Introductory Economics

Lectures 7 & 8 - Consumer Theory

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Mathematical Interlude - Differential Calculus

[Workbook Chapters 5 to 8]

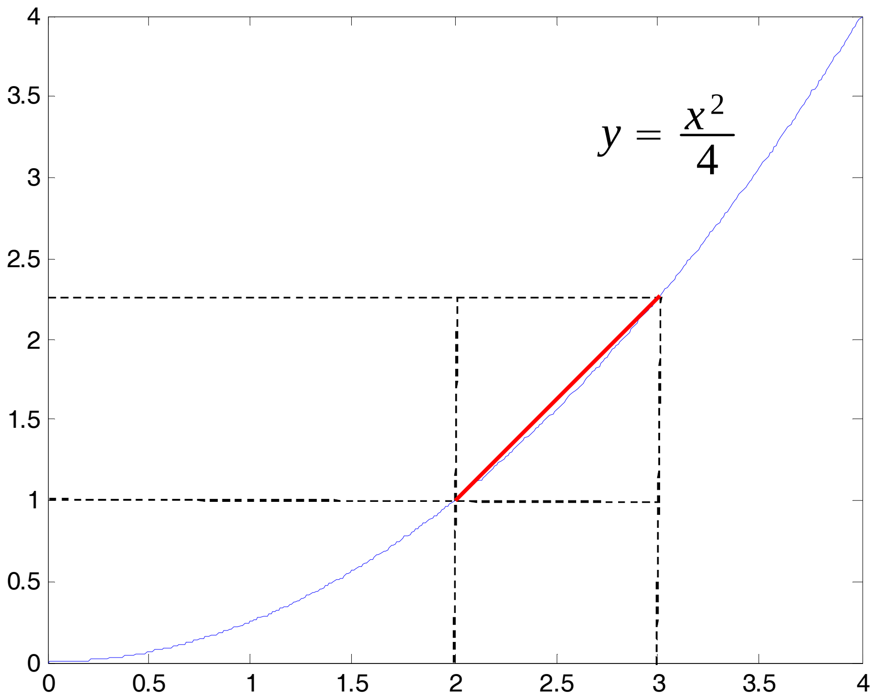
- Given a function $y(x)$ we often need to know its *gradient*.

- Gradients are often called *marginal* effects e.g. the gradient of the utility function $u(x)$ is the "marginal utility of x".

- For a linear function $y(x) = mx + c$ its graph is a straight line, with gradient equal to m

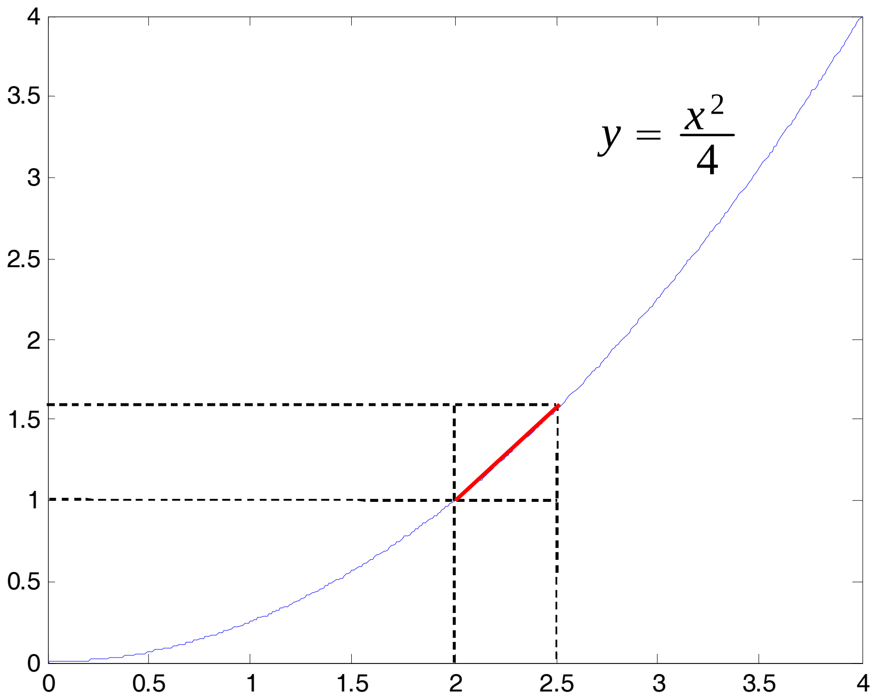
- But the graph of nonlinear functions are curves so their gradient changes with x.

Differentiation - from first principles



$$\frac{\Delta y}{\Delta x} = \frac{y(3) - y(2)}{3 - 2} = \frac{1.25}{1} = 1.25$$

Differentiation - from first principles



$$\frac{\Delta y}{\Delta x} = \frac{y\,(2.5) - y\,(2)}{2.5 - 2} = 1.125$$

Differentiation - from first principles

$$\frac{\Delta y}{\Delta x} = \frac{y(3) - y(2)}{3 - 2} = 1.25$$
$$\frac{\Delta y}{\Delta x} = \frac{y(2.5) - y(2)}{2.5 - 2} = 1.125$$
$$\frac{\Delta y}{\Delta x} = \frac{y(2.001) - y(2)}{2.001 - 2} = 1.00025$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1$$

Differentiation - from first principles

- The notation for

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

is

$$\frac{dy}{dx}$$

or

$$\frac{dy(x)}{dx}$$

or

$$y'(x)$$

- This is the *derivative* of $y(x)$.

Differentiation - from first principles

- We can always find the derivative of a function at a point (x) in the way we have just described.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{(x+h) - x} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

- This method is called *Differentiation from First Principles*.

Efficient rules for Differentiation - Sum Rule

$$y(x) = f(x) \pm g(x)$$

$$y'(x) = f'(x) \pm g'(x)$$

Efficient rules for Differentiation - Product Rule

$$
\begin{aligned}
y(x) &= f(x)\,g(x) \\
y'(x) &= f(x)\,g'(x) + f'(x)\,g(x)
\end{aligned}
$$

Efficient rules for Differentiation - Chain Rule

$$
\begin{aligned}
y(x) &= f(g(x)) \\
y'(x) &= f'(g)\, g'(x)
\end{aligned}
$$

Efficient rules for Differentiation - A Useful Rule

$$
\begin{aligned}
y(x) &= f(x)^{g(x)} \\
y'(x) &= f(x)^{g(x)} \left[g'(x) \ln f(x) + g(x) \frac{f'(x)}{f(x)} \right]
\end{aligned}
$$

Efficient rules for Differentiation - Other Rules

$$y(x) = c \quad \Rightarrow \quad y'(x) = 0$$

$$y(x) = x^n \quad \Rightarrow \quad y'(x) = nx^{n-1}$$

$$y(x) = af(x) \quad \Rightarrow \quad y'(x) = af'(x)$$

Efficient rules for Differentiation - Other Rules

$$y(x) = e^{f(x)} \quad \Rightarrow \quad y'(x) = f'(x)\, e^{f(x)}$$

$$y(x) = c^x \quad \Rightarrow \quad y'(x) = c^x \ln c \qquad (c > 0)$$

$$y(x) = \ln x \quad \Rightarrow \quad y'(x) = \frac{1}{x}$$

$$y(x) = \log_a x \quad \Rightarrow \quad y'(x) = \frac{1}{x} \log_a e \qquad (a > 0, a \neq 1)$$

Concavity again

The function $f(x)$ is concave iff (if and only if)

$$f(x_s) \leq f(x_t) + f'(x_t)(x_s - x_t)$$

where x_s and x_t are two points at which the function is evaluated and $f'(x_t)$ means the value of the derivative at x_t

This means (geometrically) that the graph of f is below any tangent line.

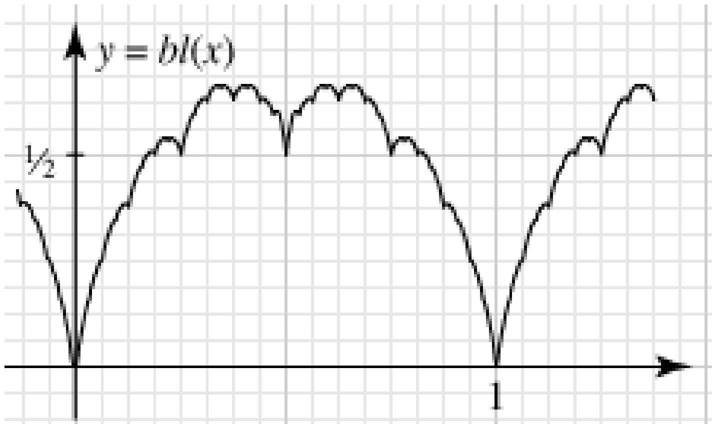
Increasingness etc

$$\begin{aligned} f'(x) &> 0 \Rightarrow f(x) \text{ is strictly increasing*} \\ f'(x) &\geq 0 \iff f(x) \text{ is increasing} \\ f'(x) &= 0 \iff f(x) \text{ is constant} \end{aligned}$$

NB the implication arrow cannot be reversed ($f(x) = x^3$ is strictly increasing but $f'(0) = 0$

Caveat: Note that not all functions are differentiable everywhere.

E.g. the Takagi fractal curve is continuous but not diffferentiable (anywhere).



E.g. the Leontief utility function $u = \text{min}\{ax_1, bx_2\}$

Example. The Marginal Rate of Substitution (MRS)

- Consider the Cobb-Douglas utility function

$$u = a \ln x_1 + (1-a) \ln x_2$$

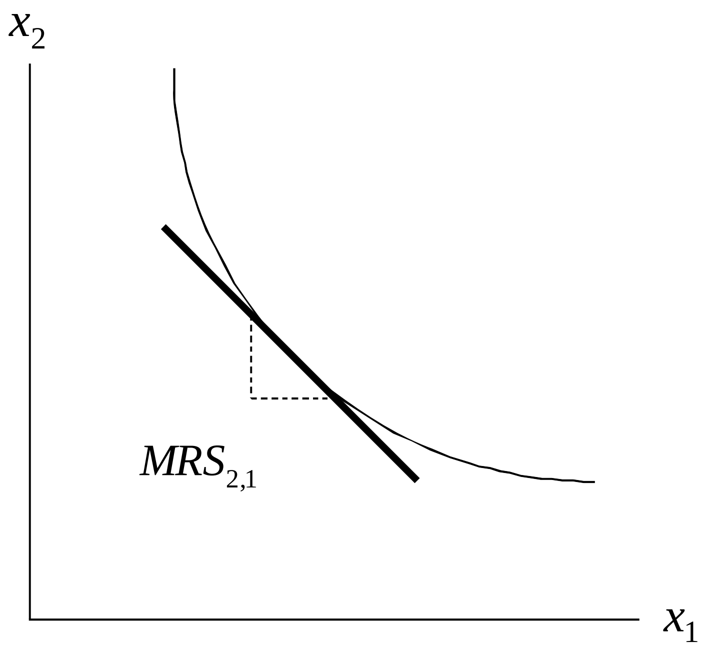
- In terms of x_2 the equation of an indifference curve is

$$x_2 = \exp\left(\frac{u - a \ln x_1}{1-a}\right)$$

- The slope is

$$MRS_{2,1} = \frac{dx_2}{dx_1} = \exp\left(\frac{u - a \ln x_1}{1-a}\right)\left(\frac{-a}{1-a}\right)\frac{1}{x_1} = -\frac{a}{1-a}\frac{x_2}{x_1}$$

Example. The Marginal Rate of Substitution (MRS)



$$MRS_{2,1} = \frac{dx_2}{dx_1} = -\frac{a}{1-a}\frac{x_2}{x_1}$$

Example 2. Price Elasticities

- Consider the demand curve

$$\ln x_1 = \alpha - \beta \ln p_1$$

- In levels it is

$$x_1 = e^{\alpha - \beta \ln p_1}$$

- The slope is

$$\frac{dx_1}{dp_1} = -\frac{\beta}{p_1} e^{\alpha - \beta \ln p_1}$$

Example. Price Elasticities

- The definition of the price elasticity is

$$\varepsilon_{x_1,p_1} = \frac{dx_1}{dp_1}\frac{p_1}{x_1}$$

- Sub in p_1 and $x_1 = e^{\alpha - \beta \ln p_1}$

$$\varepsilon_{x_1,p_1} = \left[\frac{dx_1}{dp_1}\right]\left[\frac{p_1}{x_1}\right] = \left[-\frac{\beta}{p_1}e^{\alpha-\beta\ln p_1}\right]\left[\frac{p_1}{e^{\alpha-\beta\ln p_1}}\right] = -\beta$$

Example. Income Elasticities

- Consider the Engel curve

$$w_1 = a + \beta \ln m$$

where

$$w_1 = \frac{p_1 x_1}{m}$$

- Subing in and solving for x_1

$$x_1 = (a + \beta \ln m) \frac{m}{p_1}$$

- The slope is

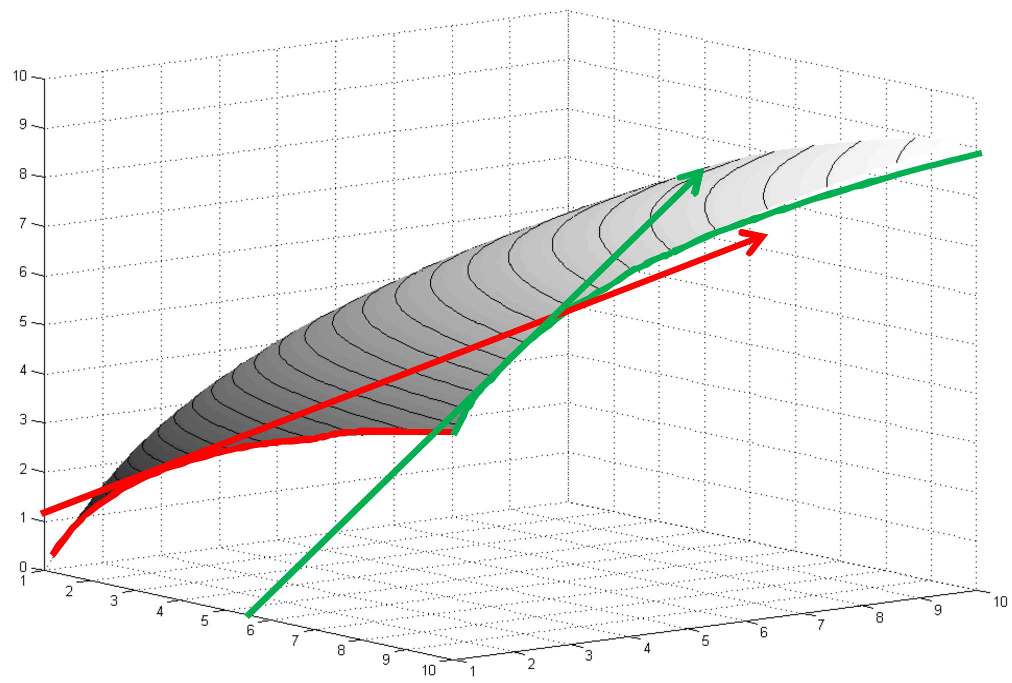
$$\frac{dx_1}{dm} = (a + \beta \ln m)\frac{1}{p_1} + \frac{\beta}{p_1}$$

- The definition of an income elasticity is

$$\varepsilon_{x_1,m} = \frac{dx_1}{dm}\frac{m}{x_1}$$

$$\begin{aligned}
\varepsilon_{x_1,m} &= \left[(a + \beta \ln m)\frac{1}{p_1} + \frac{\beta}{p_1}\right]\frac{m}{x_1} \\
\varepsilon_{x_1,m} &= (a + \beta \ln m)\frac{1}{p_1}\frac{m}{x_1} + \frac{\beta}{p_1}\frac{m}{x_1} \\
\varepsilon_{x_1,m} &= \frac{\beta}{w_1} + 1
\end{aligned}$$

Partial Differentiation



Partial Differentiation

- The calculation of the gradient of a function $y = f(w, x, z)$ with respect to (w.r.t.) one variable with the others held fixed is a process called *partial differentiation*.

$$\frac{\partial f(w,x,z)}{\partial w} = f_w = \text{The partial derivative w.r.t. } w \text{ - } (x,z) \text{ fixed}$$

$$\frac{\partial f(w,x,z)}{\partial x} = f_x = \text{The partial derivative w.r.t. } x \text{ - } (w,z) \text{ fixed}$$

$$\frac{\partial f(w,x,z)}{\partial z} = f_z = \text{The partial derivative w.r.t. } z \text{ - } (w,x) \text{ fixed}$$

- The mathematical version of the *ceteris paribus* qualification.

Total Differentiation

- If $y = f(x)$ the differential of y is

$$dy = f'(x)\,dx$$

- With multivariarate functions

$$\begin{aligned} y &= f(w, x, z) \\ dy &= \frac{\partial f(w, x, z)}{\partial w} dw + \frac{\partial f(w, x, z)}{\partial x} dx + \frac{\partial f(w, x, z)}{\partial z} dz \\ dy &= f_w dw + f_x dx + f_z dz \end{aligned}$$

logs and elasticities

- Totally differentiate $y(x) = \ln x$

$$dy = \frac{1}{x} dx$$

$$d \ln x = \frac{1}{x} dx$$

$$d \ln x = \frac{dx}{x}$$

logs and elasticities

- The definition of an elasticity is

$$\varepsilon = \frac{dy/y}{dx/x}$$

- Given $d\ln x = \frac{dx}{x}$ and $d\ln y = \frac{dy}{y}$

$$\varepsilon = \frac{d\ln y}{d\ln x}$$

Example. The price elasticity of demand

- Reconsider the demand curve

$$\ln x_1 = \alpha - \beta \ln p_1$$

- The elasticity can be obtained directly

$$\varepsilon_{x_1,p_1} = \frac{d \ln x_1}{d \ln p_1} = -\beta$$

- The log transformation makes the nonlinear function $x_1 = e^{\alpha} p_1^{-\beta}$ linear in logs and therefore easy to handle.

The Marginal Rate of Substitution

- Consider a bivariate utility function $u(x_1, x_2)$

- The total differential is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

- On an indifference curve $du = 0$

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0$$

The Marginal Rate of Substitution

- Which implies that

$$\frac{dx_2}{dx_1} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{u_1}{u_2}$$

$$MRS_{2,1} = -\frac{u_1}{u_2}$$

$$MRS_{2,1} = -\frac{\text{marginal utility of good 1}}{\text{marginal utility of good 2}}$$

- So the MRS between x_2 and x_1 is the negative of the ratio of the marginal utility of x_1 to the marginal utility of x_2.

Example: The Marginal Rate of Substitution (MRS) - again

- For our example utility function:

$$u = a \ln x_1 + (1-a) \ln x_2$$

- The total differential is

$$du = \left(\frac{a}{x_1}\right) dx_1 + \left(\frac{1-a}{x_2}\right) dx_2 = 0$$

- So the MRS is

$$MRS_{2,1} \quad = \quad \frac{dx_2}{dx_1} \quad = \quad -\frac{a}{1-a}\frac{x_2}{x_1}$$

Example: The Marginal Rate of Substitution (MRS) - again

- Using the ratio of marginal utilities:

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= u_1 = \frac{a}{x_1} \\ \frac{\partial u}{\partial x_2} &= u_2 = \frac{1-a}{x_2} \end{aligned}$$

- So the MRS is

$$MRS_{2,1} = \frac{dx_2}{dx_1} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{a}{1-a}\frac{x_2}{x_1}$$

Elasticites

- Take a general Marshallian demand function: $x_i\left(p_1, p_2, , ..., p_N, m\right)$

- The various elasticites are all partial differentials

$$\begin{aligned}
\varepsilon_{x_i,p_i} &= \frac{\partial x_i / x_i}{\partial p_i / p_i} = \frac{\partial \ln x_i}{\partial \ln p_i} \\
\varepsilon_{x_i,p_j} &= \frac{\partial x_i / x_i}{\partial p_j / p_j} = \frac{\partial \ln x_i}{\partial \ln p_j} \\
\varepsilon_{x_i,m} &= \frac{\partial x_i / x_i}{\partial m / m} = \frac{\partial \ln x_i}{\partial \ln m}
\end{aligned}$$

Example. log-linear demand elasticities

$$\ln x_1 = \alpha - \beta_1 \ln p_1 + \beta_2 \ln p_2 + \gamma \ln m$$

$$\begin{aligned}
\varepsilon_{x_1,p_1} &= \frac{\partial \ln x_1}{\partial \ln p_1} = -\beta_1 \\
\varepsilon_{x_1,p_2} &= \frac{\partial \ln x_1}{\partial \ln p_2} = \beta_2 \\
\varepsilon_{x_1,m} &= \frac{\partial \ln x_1}{\partial \ln m} = \gamma
\end{aligned}$$

Example. linear demand elasticities

$$x_1 = \alpha - \beta_1 p_1 + \beta_2 p_2 + \gamma m$$

$$\begin{aligned}
\varepsilon_{x_1,p_1} &= \frac{\partial x_1/x_1}{\partial p_1/p_1} = \frac{\partial x_1}{\partial p_1}\frac{p_1}{x_1} = -\beta_1 \frac{p_1}{x_1} \\
\varepsilon_{x_1,p_2} &= \frac{\partial x_1/x_1}{\partial p_2/p_2} = \frac{\partial x_1}{\partial p_2}\frac{p_2}{x_1} = \beta_2 \frac{p_2}{x_1} \\
\varepsilon_{x_1,m} &= \frac{\partial x_1/x_1}{\partial m/m} = \frac{\partial x_1}{\partial m}\frac{m}{x_1} = -\gamma \frac{m}{x_1}
\end{aligned}$$

The Slutsky equation

- The Slutsky equation is composed of partial differentials:

$$\frac{\partial x_i}{\partial p_j} = \left.\frac{\partial x_i}{\partial p_j}\right|_u - \frac{\partial x_i}{\partial m} x_j$$

- We can work out all of the components of the Slutsky equation from the Marshallian demand function $x_i\left(p_1, ..., p_N, m\right)$ except for the substitution effect.

- To get the substitution effect we have to back it out from the Slutsky equation.

Example. The Slutsky equation

- Suppose the Marshallian demand for a good is

$$x_1 = \frac{\alpha m}{p_1} \qquad \alpha \in [0, 1]$$

- Then

$$\begin{aligned} \frac{\partial x_1}{\partial m} &= \frac{\alpha}{p_1} \\ \frac{\partial x_1}{\partial p_1} &= -\frac{\alpha m}{p_1^2} \end{aligned}$$

Example. The Slutsky equation

- Rearranging the Slutsky equation for the substitution effect gives:

$$\left.\frac{\partial x_1}{\partial p_1}\right|_u = \frac{\partial x_1}{\partial p_1} + \frac{\partial x_1}{\partial m}x_1$$
$$= -\frac{\alpha m}{p_1^2} + \frac{\alpha}{p_1}x_1$$
$$= (\alpha - 1)\frac{x_1}{p_1} \qquad (\leq 0)$$

- The *compensated price elasticity* is a non-positive constant:

$$\left.\frac{\partial x_1}{\partial p_1}\right|_u \frac{p_1}{x_1} = (\alpha - 1)\frac{x_1}{p_1}\frac{p_1}{x_1} = (\alpha - 1)$$

Example. The Slutsky equation

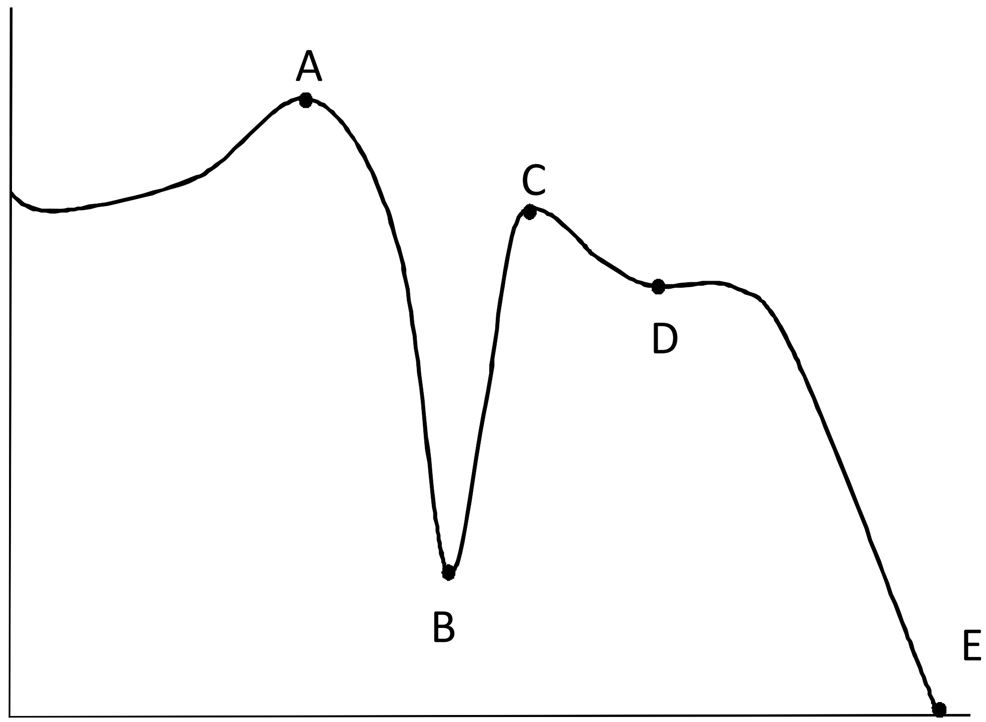
- We could also use the elasticity version of the Slutsky equation:

$$\varepsilon_{x_1,p_1} = \varepsilon_{x_1,p_1}\big|_u - \varepsilon_{x_1,m} w_1$$

$$\begin{aligned}
\varepsilon_{x_1,p_1} &= \frac{\partial x_1}{\partial p_1}\frac{p_1}{x_1} &=& -\frac{\alpha m}{p_1^2}\frac{p_1}{\alpha m/p_1} &=& -1 \\
\varepsilon_{x_1,m} &= \frac{\partial x_1}{\partial m}\frac{m}{x_1} &=& \frac{\alpha}{p_1}\frac{m}{\alpha m/p_1} &=& 1 \\
w_1 &= \frac{p_1 x_1}{m} &=& \frac{p_1\left(\alpha m/p_1\right)}{m} &=& \alpha
\end{aligned}$$

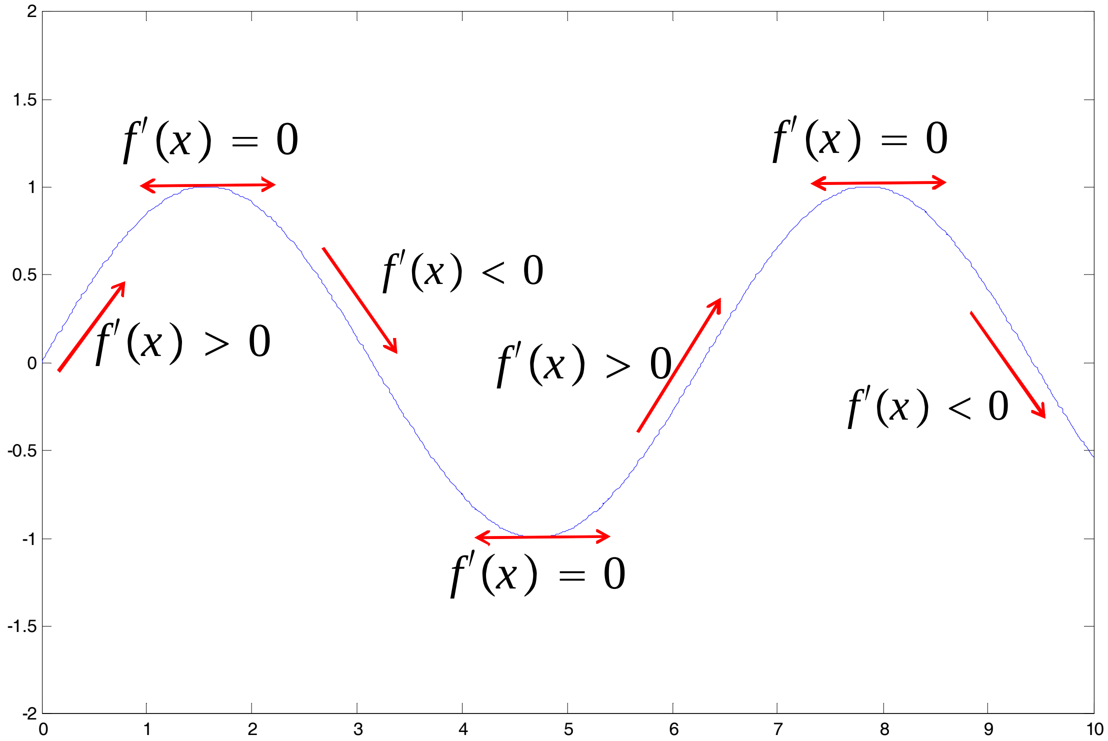
$$\varepsilon_{x_1,p_1}\big|_u = \varepsilon_{x_1,p_1} + \varepsilon_{x_1,m} w_1 = \alpha - 1$$

Optimisation



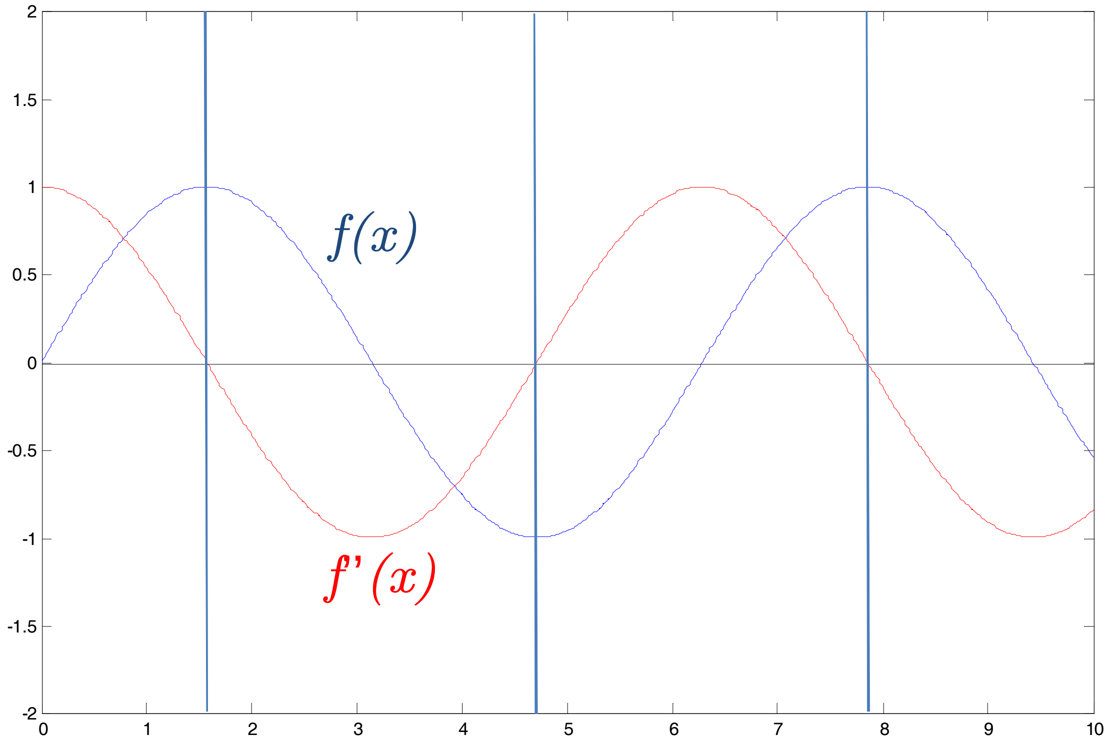
Optimisation

At turning points $f'(x) = 0$ and the gradient changes either side.



The Second Derivative

$f'(x)$ tells us whether $f(x)$ is going up, down or is flat. It is itself a function



The Second Derivative

- If we treat $f'(x)$ as a function we can work out *its* derivative.

- This is the *second derivative*.

- For a function $y = f(x)$

$$\text{1st derivative:} \qquad f'(x) \equiv \frac{df(x)}{dx} \equiv \frac{dy}{dx}$$

$$\text{2nd derivative:} \qquad f''(x) \equiv \frac{df'(x)}{dx} \equiv \frac{d}{dx}\left(\frac{df(x)}{dx}\right) \equiv \frac{d^2y}{dx^2}$$

Using the 2nd Derivative to classify stationary points.

- If $f(x)$ has a stationary point at $x = x^*$ then $f'(x^*) = 0$

- This is called the *first order condition*

- To classify it as a max or a min:

$$\begin{aligned} \text{If } f''(x^*) &> 0 \text{ then it is a min} \\ \text{If } f''(x^*) &< 0 \text{ then it is a max} \end{aligned}$$

- These are *second order conditions*.

Using the 2nd Derivative to classify stationary points.

- This extends with only a modest modification to stationary points of multivariate function: $f(x_1, ..., x_n)$

- If $f(x_1, ..., x_n)$ has a stationary point at $x_1^*, ..., x_n^*$ then

$$\frac{\partial f\left(x_1^*, ..., x_n^*\right)}{\partial x_i} = f_i = 0 \text{ for all } i \text{ variables}$$

- Unfortunately the second order conditions are very messy for 3 (and higher) dimensional problems.

- Fortunately if the function of interest is sufficiently well-behaved we don't need to worry about the second order conditions.

Calculus and Economic Theory

There is a standard approach to modelling optimising behaviour in economics

1. Formulate the economic objective function: $f(x)$
2. Differentiate it: $f'(x)$
   (a) Set $f'(x) = 0$ [1st order condition]
   (b) Check the sign of $f''(x)$ [2nd order condition]
3. Work out what $f'(x) = 0$ implies about behaviour (solve for x).

"People choose, out of the combinations of market products which they can afford, the combination which they most prefer"

- The model can be cast mathematically as

$$\max_{x_1,x_2} u(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 = m$$

- There are two way to solve this problem analytically as a simple optimisation problem.

1. Solution by substitution of the budget constraint.

2. Solution via the MRS condition (interior optima only)

1. Solution by substitution

1. Solve the budget constraint for one of the demands in terms of the other.

2. Substitute it into the utility function and this gives us a utility function in terms of *one* variable with the constraint already built into it.

3. Find its turning points and characterise the solution.

Example - solution by substitution

- Consider the Cobb-Douglas utility function once more:

$$u = a \ln x_1 + (1-a) \ln x_2 \qquad a \in [0,1]$$

- The full problem is

$$\max_{x_1, x_2} u = a \ln x_1 + (1-a) \ln x_2 \text{ subject to } p_1 x_1 + p_2 x_2 = m$$

Example - solution by substitution

1. Solve the budget constraint for x_2 in terms of x_1

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2} x_1$$

2. Substitute into the utility function for x_2

$$u(x_1) = a \ln x_1 + (1 - a) \ln \left(\frac{m}{p_2} - \frac{p_1}{p_2} x_1 \right)$$

Example - solution by substitution

3. Find and characterise its turning points.

   (a) First we need its gradient.

$$u'(x_1) = \frac{a}{x_1} - \frac{1-a}{\frac{m}{p_2} - \frac{p_1}{p_2}x_1}\left(\frac{p_1}{p_2}\right)$$

   (b) Now set $u'(x) = 0$

$$u'(x_1) = \frac{a}{x_1} - \frac{1-a}{\frac{m}{p_2} - \frac{p_1}{p_2}x_1}\left(\frac{p_1}{p_2}\right) = 0$$

   (c) Solve for x_1

$$x_1 = a\frac{m}{p_1}$$

Example - solution by substitution

- Is this a max or a min?

$$u''(x_1) = -\frac{a}{x_1^2} - \frac{1-a}{\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)^2}\left(\frac{p_1}{p_2}\right)^2$$

- You can see that $u''(x) < 0$ because

$$-\frac{a}{x_1^2} < 0 \qquad \text{and} \qquad \frac{1-a}{\left(\frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)^2}\left(\frac{p_1}{p_2}\right)^2 > 0$$

so it's a max

Example - solution by substitution

- What's the demand for x_2?

$$p_1 \left(a \frac{m}{p_1} \right) + p_2 x_2 = m$$

$$\Rightarrow x_2 = (1 - a) \frac{m}{p_2}$$

Example - solution by substitution

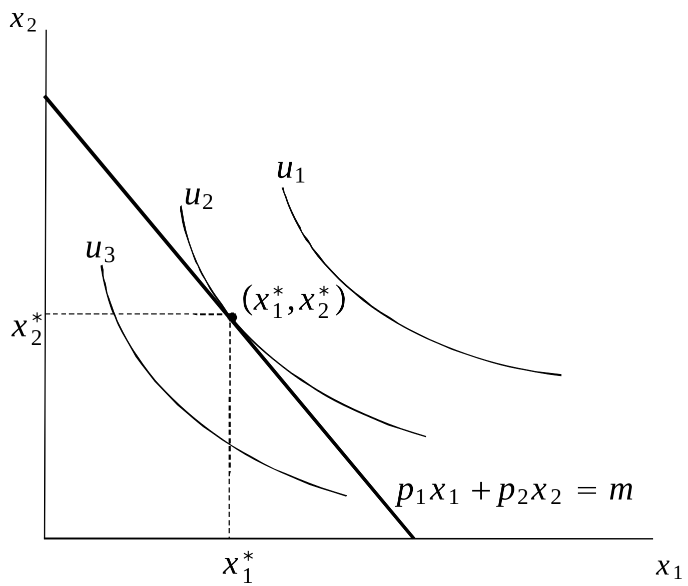
$$\max_{x_1,x_2} u = a \ln x_1 + (1-a) \ln x_2 \text{ subject to } p_1 x_1 + p_2 x_2 = m$$

$$\begin{aligned} x_1 &= a\frac{m}{p_1} \\ x_2 &= (1-a)\frac{m}{p_2} \end{aligned}$$

2. Solution using the MRS condition

This works when the optimum is an interior solution ($x_i > 0$)

$$MRS_{2,1} = -\frac{\partial u/\partial x_1}{\partial u/\partial x_2} = -\frac{u_1'}{u_2'} = -\frac{p_1}{p_2}$$



2. Solution using the MRS condition

The steps involved are:

1. Derive the marginal utilities by partial differentiation.

2. Set up the MRS condition and the budget constraint

$$MRS_{2,1} = -\frac{u_1}{u_2} = -\frac{p_1}{p_2}$$
$$p_1 x_1 + p_2 x_2 = m$$

3. Find and characterise the solution in terms of x_1 and x_2.

Example. Solution using the MRS condition

For our example preferences $u = a \ln x_1 + (1-a) \ln x_2$ zero solutions are impossible.

1. The marginal utilities are

$$\begin{aligned} \frac{\partial u}{\partial x_1} &= u_1 = \frac{a}{x_1} \\ \frac{\partial u}{\partial x_2} &= u_2 = \frac{1-a}{x_2} \end{aligned}$$

2. The MRS conditions and the budget constraint:

$$\begin{aligned} MRS_{2,1} &= -\frac{a/x_1}{(1-a)/x_2} = -\frac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 &= m \end{aligned}$$

Example. Solution using the MRS condition

3. Solve for x_1 and x_2

Solving the MRS for x_2 in terms of x_1, sub the result into the budget constraint and solving for x_1.You'll get

$$x_1 = a\frac{m}{p_1}$$

Then get x_2 from the budget constraint as we did before.