Introductory Economics

Lectures 9 & 10 - Consumer Theory

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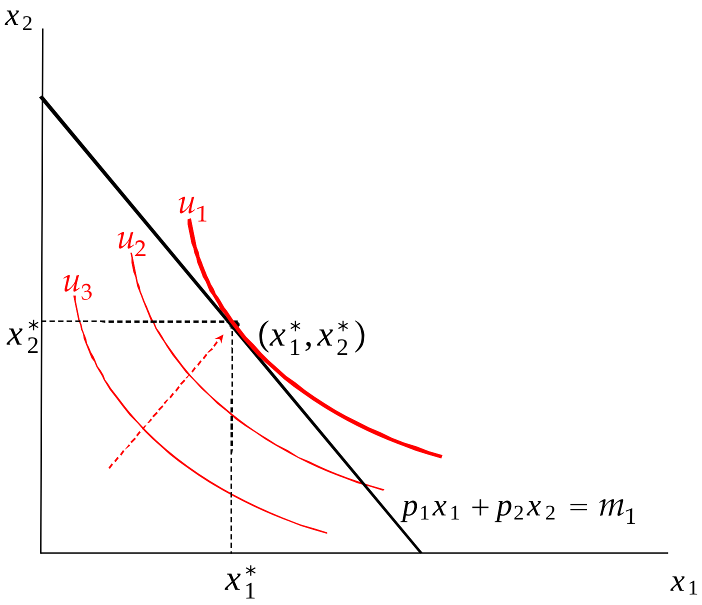
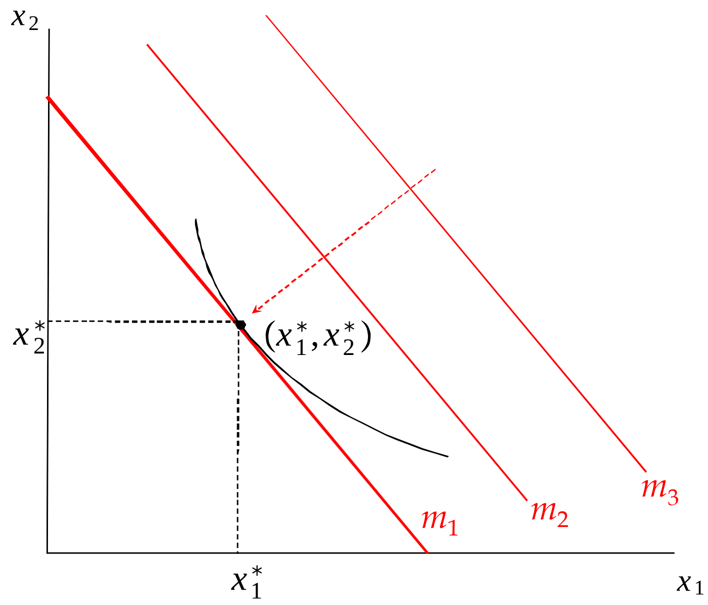
Utility Maximisation/Cost Minimisation - Dual versions of the model

"People choose, out of the combinations of market products which they can afford, the combination which they most prefer"

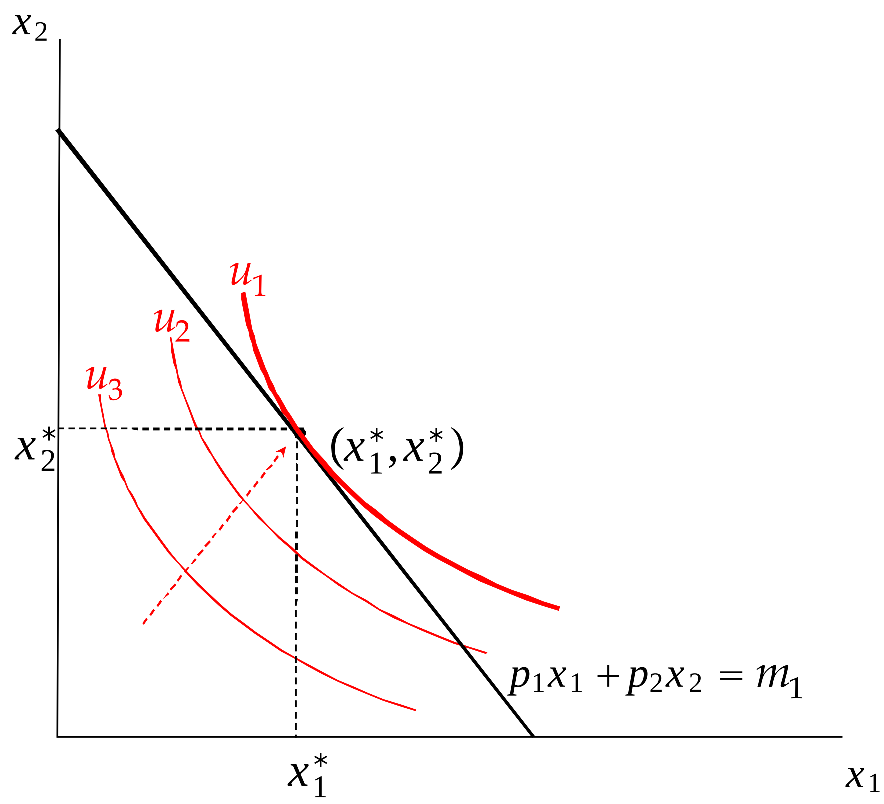
"People choose, out of the combinations of market products which guarantee a certain level of preference-satisfaction, the combination which costs the least"

Utility Maximisation/Cost Minimisation - Dual versions of the model





Utility Maximisation/Cost Minimisation - Dual versions of the model



- The maximum utility level (up to a pmt) we can reach depends on the p_1, p_2 and m

- We can vary m and p_1, p_2 continuously and note the resulting maximum utility as we go.

- This maps out a *function* relating prices and the budget to spending called the *indirect utility function*

$$v(p_1, p_2, m)$$

- It tell us the maximum u available for any combination of m and p_1, p_2

Utility Maximisation/Cost Minimisation - Dual versions of the model

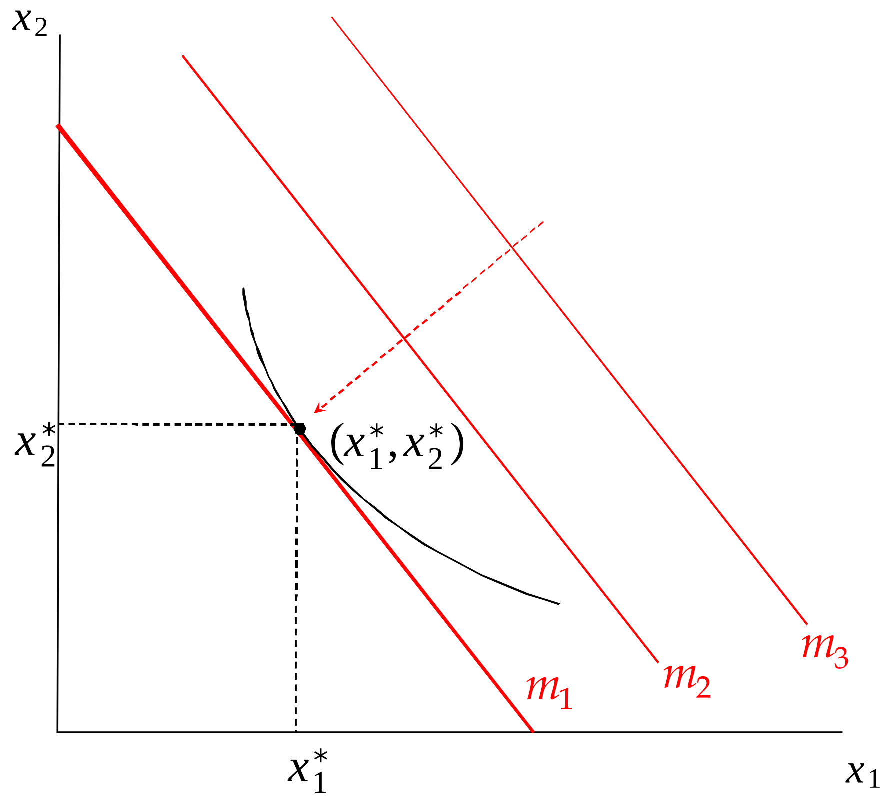
- To find it

1. take the Marshallian demands $x_1\left(p_{1,}p_2, m\right)$ and $x_1\left(p_{1,}p_2, m\right)$

2. Substitute them into the utility function

$$v\left(p_{1,}p_2, m\right) = u\left(x_1\left(p_{1,}p_2, m\right), x_2\left(p_{1,}p_2, m\right)\right)$$

Utility Maximisation/Cost Minimisation - Dual versions of the model



- The minimum cost of reaching a given utility level at given prices depends on the u and p_1, p_2

- We can vary u and p_1, p_2 continuously and note the budget needed as we go.

- This maps out a *function* relating prices and utility to spending called the *expenditure* or *cost function*

$$m(p_1, p_2, u)$$

- It tell us the minimum cost of reaching any indifference curve given any prices.

Utility Maximisation/Cost Minimisation - Dual versions of the model

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } u(x_1, x_2) = u$$

1. Solution by substitution:

   (a) Solve $u(x_1, x_2) = u$ for $x_2(u, x_1)$ and substitute into the budget constraint $m(x_1) = p_1x_1 + p_2x_2(u, x_1)$

   (b) Set $m'(x_1) = 0$ and solve for $x_1(p_1, p_2, u)$

   (c) Sub $x_1(p_1, p_2, u)$ into $u(x_1, x_2)$ and solve for $x_2(p_1, p_2, u)$

   (d) $m(p_1, p_2, u) = p_1x_1(p_1, p_2, u) + p_2x_2(p_1, p_2, u)$

Utility Maximisation/Cost Minimisation - Dual versions of the model

$$\min_{x_1,x_2} p_1x_1 + p_2x_2 \text{ subject to } u(x_1,x_2) = u$$

2. Solution by MRS condition (interior solutions only):

   (a) set up the conditions

$$\begin{aligned} MRS_{2,1} &= -\frac{u_1}{u_2} = -\frac{p_1}{p_2} \\ u &= (x_1, x_2) \end{aligned}$$

   (b) solve for $x_1(p_1,p_2,u)$ and $x_2(p_1,p_2,u)$

   (c) $m(p_1,p_2,u) = p_1x_1(p_1,p_2,u) + p_2x_2(p_1,p_2,u)$

Utility Maximisation/Cost Minimisation - Dual versions of the model

3. Inversion of the *indirect utility function*

   (a) Find the Marshallian demands $x_1(p_1, p_2, m)$and $x_2(p_1, p_2, m)$

   (b) Sub into the utility function:

   $$v(p_1, p_2, m) = u(x_1(p_1, p_2, m), x_2(p_1, p_2, m))$$

   (c) Invert to solve for $m(p_1, p_2, u)$

Utility Maximisation/Cost Minimisation - Dual versions of the model

Example:

$$u = a \ln x_1 + (1-a) \ln x_2 \qquad\qquad a \in [0,1]$$

We saw that the Marshallian demands from this problem were

$$\begin{aligned} x_1 &= a\frac{m}{p_1} \\ x_2 &= (1-a)\frac{m}{p_2} \end{aligned}$$

These represent the optimal (u-maximising) choices of $x_1$and x_2 given the prices faced and the available budget.

Utility Maximisation/Cost Minimisation - Dual versions of the model

Substitute

$$
\begin{aligned}
x_1 &= a\frac{m}{p_1} \\
x_2 &= (1-a)\frac{m}{p_2}
\end{aligned}
$$

into the utility function to get the indirect utility function

$$v(p_1, p_2, m) = a \ln\left(a\frac{m}{p_1}\right) + (1-a)\ln\left((1-a)\frac{m}{p_2}\right)$$

This tells us (up to a pmt) the maximum utility number reachable given the prices faced and the available budget.

Utility Maximisation/Cost Minimisation - Dual versions of the model

Invert the indirect utility function

$$v(p_1, p_2, m) = a \ln\left(a\frac{m}{p_1}\right) + (1-a)\ln\left((1-a)\frac{m}{p_2}\right)$$

to find the cost/expenditure function

$$m(p_1, p_2, u) = p_1 u \left(\frac{1-a}{a}\right)^{a-1}\left(\frac{p_1}{p_2}\right)^{a-1} + p_2 u \left(\frac{1-a}{a}\right)^{a}\left(\frac{p_1}{p_2}\right)^{a}$$

(Some) properties

$m(p_1, p_2, u)$ is

1. Homogenous of degree one in p_1 and p_2

2. Strictly increasing in u and weakly increasing in p_1 and p_2

$v(p_1, p_2, m)$ is

1. Homogenous of degree zero

2. Strictly increasing in m and weakly decreasing in p_1 and p_2

Three more things

Shephard's Lemma: The derivative of the cost function wrt p_i gives you the Hicksian demand for x_i

$$x_i = x_i(\mathbf{p}, u)$$

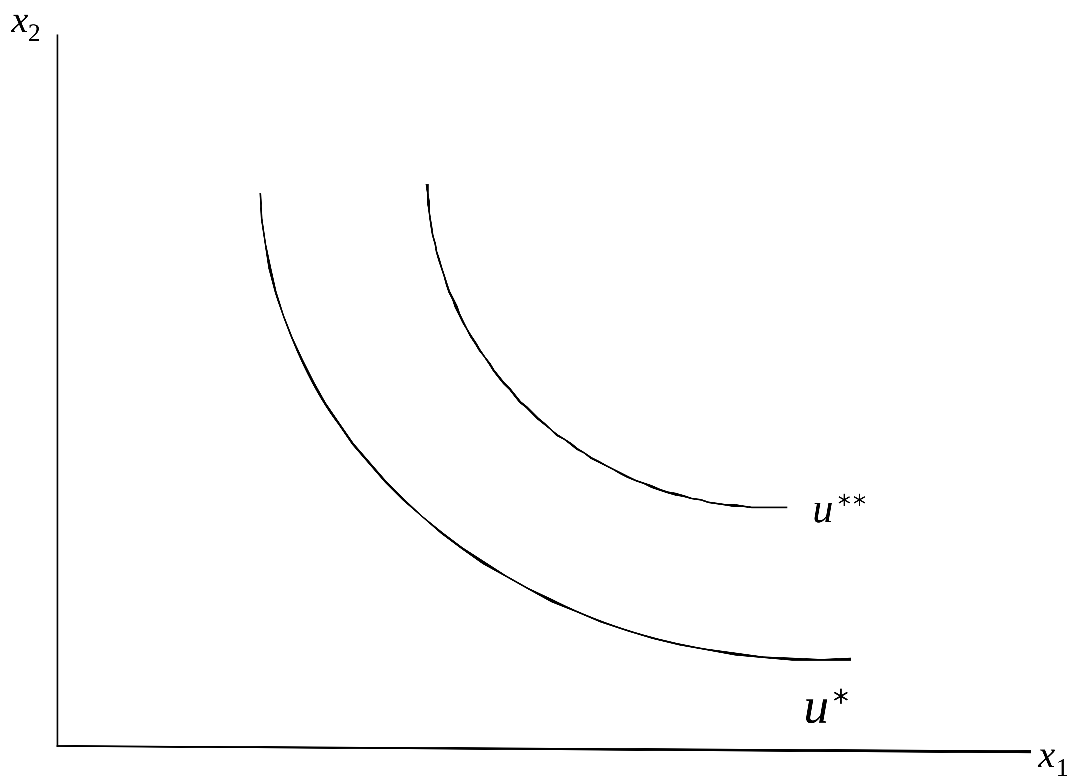
Roy's Identity: Minus the ratio of the derivative of the indirect utility function wrt p_i over its derivative wrt m gives you the Marshallian demand for x_i

$$x_i = x_i(\mathbf{p}, m)$$

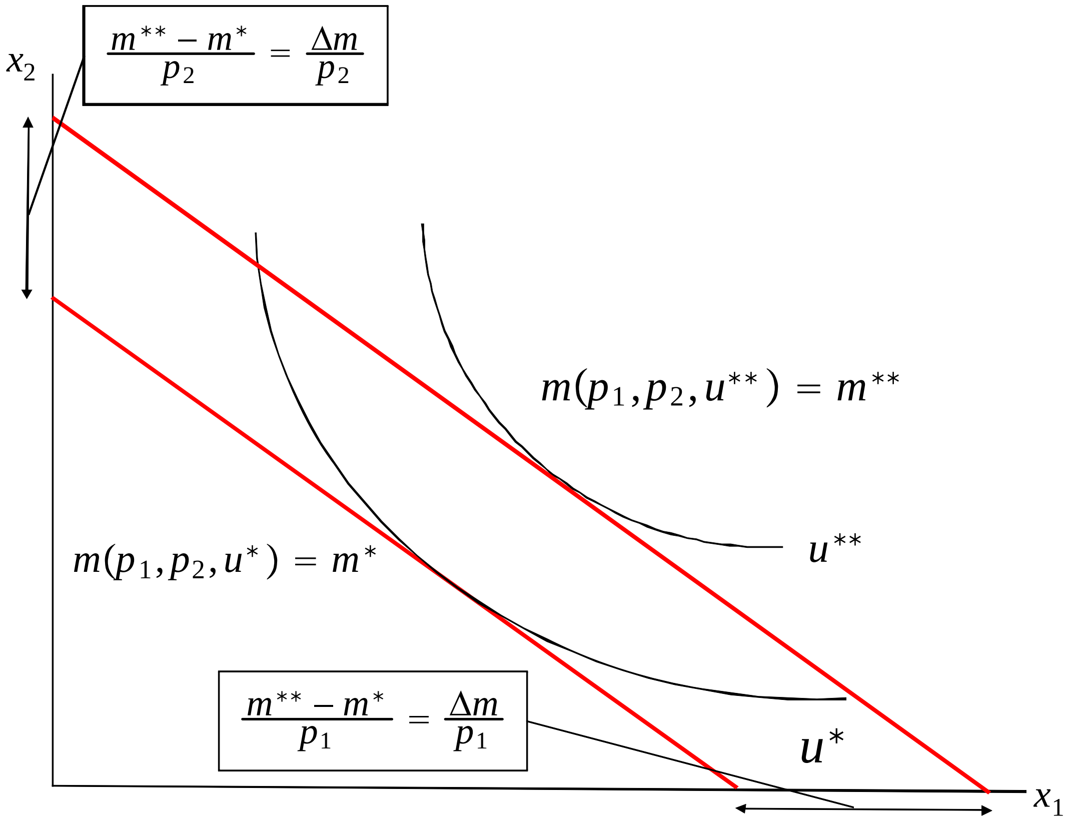
The Slutsky equation (again)

$$\frac{\partial x_i(\mathbf{p}, m)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, u)}{\partial p_j} - \frac{\partial x_i(\mathbf{p}, m)}{\partial m} x_j$$

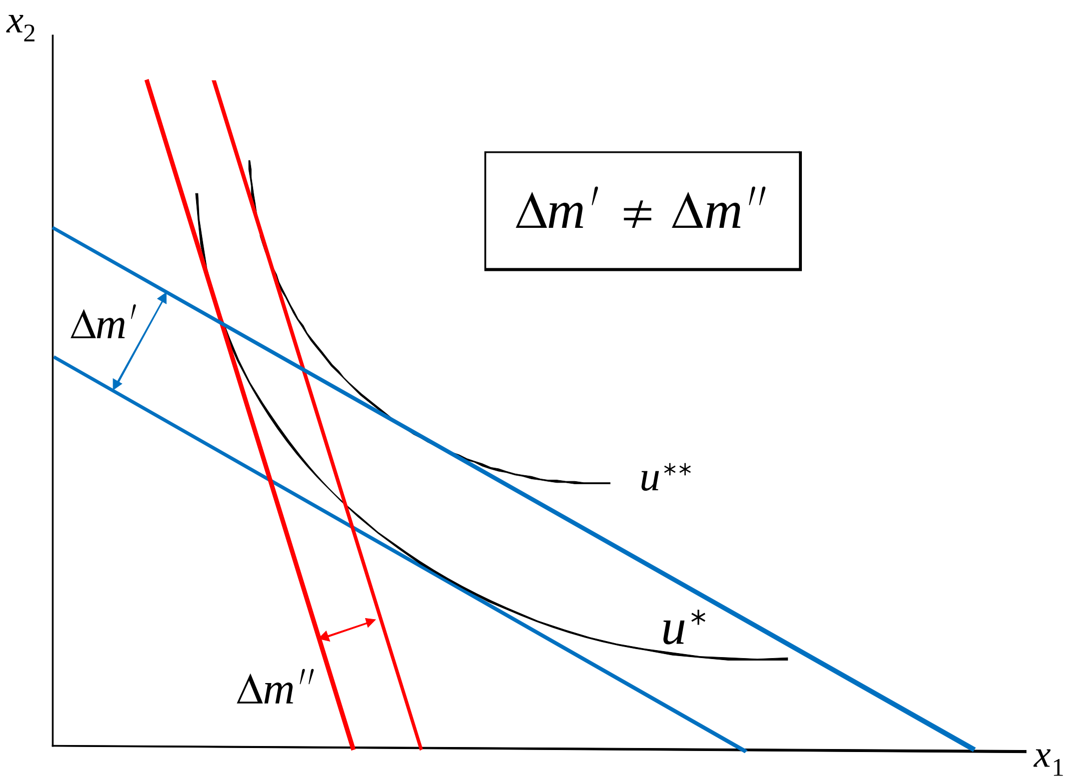
Measuring the distance between indifference curves



Measuring the distance between indifference curves



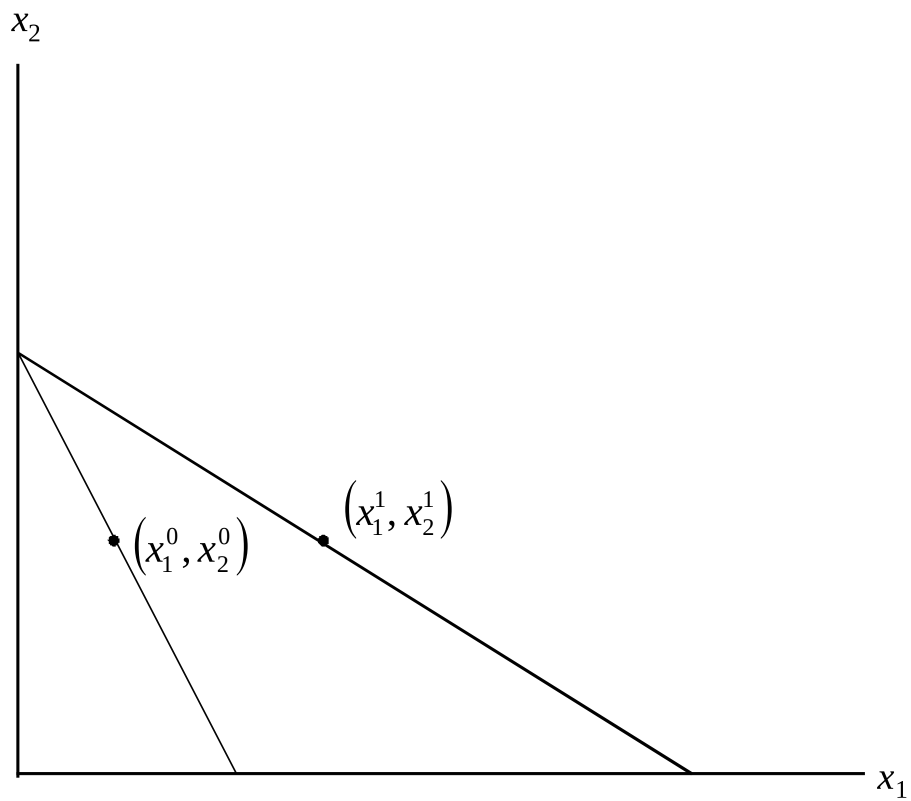
Which reference prices?



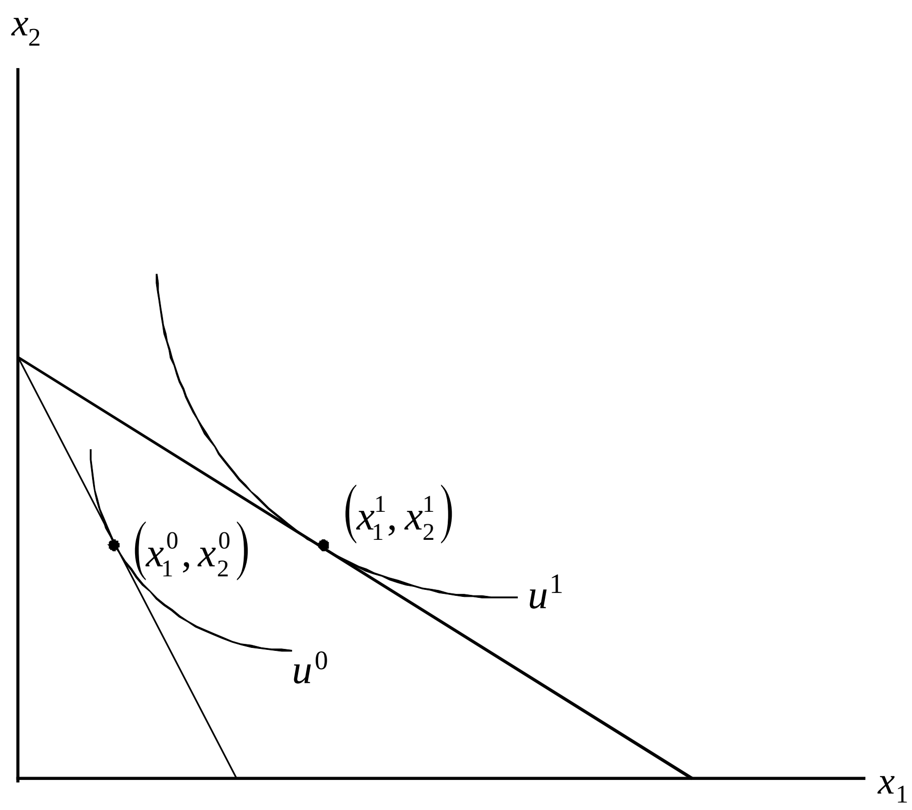
- Suppose the consumer's budget is fixed but prices change from $\left(p_1^0, p_2^0\right)$ to $\left(p_1^1, p_2^1\right)$

- At the same time we watched the consumer's demands change from $\left(x_1^0, x_2^0\right)$ to $\left(x_1^1, x_2^1\right)$.

First plot the data:



We can then add the indifference curves:



EV/CV - it's all about reference prices

- We want to measure the distance between the indifference curves
- We will express this as a cash amount.
- We now know that we are going to measure this by

$$\Delta u = \Delta m = m\left(p_1^r, p_2^r, u^1\right) - m\left(p_1^r, p_2^r, u^0\right)$$

Where $\left(p_1^r, p_2^r\right)$ are reference prices.

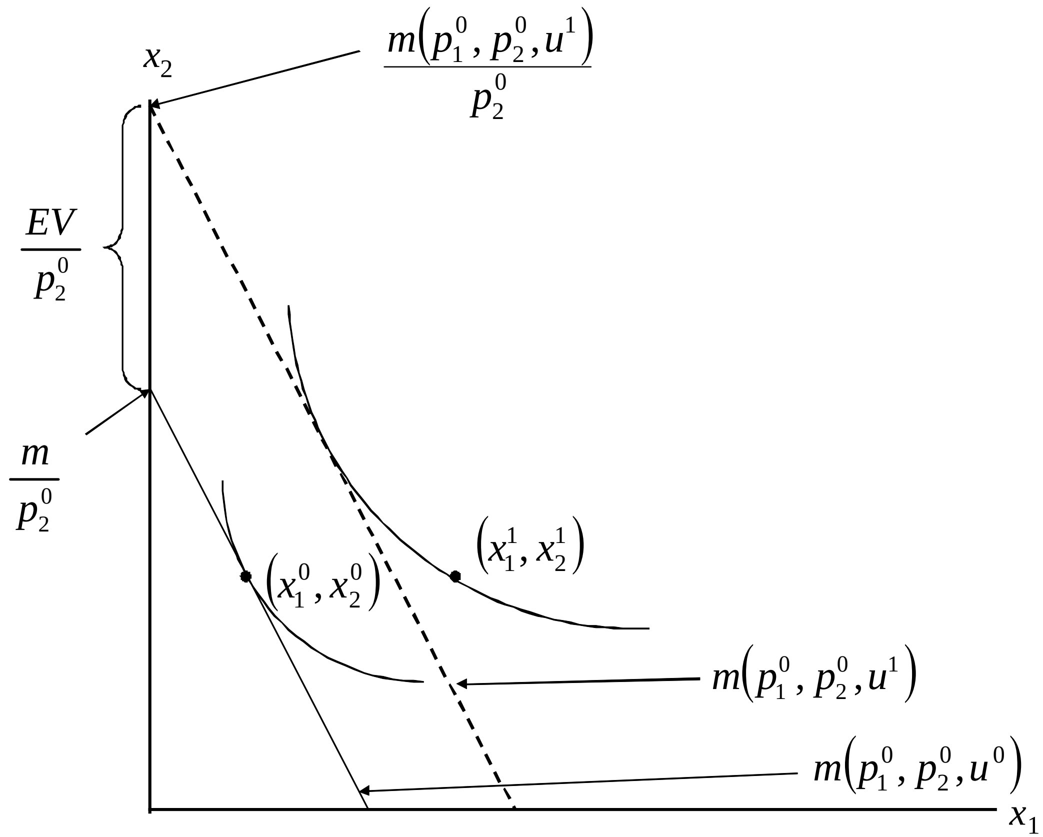
EV/CV - it's all about reference prices

- Two natural candidates for reference prices are $\left(p_1^0, p_2^0\right)$ and $\left(p_1^1, p_2^1\right)$.
- This gives us two natural measures

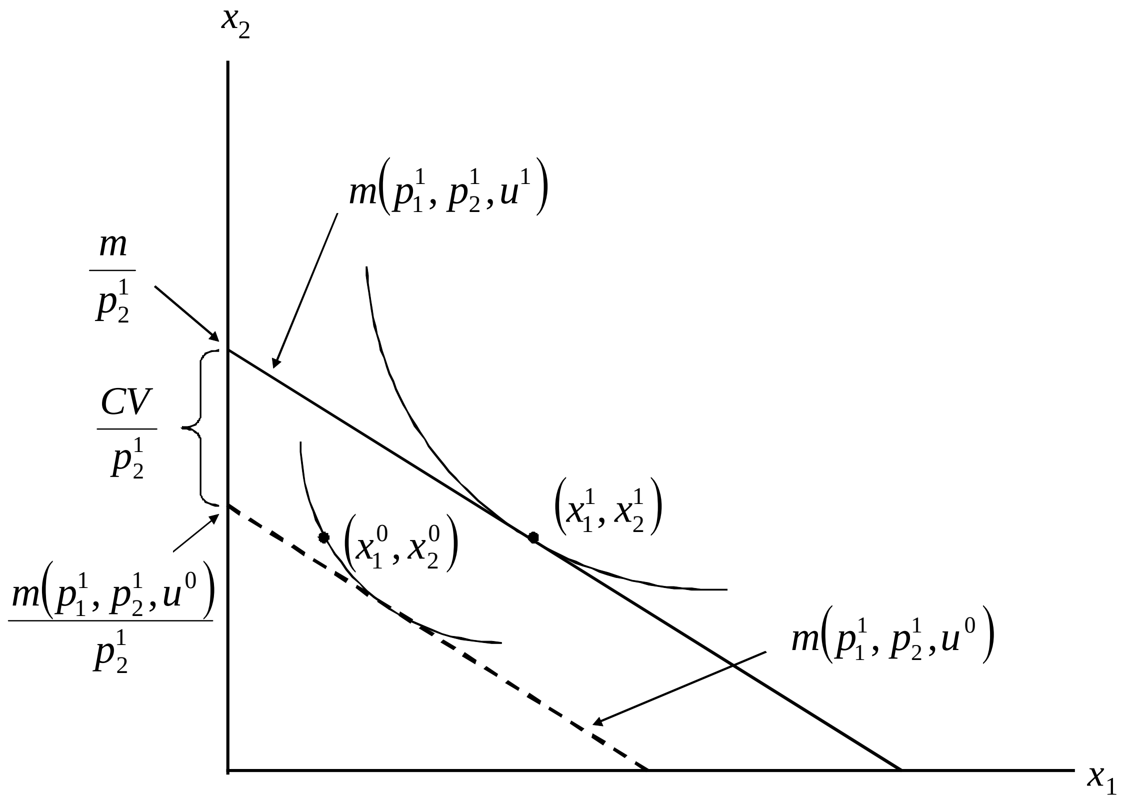
$$\text{Equivalent variation} \quad : \quad EV = m\left(p_1^0, p_2^0, u^1\right) - m\left(p_1^0, p_2^0, u^0\right)$$

$$\text{Compensating variation} \quad : \quad CV = m\left(p_1^1, p_2^1, u^1\right) - m\left(p_1^1, p_2^1, u^0\right)$$

Equivalent variation: Uses base prices $\left(p_1^0, p_2^0\right)$ and asks how much it would cost to reach u^1 compared to u^0



Compensating variation: Uses the other prices (p_1^1, p_2^1) and asks how much it would cost to reach u^1 compared to u^0



Just to confuse you ...

- You may see (e.g. *Perloff* Chapter 5.2)

$$EV = m\left(p_1^1, p_2^1, u^1\right) - m\left(p_1^0, p_2^0, u^1\right)$$

  This reads as a comparison of the cost of buying u^1 before and after the price change.

- Whereas I (and *Mas-Colell* et al Chapter 3.1) have described EV as a comparison of buying u^1 and u^0 at fixed reference prices $\left(p_1^0, p_2^0\right)$

$$EV = m\left(p_1^0, p_2^0, u^1\right) - m\left(p_1^0, p_2^0, u^0\right)$$

Just to confuse you ...

- The same is true of compensating variation.

$$CV = m\left(p_1^1, p_2^1, u^0\right) - m\left(p_1^0, p_2^0, u^0\right)$$

which again has the interpretation of measuring the cost of a fixed u-level under changing prices.

- Whereas I have written

$$CV = m\left(p_1^1, p_2^1, u^1\right) - m\left(p_1^1, p_2^1, u^0\right)$$

and the interpretation is as a comparison of the cost of different u's under fixed prices.

Just to confuse you ...

- As long as the budget is fixed the two version of EV/CV are the same in absolute value although their interpretation is different.

- If the budget changes along with the prices they are not the same, the Mas-Colell *et al* interpretation more useful as a *money metric* measure of the resulting welfare changes.

- Both CV and EV measure the difference in cost of two **different utility** levels at *fixed prices*.

- Cost-of-living indices measure the difference in cost of two **different sets of prices** at a *fixed utility level*.

- CV/EV are measured in cash amounts

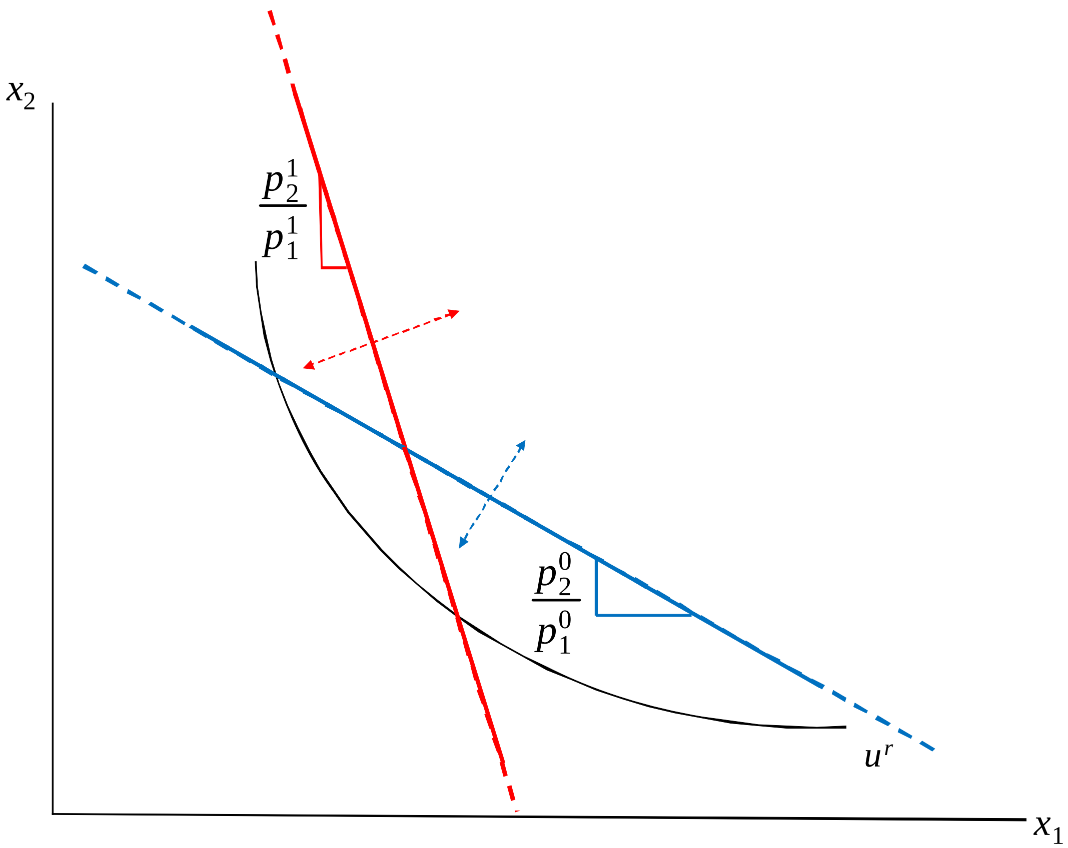
- Cost of living indices are measured as a ratio so are unit-free.

The Cost of Living Index

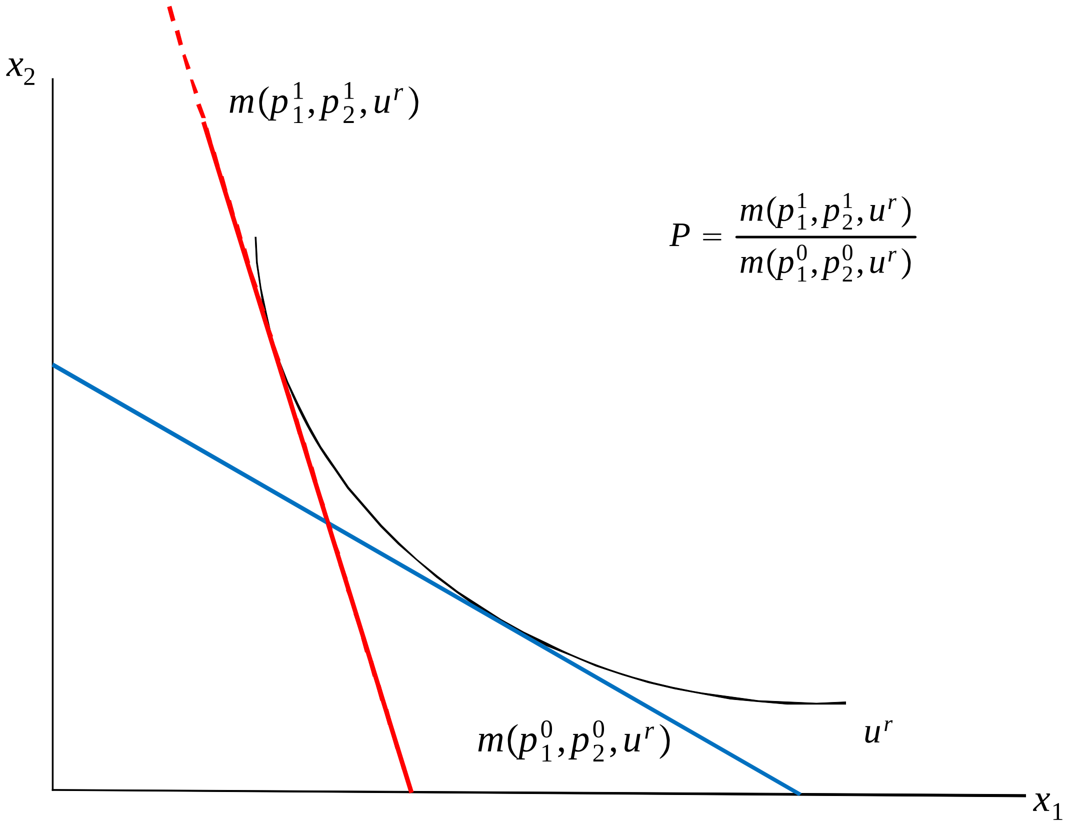
- The relative minimum cost of reaching a reference utility level u^r under prices $\left(p_1^0, p_2^0\right)$ and $\left(p_1^1, p_2^1\right)$

$$P = \frac{m\left(p_1^1, p_2^1, u^r\right)}{m\left(p_1^0, p_2^0, u^r\right)}$$

The Cost of Living Index



The Cost of Living Index



Which reference utility?

- Once again we have to make a choice as, in general, the cost-of-living index will be different according to the welfare level* - this implies that rate of inflation for rich and poor is *heterogeneous*.

- When we are comparing the rate of inflation between two observed situations two natural reference points are u^0 and u^1 :

Generally

$$P_0 = \frac{m\left(p_1^1, p_2^1, u^0\right)}{m\left(p_1^0, p_2^0, u^0\right)} \neq \frac{m\left(p_1^1, p_2^1, u^1\right)}{m\left(p_1^0, p_2^0, u^1\right)} = P_1$$

*The circumstances under which it is independent of u^r are, unsurprisingly, similar to those for price independence of the EV/CV measures (and are equally empirically suspect).

Practical Measurement

- If we know the utility function all of this is easy to apply. But we generally don't.

- We only observe prices, budget and demands.

- We can either:

1. infer the utility function from the data (can be tricky)

2. use approximations (very easy)

Practical Measurement

- Consider the EV

$$EV = m\left(p_1^0, p_2^0, u^1\right) - m\left(p_1^0, p_2^0, u^0\right)$$

- $m\left(p_1^0, p_2^0, u^0\right)$ is easy to quantify.

- Given optimising behaviour we know that $m\left(p_1^0, p_2^0, u^0\right) = p_1^0 x_1^0 + p_2^0 x_2^0$

Practical Measurement

- $m\left(p_1^0, p_2^0, u^1\right)$ is the minimum cost of buying the utility level u^1 at prices $\left(p_1^0, p_2^0\right)$

- You can always buy u^1 by buying $\left(x_1^1, x_2^1\right)$ because $u\left(x_1^1, x_2^1\right) = u^1$

- At the initial prices buying $\left(x_1^1, x_2^1\right)$ would cost $p_1^0 x_1^1 + p_2^0 x_2^1$

- This might not the be the least-cost way of buying but it can't be less than the least-cost way.

- Therefore $p_1^0 x_1^1 + p_2^0 x_2^1 \geq m\left(p_1^0, p_2^0, u^1\right)$.

Approximations

- Given optimising behaviour

$$\begin{aligned} m\left(p_1^0, p_2^0, u^0\right) &= p_1^0 x_1^0 + p_2^0 x_2^0 \\ m\left(p_1^0, p_2^0, u^1\right) &\leq p_1^0 x_1^1 + p_2^0 x_2^1 \end{aligned}$$

- We can therefore approximate EV

$$EV = m\left(p_1^0, p_2^0, u^1\right) - m\left(p_1^0, p_2^0, u^0\right)$$

$$EV \leq \left(p_1^0 x_1^1 + p_2^0 x_2^1\right) - \left(p_1^0 x_1^0 + p_2^0 x_2^0\right)$$

Approximations

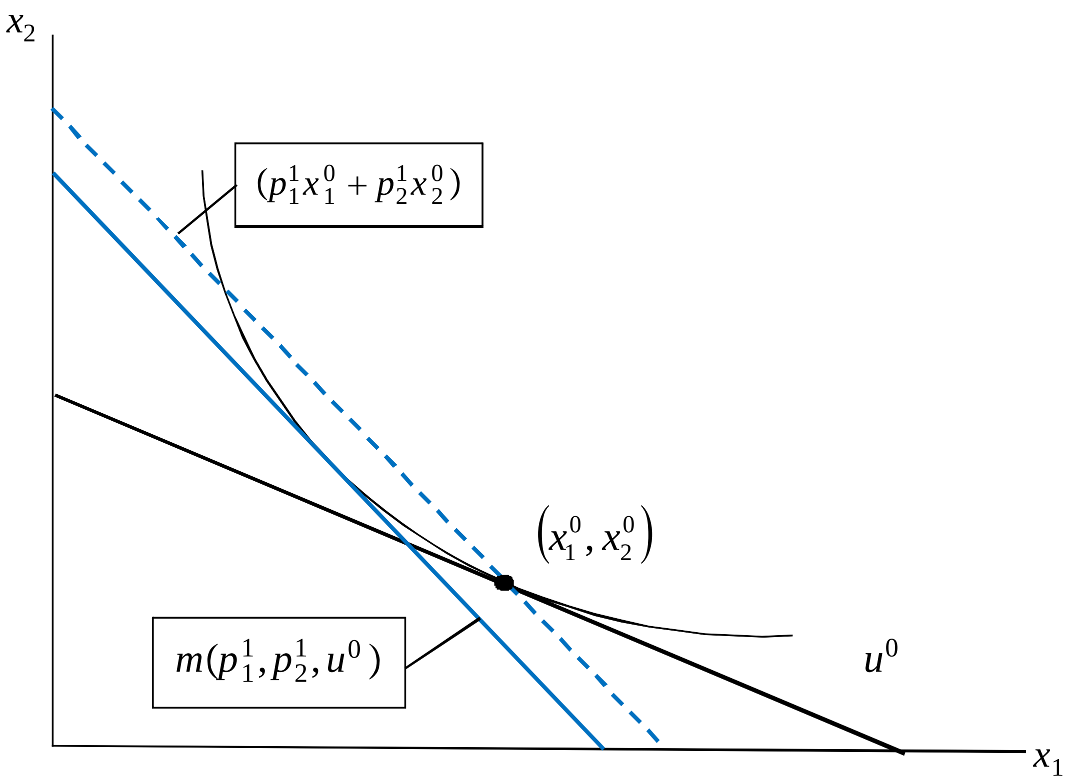
$$EV \leq \left(p_1^0 x_1^1 + p_2^0 x_2^1\right) - \left(p_1^0 x_1^0 + p_2^0 x_2^0\right)$$

$$CV \geq \left(p_1^1 x_1^1 + p_2^1 x_2^1\right) - \left(p_1^1 x_1^0 + p_2^1 x_2^0\right)$$

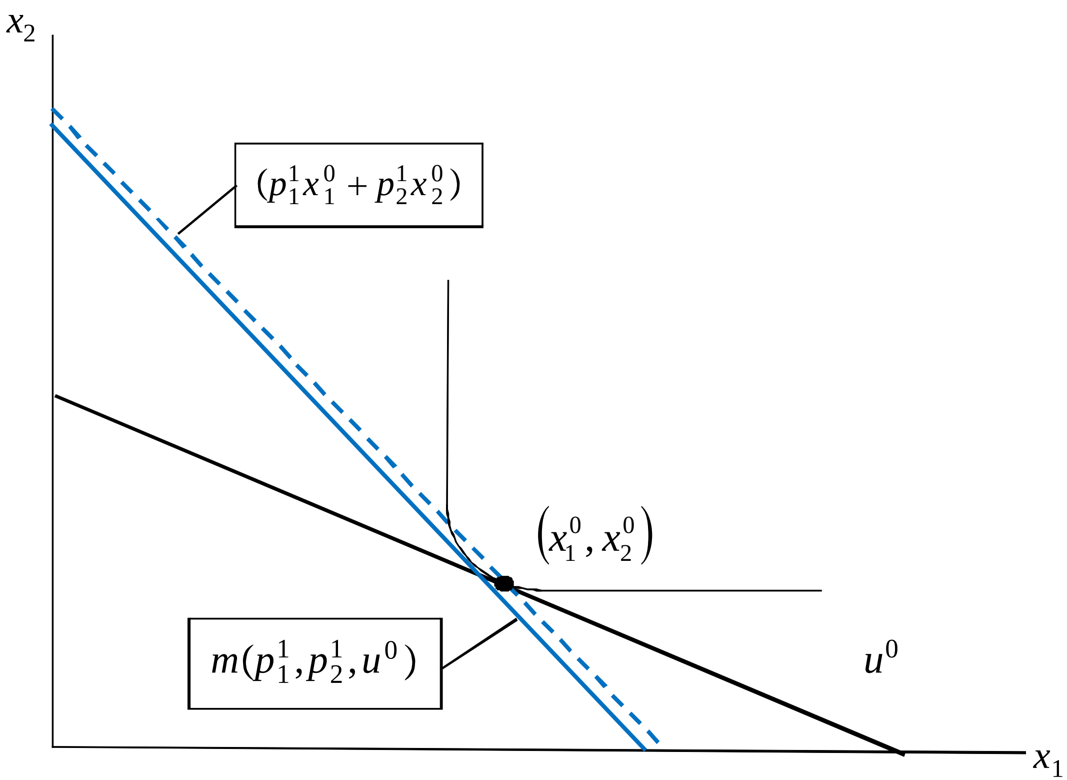
$$P_0 \leq \frac{\left(p_1^1 x_1^0 + p_2^1 x_2^0\right)}{\left(p_1^0 x_1^0 + p_2^0 x_2^0\right)}$$

$$P_1 \geq \frac{\left(p_1^1 x_1^1 + p_2^1 x_2^1\right)}{\left(p_1^0 x_1^1 + p_2^0 x_2^1\right)}$$

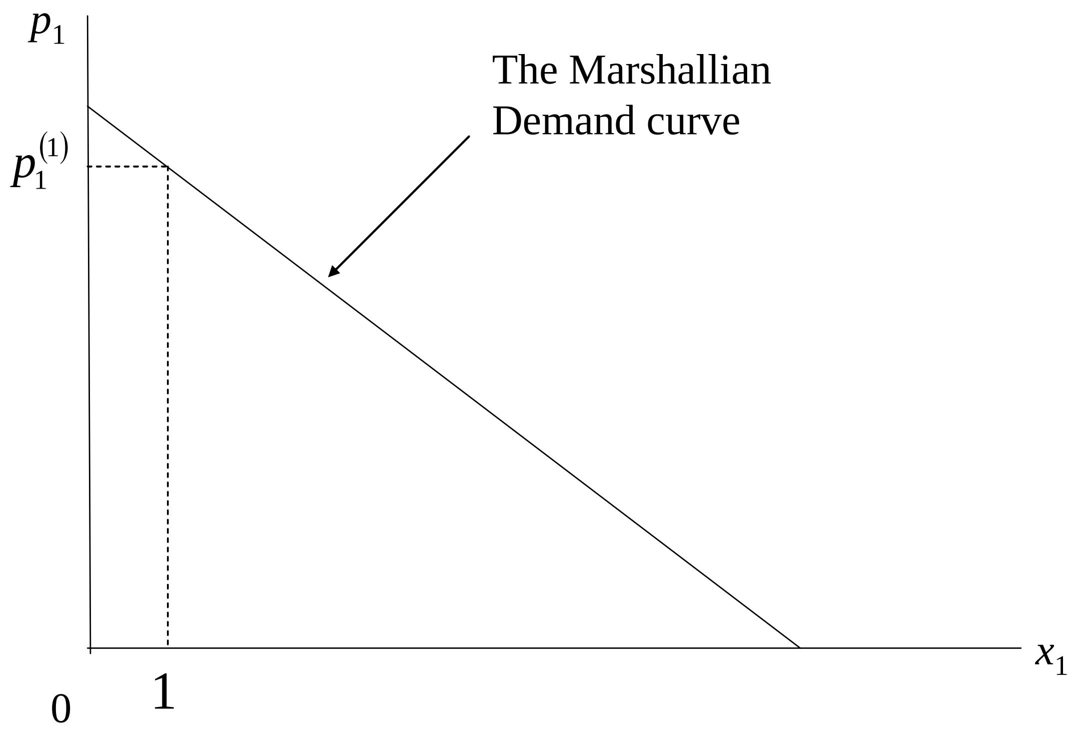
Approximations



Approximations



Consumer Surplus



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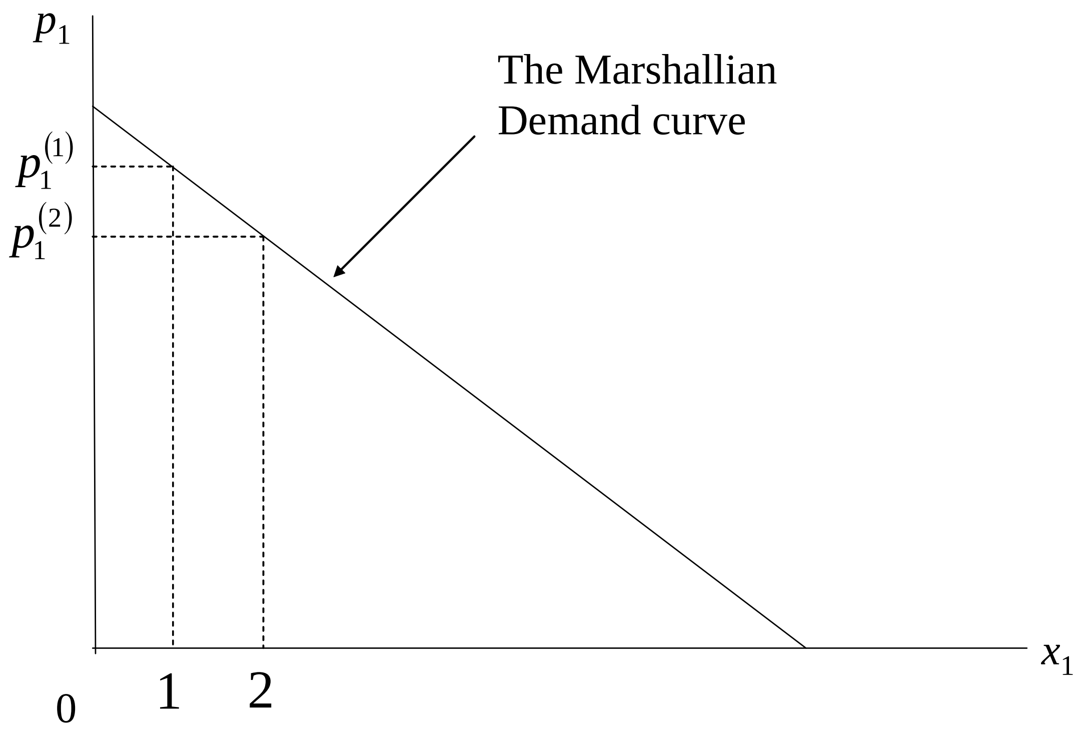

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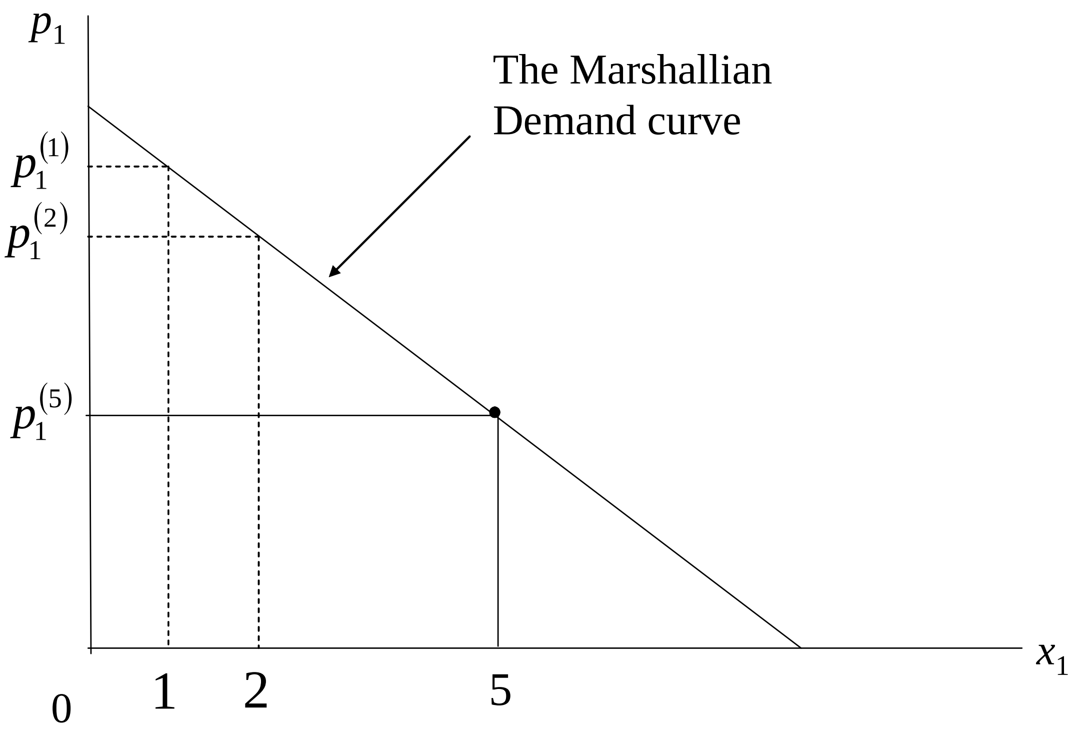

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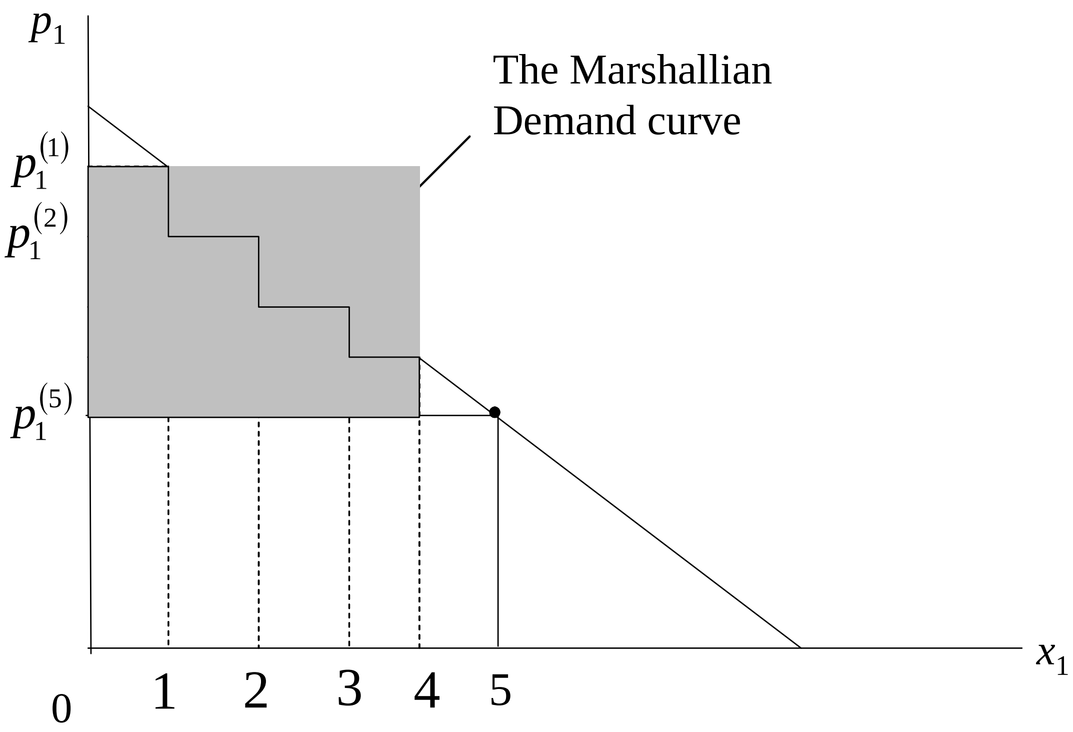

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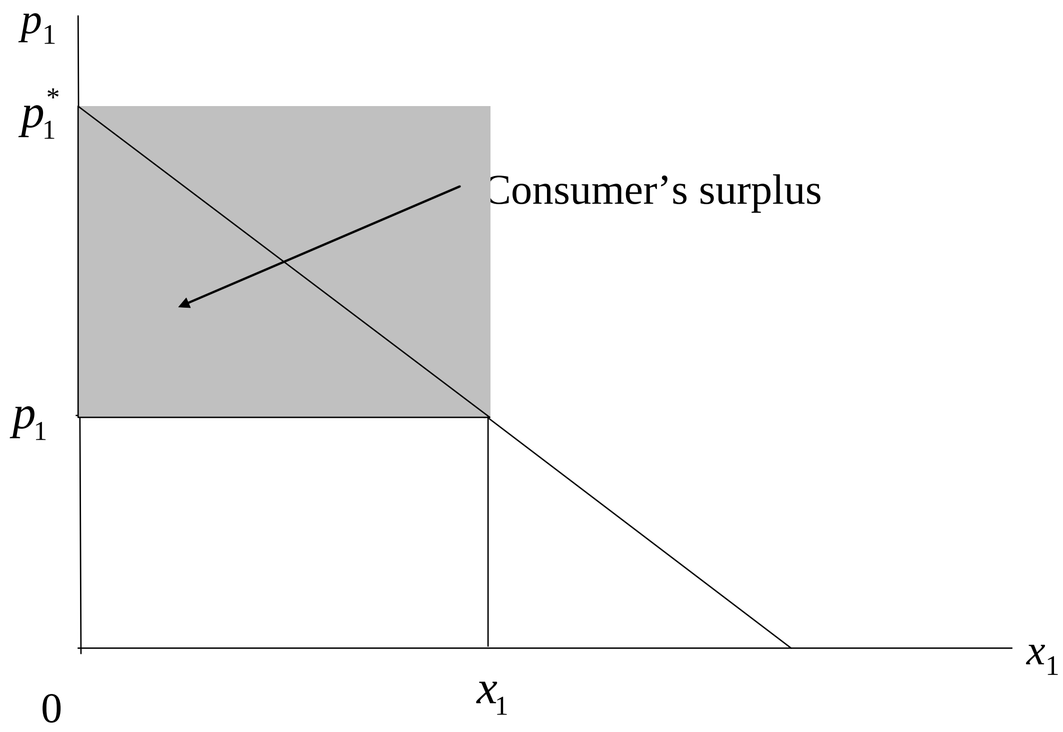

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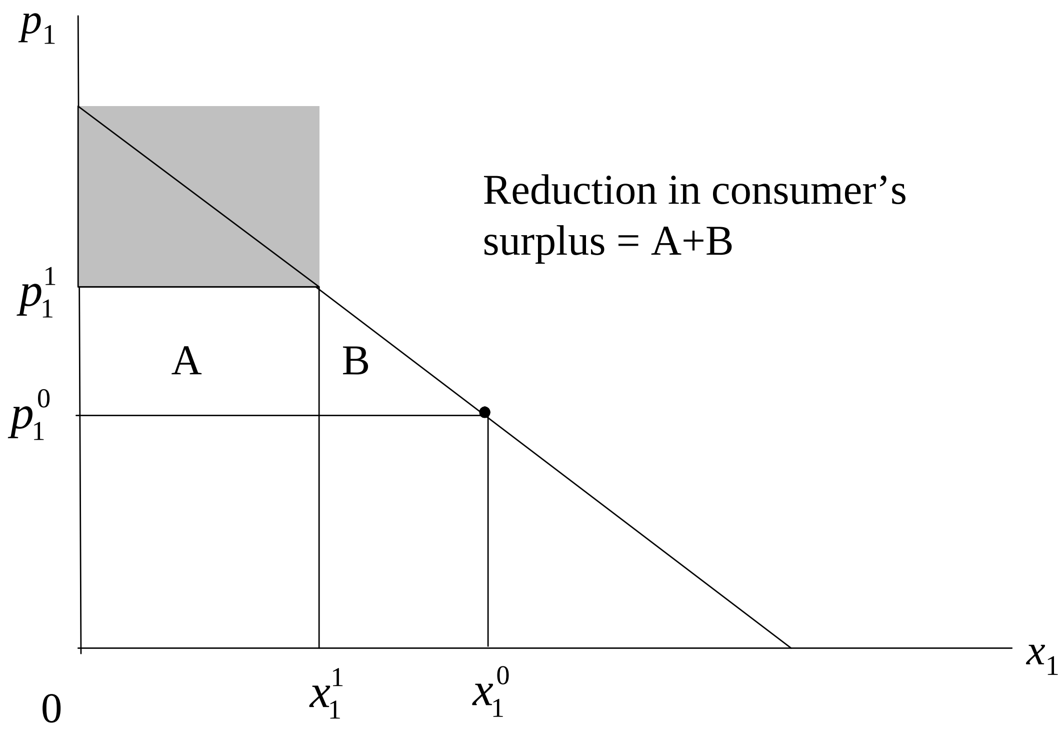

Summary - Util counting

- Policy analysis, beyond finding Pareto improvements†, is a matter of trade-offs: new policies generally create winners and losers.

- Without a way to weigh up the gains and losses, useful policy is impossible.

- This is going to have to involve making interpersonal comparisons of welfare.

- The methods we have looked at are some schemes for doing that.

†There aren't many of those left!

Mathematical Interlude - Constrained Optimisation

[Workbook Chapters 9 & 10]

$$\max_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n) \quad \text{subject to} \quad c(x_1, \ldots, x_n) = c$$

"Maximise, by choice of the variables $x_1, \ldots, x_n$, the

objective $f(.)$ subject to the constraint that $c(.) = c$"

Lagrange's Theorem[‡]: *If* $(x_1^*, ..., x_n^*)$ *solves this problem then*

$$f_i = \lambda c_i \text{ for each } i$$

[‡]This result could do with a name. At the risk of confusion I'll call it *Lagrange's Theorem* [there are at least two other theorems in mathematics also called Lagrange's Theorem!].

The Method of Lagrange Multipliers (Single Constraint)

$$\max_{x_1, x_2, \ldots, x_n} f(x_1, \ldots, x_n) \quad \text{subject to} \quad c(x_1, \ldots, x_n) = c$$

1. Form an auxiliary function called a *Lagrangian*:

$$L\left(x_1, \ldots, x_n, \lambda\right) = f(x_1, \ldots, x_n) - \lambda \left[c(x_1, \ldots, x_n) - c\right]$$

The Method of Lagrange Multipliers (Single Constraint)

2. Solve the problem

$$\max_{x_1,...,x_n,\lambda} L(x_1, ..., x_n, \lambda)$$

(a) Find the $n+1$ first order conditions for a max.

$$L_i = 0 \text{ for each } i$$
$$L_\lambda = 0$$

(b) Solve for the objects of interest ($x_1, ..., x_n$ and sometimes λ) which satisfy (a), simplify and interpret.

How does this relate to Lagrange's Theorem?

The first order conditions of the Lagrangian with respect to $(x_1, ..., x_n)$ are

$$L_i = 0 \text{ for each } i$$

Which (from the definition of the Lagrangian) can be written as:

$$f_i - \lambda c_i = 0 \text{ for each } i$$

So

$$f_i = \lambda c_i \text{ for each } i$$

which is exactly the condition in Lagrange's Theorem.

Multiple Constraints.

To simplify the notation let bold $\mathbf{x}$ denote the list of variables $(x_1, x_2, ..., x_n)$.and $\boldsymbol{\lambda}$ denote the list of miltipliers $(\lambda_1, \lambda_2, ..., \lambda_K)$. Take the problem:

$$\max_{x_1, x_2, ..., x_n} f(\mathbf{x}) \text{ subject to } c_1(\mathbf{x}) = c_1,\ c_2(\mathbf{x}) = c_2, ...,\ c_K(\mathbf{x}) = c_K$$

The Lagrangian simply expands to accommodate the extra constraints

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \lambda_1 [c_1(\mathbf{x}) - c_1] - ... - \lambda_K [c_K(\mathbf{x}) - c_K]$$

The Method of Lagrange Multipliers (Multiple Constraints)

$$\max_{x_1, x_2, \ldots, x_n} f(\mathbf{x}) \text{ subject to } c_1(\mathbf{x}) = c_1,\ c_2(\mathbf{x}) = c_2, \ldots,\ c_K(\mathbf{x}) = c_K$$

1. Form the *Lagrangian*:

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \lambda_1 [c_1(\mathbf{x}) - c_1] - \ldots - \lambda_K [c_K(\mathbf{x}) - c_K]$$

The Method of Lagrange Multipliers (Multiple Constraints)

2. Solve the problem

$$\max_{x_1,\ldots,x_n,\lambda_1,\ldots,\lambda_K} L\left(\mathbf{x}, \boldsymbol{\lambda}\right)$$

(a) Find the $n + K$ first order conditions for a max.

$$L_i = 0 \text{ for each variable } i$$
$$L_k = 0 \text{ for each constraint } k$$

(b) Solve for the objects of interest ($x_1, \ldots, x_n$ and sometimes $\lambda_1, \ldots, \lambda_K$), simplify and interpret

Further Issues: Maxima/Minima

- Minimisation and maximisation problems are approached in the same way but how do you know you've found a max or a min?

- In general you need to look at the second-order conditions, but the SOCs for constrained optimisation problems are moderately complicated.

- But, we can often manage without SOCs.

- If the objective function is *well-behaved* (e.g. if the utility function is increasing, with convex indifference curves) then the problem can be solved by the Lagrangian method and it will deliver the min/max as required.

Further Issues: Inequality Constraints

- Inequality constraints are also sometimes important. e.g. demands must be non-negative $x_i \geq 0$.

- A somewhat generalised version of Lagrange's Theorem called the *Kuhn-Tucker Conditions* are used to solve this sort of problem.

- The KT-conditions involve an extension of those which arise when we use the method of Lagrange Multipliers - it's just a question of allowing constraints either to bind or to remain slack (fiddly, but straightforward).

Constrained Optimisation

We can use the method of *Lagrange Multipliers* to solve equality-constrained optimisation problems with any number of variables and any number of constraints.

The steps involved are:

1. Set up the Lagrangian.

2. Characterise its turning points (i.e. derive the 1st order conditions).

3. Solve for the values of the variables which satisfy those conditions.

Example 1. Consumer Choice - utility maximisation version

Suppose we can represent a consumer's preferences by the utility function

$$u = a \ln x_1 + (1 - a) \ln x_2$$

The problem we need to solve is

$$\max_{x_1, x_2} \; a \ln x_1 + (1 - a) \ln x_2 \text{ subject to } p_1 x_1 + p_2 x_2 = m$$

Example 1. Consumer Choice - utility maximisation version

1. Form the Lagrangian

$$L = a \ln x_1 + (1 - a) \ln x_2 - \lambda \left[p_1 x_1 + p_2 x_2 - m \right]$$

Example 1. Consumer Choice - utility maximisation version

2. FOC's

$$L_1 = \frac{a}{x_1} - \lambda p_1 = 0 \tag{1}$$

$$L_2 = \frac{1-a}{x_2} - \lambda p_2 = 0 \tag{2}$$

$$L_\lambda = p_1 x_1 + p_2 x_2 - m = 0 \tag{3}$$

[Interpretation: The ratio of (1) and (2) say that "the MRS between the goods equals the ratio of their prices". (3) Says "the budget constraint is satisfied"]

Example 1. Consumer Choice - utility maximisation version

3. We have three simultaneous equations and two unknowns[§] $\{x_1, x_2\}$

$$\frac{a}{x_1} - \lambda p_1 = 0 \quad (1)$$

$$\frac{1-a}{x_2} - \lambda p_2 = 0 \quad (2)$$

$$p_1 x_1 + p_2 x_2 - m = 0 \quad (3)$$

Solve for Marshallian demands in terms of parameters

$$x_1 = a\frac{m}{p_1} \qquad x_2 = (1-a)\frac{m}{p_2}$$

[§].. which we care about: λ is an unknown too but in this problem we don't care about it. In other problems we might.

Example 2. Consumer Choice - cost minimisation version

Take the same utility function but model consumer choice as the cost-minimisation problem:

$$\min_{x_1,x_2} \; p_1 x_1 + p_2 x_2 \text{ subject to } a \ln x_1 + (1-a) \ln x_2 = u$$

Example 2. Consumer Choice - cost minimisation version

1. Form the Lagrangian

$$L = p_1 x_1 + p_2 x_2 - \lambda \left(a \ln x_1 + (1 - a) \ln x_2 - u \right)$$

Example 2. Consumer Choice - cost minimisation version

2. FOC's

$$
\begin{aligned}
L_1 &= p_1 - \lambda \frac{a}{x_1} = 0 && (1) \\
L_2 &= p_2 - \lambda \frac{(1-a)}{x_2} = 0 && (2) \\
L_\lambda &= a \ln x_1 + (1-a) \ln x_2 - u = 0 && (3)
\end{aligned}
$$

As before we have three simultaneous equations which we solve for x_1 and x_2 in terms of parameters. These give the *Hicksian* demands:

$$
\begin{aligned}
x_1 &= u \left(\frac{1-a}{a} \right)^{a-1} \left(\frac{p_1}{p_2} \right)^{a-1} \\
x_2 &= u \left(\frac{1-a}{a} \right)^{a} \left(\frac{p_1}{p_2} \right)^{a}
\end{aligned}
$$

Duality - the connection

Cost function plug the Hicksian demands into the budget constraint:

$$m = p_1 x_1 + p_2 x_2$$

$$m(p_1, p_2, u) = p_1 u \left(\frac{1-a}{a}\right)^{a-1} \left(\frac{p_1}{p_2}\right)^{a-1} + p_2 u \left(\frac{1-a}{a}\right)^{a} \left(\frac{p_1}{p_2}\right)^{a}$$

Indirect utility function plug the Marshallian demands into the utility function:

$$u = a \ln x_1 + (1-a) \ln x_2$$

$$u(p_1, p_2, m) = a \ln\left(\frac{am}{p_1}\right) + (1-a) \ln\left(\frac{(1-a)\, m}{p_2}\right)$$