\right|_u = u\beta\left(\frac{p_1}{p_2}\frac{1-\beta}{\beta}\right)^{\beta-1}\left(-\frac{p_1}{p_2^2}\frac{1-\beta}{\beta}\right) = \frac{-\beta u\left(\frac{p_1}{p_2}\frac{1-\beta}{\beta}\right)^{\beta}\left(\frac{p_1}{p_2^2}\frac{1-\beta}{\beta}\right)}{\left(\frac{p_1}{p_2}\frac{(1-\beta)}{\beta}\right)}$

$= \boxed{\frac{-\beta x_2}{p_2}}$

3. $m(p_1,p_2,u) = p_1u\left(\frac{p_2}{p_1}\frac{\beta}{1-\beta}\right)^{1-\beta} + p_2u\left(\frac{p_1}{p_2}\frac{1-\beta}{\beta}\right)^{\beta}$

4. $m(p_1, p_2 u) = p_1 u \left(\frac{p_2}{p_1}\right)^{1-\beta}\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + p_2 u \left(\frac{p_1}{p_2}\right)^{\beta}\left(\frac{1-\beta}{\beta}\right)^{\beta}$

$$= p_1 u \frac{p_2^{1-\beta} p_1^{\beta}}{p_1}\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + p_2 u \frac{p_1^{\beta}}{p_2^{\beta}}\left(\frac{1-\beta}{\beta}\right)^{\beta}$$

$$= u p_2^{1-\beta} p_1^{\beta}\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + u p_2^{1-\beta} p_1^{\beta}\left(\frac{1-\beta}{\beta}\right)^{\beta}$$

$$= u p_1^{\beta} p_2^{1-\beta}\left[\left(\frac{\beta}{1-\beta}\right)^{1-\beta} + \left(\frac{1-\beta}{\beta}\right)^{\beta}\right]$$

$$= u p_1^{\beta} p_2^{1-\beta}\left[\frac{\beta}{1-\beta}\frac{(1-\beta)^{\beta}}{\beta^{\beta}} + \frac{(1-\beta)^{\beta}}{\beta^{\beta}}\right]$$

$$= u p_1^{\beta} p_2^{1-\beta}\left[\frac{(1-\beta)^{\beta}}{\beta^{\beta}}\left(\frac{\beta}{1-\beta} + 1\right)\right]$$

$$= u p_1^{\beta} p_2^{1-\beta}\left[\frac{(1-\beta)^{\beta}}{\beta^{\beta}}\left(\frac{1}{1-\beta}\right)\right] = u p_1^{\beta} p_2^{1-\beta}\left[\frac{(1-\beta)^{\beta-1}}{\beta^{\beta}}\right]$$

$$= u p_1^{\beta} p_2^{1-\beta}\left[\frac{1}{\beta^{\beta}(1-\beta)^{1-\beta}}\right] = u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta}$$

So $m(p_1, p_2, u) = u\left(\frac{p_1}{\beta}\right)^{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta}$

Solve for u:

$$v(p_1, p_2, m) = m\left(\frac{\beta}{p_1}\right)^{\beta}\left(\frac{1-\beta}{p_2}\right)^{1-\beta}$$

Shephard's Lemma $\quad \frac{\partial m(p_1, p_2, u)}{\partial p_i} = x_i(p_1, p_2, u)$

$$\frac{\partial m(p_1, p_2, u)}{\partial p_1} = u\beta\left(\frac{p_1}{\beta}\right)^{\beta-1}\frac{1}{\beta}\left(\frac{p_2}{1-\beta}\right)^{1-\beta} = u\left(\frac{p_2}{p_1}\right)^{1-\beta}\left(\frac{\beta}{1-\beta}\right)^{1-\beta} = x_1(p_1, p_2, u)$$

$$\frac{\partial m(p_1, p_2, u)}{\partial p_2} = u\left(\frac{p_1}{\beta}\right)^{\beta}(1-\beta)\left(\frac{p_2}{1-\beta}\right)^{1-\beta-1}\frac{1}{(1-\beta)} = u\left(\frac{p_1}{p_2}\right)^{\beta}\left(\frac{1-\beta}{\beta}\right)^{\beta} = x_2(p_1, p_2, u)$$