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Unemployment, Participation and Market Size

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Abstract

We construct an equilibrium random matching model of the labour market, with endogenous market participation and a general matching technology that allows for market size effects: the job-finding rate for workers and the incentives for participation change with the level of unemployment. In comparison to standard models with constant returns to scale in matching, agent behaviour is more complex – the model generates plausible joint dynamics of employment, unemployment and participation with heterogeneity in search behaviour for workers with different degrees of attachment to the labour market. Techniques are developed to reduce the dimensionality of the problem to establish local and global stability; a complicating factor is the possibility of multiple equilibria, welfare-ranked by market size. A Hosios-type condition internalises search externalities.

JEL Classification Numbers: J41, J64

Keywords: Unemployment, Participation, Job Search, Matching Function, Returns to Scale, Multiple Equilibria, Stability, Coordination, Search Externalities.

1 Introduction

In labour markets affected by frictions it takes time for potential trading partners to find each other and agree to trade. Random matching models of the labour market do not model explicitly the heterogeneities and information problems that give rise to frictions, but attempt to capture the implications of the costly trading process by describing the rate at which workers and firms meet as a function of the numbers of agents on each

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side of the market. A natural assumption for the matching function is that the relative number of agents matters: for example the job-finding rate for an unemployed worker increases with market tightness. It is also possible that the absolute number of agents in the market affects the rate at which individuals match; a market could be too small to support effective mechanisms for exchanging information about opportunities for trade, or so large that agents suffer from information overload.

Market size effects were the focus of Diamond's (1982) model of multiple equilibria and co-ordination failure, but have since received little attention in the extensive matching literature. The dominant equilibrium random matching model of the labour market, most fully described by Pissarides (2000), rules them out by assuming a matching function with everywhere-constant returns to scale, usually justified by an appeal to empirical evidence. But the evidence for constant returns is not compelling: at the aggregate level studies for some countries have found significantly increasing or decreasing returns,[1] and substantial departures from constant returns have been found for regional and occupational markets.[2]

The constant returns assumption buys tractability, but risks over-simplification: it has stark implications for the dynamic behaviour of the Pissarides model. Away from the steady state unemployment evolves slowly, but reservation wages, search intensity, and market tightness remain constant at their equilibrium values. Since market size does not matter, unemployed workers are indifferent to the level of unemployment, and fluctuations in unemployment have no effect on labour force participation.

In this article we present an equilibrium matching model of the labour market, similar to that of Pissarides (2000) but with a general form for the matching function and endogenous participation, allowing us to investigate the determination of market size and the properties of equilibria with non-constant returns. Steady state equilibria have simple properties, but the model provides a richer environment for the analysis of dynamics. Since the job-finding rate changes with the level of unemployment, workers adjust their

[1] Petrongolo and Pissarides (2001) report p-values for a test of constant returns to scale for thirteen estimates, from a variety of studies, of a Cobb-Douglas aggregate matching function; six of these reject constant returns at the 5% level; eight do so at the 10% level.

[2] The evidence for disaggregated markets is mixed. For travel-to-work areas in Britain, the results of Coles and Smith (1996) strongly support constant returns, but Burgess and Profit (2001) allow for area fixed effects and find decreasing returns. Kano and Ohta (2005) also obtain decreasing returns for Japanese regions. Several studies of local labour markets have found increasing returns: Kangasharju, Pehkonen and Pekkala (2005) for Finland; Baker, Hogan and Ragan (1996) for Canada; and Munich, Svejnar and Terrell (1999) for the Czech Republic. Fahr and Sunde (2004) find increasing returns for craft and technical occupations, and substantially decreasing returns for white collar occupations and for highly educated workers.

behaviour to allow for the current and expected future state of the market. We find that workers who have higher outside options are less attached to the market; while participating they set higher reservation wages and search less intensively. When the market is expected to improve in future, it can be optimal for participating workers to withdraw temporarily, to wait for more favourable conditions.

We obtain conditions for existence and local stability of steady-state equilibria; when multiple equilibria exist they are welfare-ranked according to market size, irrespective of the shape of the matching function and the elasticity of labour supply. Equilibria with decreasing returns are locally saddle-path stable; those with increasing returns can be stable provided that returns to scale do not increase steeply, and labour supply is sufficiently inelastic. Following a shock that increases the level of unemployment, reservation wages either rise or fall depending on the direction of the market size effect, and participation, search intensity, reservation wages and market tightness adjust along the saddle-path. We also consider global stability: as in other rational expectations models with multiple equilibria, when agents' behaviour depends on their expectations of the future path of the economy it is possible for a sub-optimal steady state to be selected simply because they believe that it will be.

Finally we compare decentralised equilibria with the efficient allocation, and show that a generalisation of the well-known Hosios condition for efficiency holds: as for the constant returns case, the decentralised equilibrium can be efficient if the shares of the surplus obtained by workers and firms are equal to the elasticities of the matching function with respect to unemployment and vacancies. Thus efficiency requires an employment tax with decreasing returns, or a subsidy with increasing returns.

1.1 Labour Market Matching Models with Constant Returns

Equilibrium random matching models are now well-established as a framework for understanding unemployment. In the textbook random matching model of Pissarides (2000), ex-ante identical workers and firms search in the labour market and meet according to a matching technology that is linearly homogeneous in the stocks of unemployed workers and vacancies. Each match produces a flow of output, with constant returns in production, and the wage is determined through bilateral bargaining. When a match is destroyed as a result of an exogenous productivity shock the worker re-enters the market. The size of the labour force is fixed, and a zero-profit condition determines the stock of vacancies.

The search-theoretic approach is consistent at a microeconomic level with many features of observed labour market behaviour (Rogerson, Shimer and Wright, 2005). At the aggregate level, the model provides a theory of equilibrium unemployment. Allowing for endogenous job destruction and aggregate productivity shocks it predicts labour market flows broadly consistent with business cycle evidence (Mortensen and Pissarides, 1994; Mortensen and Pissarides, 1999; Cole and Rogerson, 1999). Incorporating a matching model of the labour market has been shown to improve the ability of RBC models to reproduce macroeconomic stylised facts (Merz 1995, 1999; Andolfatto, 1996; and Den Haan, Ramey and Watson, 2000).

A number of recent papers have highlighted problems, however. Shimer (2005) finds that the Mortensen-Pissarides model cannot account for the strong procyclicality of the job-finding rate and the vacancy-unemployment ratio. Cole and Rogerson (1999) argue that the model needs to allow for heterogeneity in search intensity, since consistency with the evidence requires workers classified as out of the labour force to be treated as low-intensity searchers. Veracierto (2004) points out that neither standard RBC models (in which all non-employed workers are non-participants), nor those augmented by matching frictions (which have assumed a fixed labour force) can explain the joint dynamics of employment, unemployment and labour force participation.

The conventional assumption of a fixed labour force can be relaxed (see Pissarides, 2000, chapter 7) to allow for upward-sloping labour supply to the matching market and hence make participation endogenous. However in the standard constant returns model this gives no insight into participation dynamics. Since the reservation wage – which captures the value of participation – always remains constant at its equilibrium level, there is no variation in participation: during a period of adjustment workers do not enter or leave the labour force as unemployment changes.

The efficiency properties of matching models with constant returns are well-understood. In addition to the inevitable losses due to the existence of frictions, search externalities arise because each agent affects the matching rate for other agents. When wages are determined ex-post, the market equilibrium will not in general be constrained-efficient. For the standard model with constant returns and homogeneous agents, Hosios (1990) showed that externalities are internalised if the bargaining share of workers is equal to the elasticity of matching with respect to unemployment. In this environment, all external effects act through market tightness (the v-u ratio), and a single condition for relative surplus shares is sufficient to ensure that agents make efficient search and matching decisions.

1.2 Non-Constant Returns, Multiple Equilibria and Dynamics

In the elegant stylised model of Diamond (1982) there is only one type of searching agent. Trading opportunities occur when they meet each other, and the meeting rate – and therefore the incentive to search – increases with the number of searchers. Increasing returns to scale in matching generates multiple equilibria and potential coordination failure. Diamond and Fudenberg (1989) analyse the dynamics of the Diamond (1982) model, emphasising the self-fulfilling expectations property. For some initial conditions there are rational expections equilibrium paths to both high and low activity equilibria, and there may be cyclical equilibrium paths. Boldrin, Kiyotaki and Wright (1993) provide a full analysis of dynamics in a generalisation of the Diamond model with differentiated commodities, focusing in particular on the existence of limit cycles.

The assumption of a single type of agent limits the applicability of the Diamond model, although Camera (2000) uses this approach to obtain multiple equilibria in a monetary search model. There are rather few analyses of the implications of non-constant returns to scale when the matching function depends separately on the numbers of buyers and sellers, as in the conventional labour market approach. Pissarides (1984) shows that with a general matching technology equilibria are inefficient – the Hosios (1990) condition cannot hold without constant returns. Howitt and McAfee (1987) use a specific increasing returns technology to introduce Diamond's thin-market externality and hence multiple equilibria into a labour market model, again emphasising the inefficiency of all equilibria in this case. Hyde (1997) constructs a bargaining model with homogeneous buyers and heterogeneous sellers, in which an increasing returns matching technology gives rise to both high and low participation equilibria. Burdett and Wright (1998) model a labour market with fixed and equal numbers of agents on both sides, and stochastic match productivity, and show that multiple equilibria may occur either as a result of increasing returns in the matching function or when utility is not perfectly transferable within a match. In both cases multiple equilibria are ruled out when the distribution of match productivity is log-concave.

None of the labour market models considers dynamics with non-constant returns to scale in matching, although Mortensen (1989) suggests that dynamic behaviour will be similar to that in a labour market matching model with aggregate increasing returns in production (Drazen, 1988; Mortensen, 1999) which, like the Diamond model, has multiple Pareto-ranked equilibria, self-fulfilling expectations and limit cycles.

Multiple equilibria in matching models may of course arise for reasons other than the matching technology. Non-transferable utility and increasing returns in production have already been mentioned; in addition, ex-ante heterogeneity or an ex-ante investment decision can generate strategic complementarity between agents on either side of the market, so that, for example, when workers set higher reservation wages firms create more high productivity jobs (Acemoglu, 1997, 2001; van den Berg, 2003).

2 The Model

2.1 The Matching Technology

Unemployed workers and potential employers meet randomly according to a matching technology that determines the meeting rate, M, as a function of the mass of unemployed workers, u, and the mass of vacant jobs, v:

$$M = M(\bar{s}u, v)$$

$\bar{s}$ is the average search intensity of unemployed workers, so $\bar{s}u$ is their total effective search. The rate of meeting per unit of search, which we call the job-finding rate, is:

$$\lambda = \frac{M}{\bar{s}u}$$

and a worker searching at intensity s meets firms at rate $s\lambda$. The search intensity of employers is exogenous and normalised to one;[3] hence a vacancy encounters potential employees at rate M/v. We assume that $M(0, v) = M(\bar{s}u, 0) = 0$, and that M is strictly increasing in both arguments, homothetic, and quasi-concave. Hence it can be expressed:

$$M = \Phi(m(\bar{s}u, v))$$

where Φ is a strictly increasing function, $\Phi(0) = 0$, and $m(\bar{s}u, v)$ is homogeneous of degree one with strictly convex isoquants (so m is concave). We can normalise this decomposition by assuming that $m(1, 1) = 1$.

It is helpful to think of $m(\bar{s}u, v)$ as the level of *aggregate search activity* in the market, and $\Phi(m)$ as converting activity into matching. Activity increases linearly with the stocks of searching workers and firms. We define the elasticity of matching with respect to activity, $\eta(m)$, and the average matching rate, $\phi(m)$, by:

$$\eta(m) \equiv \frac{m\Phi'(m)}{\Phi(m)} \quad \text{and} \quad \phi(m) \equiv \frac{\Phi(m)}{m}$$

The matching function has locally decreasing, constant, or increasing returns to scale when the elasticity η is, respectively, less than, equal to, or greater than 1. The average matching rate is a measure of the effectiveness of matching. For a constant returns matching function it is constant; otherwise $\phi(m)$ rises or falls where returns to scale are increasing or decreasing respectively. In the example in Figure 1a (see section 3.1), there are locally constant returns at m^*, where $\eta(m) = 1$ and the average matching rate $\phi(m)$ is maximised; if $m < m^*$ there are increasing returns, the marginal matching rate is greater than the average matching rate, and $\eta > 1$; above m^* the converse conditions

[3]With a zero-profit condition for firms (section 2.2) endogenising their search intensity has little effect.

hold. Intuitively, m^* is the point where search activity leads most effectively to matching.

Since activity $m(su, v)$ is homogeneous of degree 1 its elasticities with respect to unemployment and vacancies, denoted by $1-\alpha$ and α respectively, are functions only of market tightness, $\theta \equiv v/\bar{s}u$, (as are average and marginal activity with respect to u and v). The elasticities of the matching rate $M = \Phi(m)$ with respect to unemployment and vacancies are given by:

$$\eta_u = (1-\alpha)\eta \quad \text{and} \quad \eta_v = \alpha\eta$$

The elasticity of substitution between unemployment and vacancies also depends on θ only, and is the same for M and m. It is convenient to define $\mu(\theta)$ as activity per unit of worker search:

$$\mu(\theta) \equiv m(1, \theta) = \frac{m(\bar{s}u, \bar{s}u\theta)}{\bar{s}u}$$

The function $\mu(\theta)$ also has elasticity α. Then the job-finding rate λ, which in the standard model depends on market tightness only, is proportional to the average matching rate:

$$\lambda = \phi(m)\mu(\theta)$$

2.2 Participation, Search, and Employment

All agents are infinitely lived with common discount rate ρ. A worker is in one of three states at any instant: employed, unemployed and searching for a job, or "inactive" – that is to say, outside the market.

Workers can move instantaneously between unemployment and inactivity, and may differ in the opportunities available to them outside the market: a worker of type q obtains a constant flow income q while inactive that he must forgo while participating. The outside income q can be interpreted as the value of leisure, or home production, or of participation in a different market. Labour supply to the market is described by an increasing function $L(q)$, the number of workers with outside income less than or equal to q, and the elasticity of labour supply is denoted by γ. A worker searching at intensity s receives a flow of utility $b(s)$, which is decreasing and concave with constant elasticity $1 + 1/\epsilon$. The parameter $\epsilon > 0$ captures the responsiveness of search intensity to market conditions. The utility of a worker searching at zero intensity, $b(0)$, is normalised to zero.

There are many firms, each with a single potential job.[4] A firm with no employee will create and maintain a vacancy whenever the present value of doing so is greater than zero. Thus the supply of vacancies is perfectly elastic at zero profit. While maintaining a vacancy the firm incurs a constant flow search cost c.

[4]Equivalently, firms are of indeterminate size and have constant returns in production.

Match productivity is stochastic: an employed worker produces a constant flow of output y, which is a random variable realised when the worker and firm meet, with distribution function $F(y)$ and supremum $\bar{y}$. Matches are destroyed at constant rate δ, in which case the agents may search for a new match, or leave the market. There is no divorce: once a match has been formed agents cannot leave until it is destroyed. This assumption – which substantially increases tractability – is a simple way of capturing turnover costs. If agents incur a sunk cost on match formation, and y represents productivity net of this (amortised) cost, the ex-post value of a match will be high enough that relatively small changes in market conditions do not induce re-entry into unemployment.

Potential matches are not necessarily consummated, since if productivity is low the agents may prefer to search for a better match. If agents have a joint reservation productivity level z, the probability that a match is acceptable is:

$$\pi(z) = \mathrm{P}(y \geq z) = 1 - F(z)$$

and the expected productivity of an acceptable match is given by:

$$\mathrm{E}(y \mid y \geq z) = z + h(z)$$

where h is the expected surplus, defined by:

$$h(z) = \frac{1}{\pi(z)} \int_z (1 - F(y))\, dy$$

Intuitively one might expect the surplus $h(z)$ to decrease with z, and it is often useful in search models to assume that the distribution of productivity is log-concave in order to guarantee this property. We will assume only that the surplus does not increase too fast – specifically that the elasticity $zh'(z)/h$ is less than one.[5]

2.3 Wages and Surplus-Sharing

Wages are determined when the worker and firm meet, and remain constant while the match lasts. The worker receives a wage w_1, and the firm pays a wage w_2; to allow for the possibility of an employment tax or subsidy we do not assume that they are equal. The firm's profit flow is then $p = y - w_2$.

If z is the worker's reservation wage, and x is the firm's reservation profit flow,[6] the flow surplus from the match is $y - (z + x)$. We will assume that wages are set, either exogenously or through a bargaining process, so that the worker and firm receive non-

[5] The analysis does not rely heavily on this assumption, but it reduces the number of cases we need to consider. It is of course implied by, but weaker than, log-concavity.

[6] Intuitively, free entry implies that $x = 0$; this is established formally below.

negative shares β_1 and β_2 of the surplus:

$$w_1 = z + \beta_1(y - z - x) \quad \text{and} \quad y - w_2 = x + \beta_2(y - z - x)$$

2.4 The Decisions of Individual Workers and Firms

In this section we determine the reservation wage z and search intensity s of a worker of type q, and the reservation profit level x of a firm, taking the behaviour of other agents in the market as given.

For workers, market opportunities depend on the job-finding rate λ, and the distribution of wages. Let the expected present value of income be $U(t)$ for a worker who is not currently employed (he may be unemployed or inactive), and $W(w_1, t)$ for a worker employed at wage w_1. Similarly $V(t)$ and $J(p, t)$ are the values for a firm of a vacant and filled job. The worker remains employed until the match is exogenously destroyed, so $W(w_1, t)$ satisfies:

$$\rho W = w_1 + \delta(U - W) + \frac{\partial W}{\partial t} \tag{1}$$

Since an increase in the wage, dw_1, raises the worker's income by this amount for the duration of the match, its effect on his valuation of the match is given by:

$$\frac{\partial W}{\partial w_1} = \frac{1}{\rho + \delta} \tag{2}$$

Consider the decisions of a worker who is not currently employed. Over a time period of length dt he can take the outside income $q\,dt$, after which he will still be non-employed. Alternatively he can participate in the market, choosing his current search intensity s optimally, in which case he may remain non-employed, but with probability $s\lambda\,dt$ he will encounter a potential match with present value W. If a potential match arises, it is acceptable if and only if $W \geq U$. Hence U satisfies:

$$\rho U = \max\left\{q, \max_s \left(b(s) + s\lambda \mathrm{E}\left[\max\left\{W - U, 0\right\}\right]\right)\right\} + \frac{dU}{dt} \tag{3}$$

Since W increases with the wage, he will accept any wage greater than z, where:

$$W(z, t) = U(t) \tag{4}$$

The reservation wage z satisfies this equation continuously, so:

$$\frac{1}{\rho + \delta}\frac{dz}{dt} + \frac{\partial W}{\partial t} = \frac{dU}{dt} \tag{5}$$

Using (4), (5), and (1) evaluated at z, we can eliminate U from (3) to obtain for the optimal reservation wage z:[7]

$$\begin{aligned} z &= \max\{q, a(z,\lambda)\} + \frac{\dot{z}}{\rho+\delta} & (6) \\ \text{where} \quad a(z,\lambda) &\equiv \max_s (b(s) + s\lambda \mathrm{E}[\max\{W(w_1) - W(z), 0\}]) & (7) \end{aligned}$$

The value functions for the firm are obtained in the same way. But free entry implies that V is identically zero, from which it follows that the firm's reservation profit level is also zero and the value $J(p)$ of a filled job satisfies:

$$0 = -c + \frac{\lambda}{\theta}\mathrm{E}[\max\{J(p), 0\}] \quad (8)$$

$J(p)$ is constant over time, with $dJ/dp = 1/(\rho+\delta)$.

With these reservation values the joint flow surplus of a potential match is $y - z$. The worker's expected surplus from a meeting with a firm is:

$$\mathrm{E}[\max(W(w_1) - W(z), 0)] = \int_z (W(w_1(y)) - W(z))\, dF(y) = \beta_1 \frac{\pi(z)h(z)}{\rho+\delta} \equiv \beta_1 S(z)$$

(where (2) is used to evaluate the integral). Note that $S(z)$, the present value of the joint surplus, decreases with the worker's reservation wage. Similarly the firm's expected surplus from a meeting with a worker with reservation wage z is $\beta_2 S(z)$, and its expected surplus from a random meeting is $\beta_2 \bar{S}$, where $\bar{S}$ is the expectation of $S(z)$ allowing for differences across worker types. Using these expressions for the surplus gives for workers:

$$\begin{aligned} z &= \max\{q, a(z,\lambda)\} + \frac{\dot{z}}{\rho+\delta} & (9) \\ \text{where } a(z,\lambda) &= \max_s (b(s) + s\lambda\beta_1 S(z)) & (10) \end{aligned}$$

and for firms:

$$0 = -c + \frac{\lambda}{\theta}\beta_2 \bar{S} \quad (11)$$

$a(z,\lambda)$ can be interpreted as the current market income available to the worker; he participates when this is greater than his alternative income q. Current income a increases with the job-finding rate λ, but it decreases with z because with a higher reservation wage the worker is less likely to find an acceptable job.

[7] z is a function $z(q,t)$. We suppress arguments, and in particular time dependence, where this can be done without confusion. $\dot{z}$ denotes the derivative with respect to time.

2.5 Aggregate Unemployment and Participation

To determine aggregate variables we need to keep track of the distribution of worker-types across employment, unemployment and non-participation. Let $N(q,t)$ be the current number of non-employed workers with outside income less than q. $N(q,t)$ is the effective labour supply, instantaneously available to the matching market at time t. The number of *employed* workers with outside income less than q is $L(q) - N(q,t)$. From (9) we can describe participation by a function $p(q,t)$:

$$p(q) = \begin{cases} 1 & \text{if} \quad a(z(q), \lambda) \geq q \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Aggregate variables can then be expressed in terms of N and p. Assuming that we can write $N(q) = \int^q dN(q) = \int^q n(q)\, dq$ for some non-negative function $n(q)$, unemployment, average search intensity, and the average expected surplus from a random meeting are:

$$u = \int p(q)\, dN(q) \tag{13}$$

$$\bar{s} = \frac{1}{u} \int s(q)p(q)\, dN(q) \tag{14}$$

$$\bar{S} = \frac{1}{\bar{s}u} \int S(z(q))s(q)p(q)\, dN(q) \tag{15}$$

Finally, we can describe the evolution of the instantaneous supply of labour $N(q)$. The change in the distribution of non-employed workers is the difference between the inflow from employment as jobs are destroyed, and the outflow to employment, allowing for the search intensity and reservation wages of each type:

$$\frac{\partial N(q)}{\partial t} = \delta(L(q) - N(q)) - \lambda \int^q s(q)\pi(z(q))p(q)\, dN(q) \tag{16}$$

This equation, together with (9) to (11) for individual behaviour and (12) to (15) for aggregate variables, completes the description of an evolving market. In the next section we look for steady-state equilibria, before analysing the dynamics in more detail.

3 The Steady State

In a steady state, the numbers of participants are constant, as are their reservation values. From (9) and (10) we can see that all participating workers will have the same reservation wage z and search intensity s, so all matches have the same expected surplus $S(z)$. Since z is constant, it can be interpreted as the steady-state income of unemployed workers, and workers participate if they have outside income below z: $p(q) = 1$ if and only if $q \leq z$.

The active labour force is $L(z)$ and unemployment is $u = N(z)$. From equations (9) to (11) and (16), a steady-state equilibrium $(z^*, s^*, \theta^*, u^*)$ satisfies:[8]

$$\begin{aligned}
z &= b(s) + s\lambda\beta_1 S(z) && (17)\\
0 &= b'(s) + \lambda\beta_1 S(z) && (18)\\
0 &= -c + \frac{\lambda}{\theta}\beta_2 S(z) && (19)\\
\Phi(m)\pi(z) &= \delta(L(z) - u) && (20)\\
\text{where} \quad \lambda \equiv \phi(m)\mu(\theta) & \qquad \text{and} \qquad m \equiv su\,\mu(\theta) && (21)
\end{aligned}$$

and the distribution of non-employed workers is:

$$N(q) = \begin{cases} u + L(q) - L(z) & \text{if } q \geq z \\ \frac{u}{L(z)} L(q) & \text{if } q \leq z \end{cases} \qquad (22)$$

3.1 Existence of Equilibria

To establish existence, equations (17) to (20) can be reduced to a pair of equations in the reservation wage z and market activity m, as follows:

Lemma 1 *(i) For all m such that $\phi(m) \geq \phi_0$ (a positive constant) equations* (17) *to* (19) *have a unique solution:*

$$\begin{aligned}
z &= \zeta_1(m) \quad 0 \leq \zeta_1(m) < \bar{y},\ \zeta_1(m) = 0 \textit{ if } \phi(m) = \phi_0,\ \zeta_1' \overset{\text{sgn}}{=} \eta(m) - 1\\
s &= s_1(z) \quad s_1(0) = 0,\ s_1' > 0\\
\theta &= \theta_1(z) \quad \theta_1(0) = 0,\ \theta_1' > 0
\end{aligned}$$

(ii) Equation (20)*, together with $s = s_1(z)$ and $\theta = \theta_1(z)$, has a unique solution $\forall\, m \geq 0$:*

$$z = \zeta_2(m) \quad \zeta_2(0) = 0,\ \zeta_2' > 0,\ \textit{and } \zeta_2(\bar{m}) = \bar{y} \textit{ for some } \bar{m} < \infty$$

Proof: See Appendix A. ■

The solution to (17) to (19), for the choices of individual agents, depends on the level of market activity through the average matching rate, $\phi(m)$. If this is low – below a constant ϕ_0 that depends on the parameters – workers will not search and there is no solution. Above ϕ_0, the income z that the market can provide for workers increases with the average matching rate, so $z = \zeta_1(m)$ rises or falls with market activity m according to whether returns to scale in matching are increasing or decreasing. In the case of a constant

[8] It can be verified that there is always an interior solution for s.

returns matching function there is a unique income level for workers, irrespective of the level of activity in the market.

Equilibria can be found by looking for levels of market activity where $\zeta_1(m) = \zeta_2(m)$. Both functions are continuous, and $\zeta_2(m)$ is strictly increasing; if the matching function has everywhere non-increasing returns to scale $\zeta_1(m)$ is non-increasing and there is at most one equilibrium. Otherwise there may be several equilibria. By considering the properties of ζ_1 and ζ_2 we can show that:

Proposition 1 *A steady-state equilibrium (z, s, θ, u) with activity $m > 0$ exists for any $\beta_1, \beta_2 > 0$, provided that $M_v(1, 0)$ is sufficiently large. It is unique if $\eta(m) \leq \eta_0$ for all m, where*

$$\eta_0 \equiv \frac{1+\gamma}{\frac{\epsilon+\alpha}{\epsilon+1} + \gamma}$$

PROOF: See Appendix A. ■

Figure 1b illustrates possible equilibria for a matching function with initially increasing and subsequently decreasing returns to scale. For this form of matching function, at most one equilibrium – the one with the highest activity level – can occur in the region of decreasing returns. More generally, the equilibrium with highest activity always occurs at a point where $\zeta_2' > \zeta_1'$. Note that, since the function $\zeta_2(m)$ is increasing, equilibria with higher activity levels have higher average matching rates and higher levels of worker income.

As in the standard model a sufficient (but not necessary) condition for existence is that there are incentives for vacancy creation even when the market is small. The sufficient condition for uniqueness is obtained from a single-crossing condition: $\zeta_2' > \zeta_1'$ at any equilibrium where $\eta(m) \leq \eta_0$. The uniqueness condition may be compared with the result of Burdett and Wright (1998) that in the special case of fixed and identical numbers of workers and firms and exogenous search intensity, log-concavity of the productivity distribution is sufficient to guarantee uniqueness with increasing returns. Here log-concavity is not enough: when agents are able to respond to market conditions we need an upper bound on the degree of returns to scale, which becomes more restrictive with increasing elasticity of labour supply, γ, or responsiveness of search intensity, ϵ.

Later we will consider the dynamic stability of equilibria, but we can show immediately that equilibria like (2) where $\zeta_2' < \zeta_1'$ are implausible, in that they are not robust to any variation in the number of vacancies.

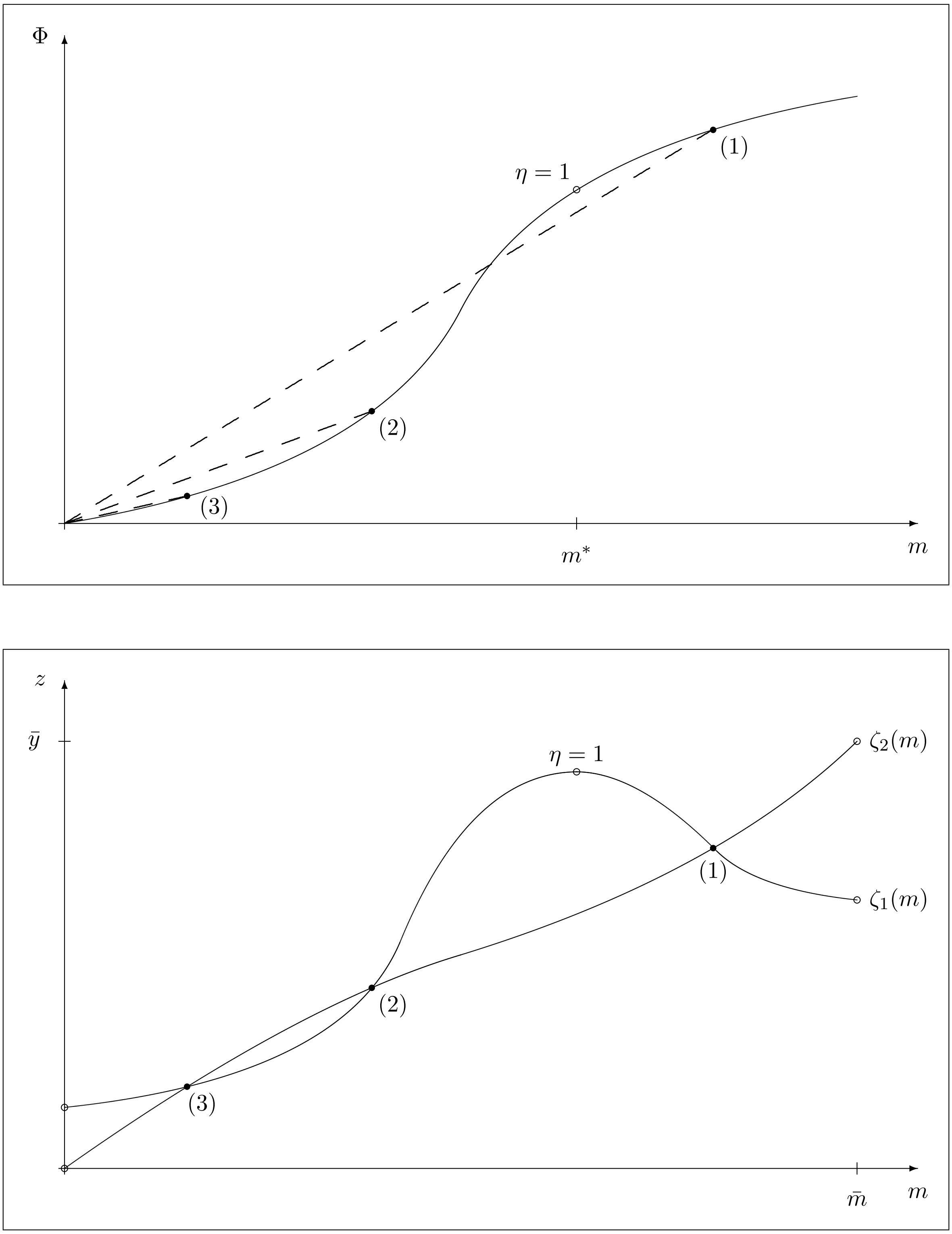

Figure 1: (a) A matching function with initially increasing and then decreasing returns to scale; (b) corresponding equilibria at the intersection of ζ_1 and ζ_2.

3.2 Incentives for Vacancy Creation

We have used the conventional assumption that the number of vacancies is determined by a zero-profit condition. Behind this is an implicit assumption that if profits ever rose above zero more firms would enter, reducing profit to zero again; similarly a fall in profit would be reversed by exit of firms. However, there is nothing in the model to ensure that this mechanism will operate. It is possible that at a solution of the steady-state equilibrium conditions with increasing returns to market activity, creation of additional vacancies would lead to a rise in profit and further incentives to create vacancies. This would be an implausible equilibrium – effectively a profit-minimising point for firms.

A minimal requirement for a market equilibrium is therefore that it should be *stable with respect to vacancy creation* – that a rise in vacancies should lead to a fall in the profit flow of firms (and vice-versa). This requires a bound on the rate of increasing returns at the equilibrium, and the condition fails at alternate equilibria:

Proposition 2 *With one or more steady-state equilibria, numbered in order of decreasing market size, even-numbered equilibria are unstable with respect to vacancy creation. Odd-numbered equilibria are stable with respect to vacancies if and only if:*

$$\eta < \frac{1+\bar{\gamma}}{\frac{\epsilon+\alpha}{\epsilon+1}+\bar{\gamma}} \equiv \bar{\eta}_0 \quad \textit{where} \quad \bar{\gamma} = \frac{L}{u}\gamma$$

Proof: Suppose that, in a steady-state equilibrium satisfying (17) to (20), v increases instantaneously by dv. This has no effect on reservation wages because agents expect to be in equilibrium in future when matches are formed. However the change in v changes the current activity level m, and hence also the current income of workers. They respond by changing their current search intensity s, and some workers enter or leave the market temporarily. The change in activity level $m(su, v)$ is:

$$\frac{dm}{m} = (1-\alpha)\left(\frac{du}{u}+\frac{ds}{s}\right)+\alpha\frac{dv}{v}$$

From (18) and (19) the changes in search intensity and instantaneous profit x satisfy:

$$\begin{aligned}\left(1+\frac{1}{\epsilon}\right)\frac{ds}{s} &= \eta\frac{dm}{m}-\frac{du}{u}\\ \frac{dx}{c} &= \eta\frac{dm}{m}-\frac{dv}{v}\end{aligned}$$

Finally, the instantaneous change in unemployment is $du = n(z)da$. Calculating the change in current income a using (10) gives:

$$\frac{du}{u} = (1+\epsilon)\frac{n(z)z}{u}\left(\eta\frac{dm}{m}-\frac{du}{u}-\frac{ds}{s}\right)$$

Solving these four equations for dx gives the overall effect on profit:

$$\frac{v}{c}\frac{dx}{dv} \overset{\text{sgn}}{=} \eta\left(\frac{\epsilon+\alpha}{1+\epsilon}+\frac{n(z)z}{u}\right)-\left(1+\frac{n(z)z}{u}\right)$$

Stability requires this expression to be negative. From (22), $n(q)$ is discontinuous at z: if z rises the change in labour supply is $n(z) = L'(z)$ whereas if z falls $n(z) = L'(z)u/L$. We have stability in both directions if and only if this expression is negative when $n = L'$; that is, if and only if $\eta < \bar{\eta}_0$. Since $\bar{\eta}_0 < \eta_0$, this condition implies that $\zeta_2' > \zeta_1'$ (see the proof of Proposition 1). Hence it cannot hold at equilibria where $\zeta_2' < \zeta_1'$ – that is, at even-numbered equilibria – but may do so at odd-numbered equilibria. ■

The interpretation of the condition $\eta(m) < \bar{\eta}_0$ is straightforward: for an equilibrium to be stable with respect to vacancy creation, it must have decreasing returns to vacancies, allowing for the responses of workers and firms. With fixed participation of workers ($\gamma = \epsilon = 0$) the condition is simply that the elasticity of matching with respect to vacancies, $\alpha\eta$, is less than one. The upper bound $\bar{\eta}_0$ falls as γ or ϵ rises; in the limiting case of a perfectly elastic supply of workers stability requires decreasing returns to scale at the equilibrium: $\bar{\eta}_0(m) = 1$.

Stability depends on $\bar{\gamma}$, an upper bound on the elasticity of *instantaneous* labour supply $N(z)$, which may be higher than the elasticity of total labour supply γ. At an equilibrium the instantaneous supply of labour is given by (22). When worker income rises, $N(z)$ is more elastic than total labour supply $L(z)$ because the number of workers who enter is *all* of those with outside income just above z, while the unemployed are only a proportion of those with outside income below z. When worker income falls, however, the elasticity of instantaneous labour supply is equal to that of total labour supply.

3.3 Steady State Welfare

In a steady-state equilibrium, social welfare Ω, which we take to be the present value of aggregate net income, is given by:

$$\rho\Omega = (L(z)-u)\,(z+h(z)) + ub(s) + \int_z q\,dL(q) - su\theta c$$

– the flow of output, plus the utility of unemployed workers, plus the outside income of non-participants, less the cost of vacancies. Using the equilibrium conditions:

$$\rho\Omega = \int \max\{z,q\}\,dL(q) + \Phi(m)\pi(z)h(z)\left(\frac{1}{\delta}-\frac{\beta_1+\beta_2}{\rho+\delta}\right)$$

If there are several equilibria, we know that since they satisfy $z = \zeta_2(m)$, equilibria with higher market size provide higher income z for workers. They also satisfy $z = \zeta_1(m)$, from which it can be shown that $\Phi(m)\pi(z)h(z)$ increases with m. Hence a comparison of welfare gives an unambiguous ranking:

Proposition 3 *When multiple equilibria exist, steady-state welfare increases with equilibrium market size if* $\beta_1 + \beta_2 \leq 1 + \rho/\delta$.

Proof: See Appendix A. ■

Thus in a decentralised equilibrium with no employment subsidy,[9] the equilibrium with the highest market size is welfare-superior: the average matching rate, the expected income z for unemployed workers, labour supply $L(z)$, and the aggregate surplus from employment $(L(z) - u)h(z)$ are all higher than at other equilibria.

4 Participation and Reservation Wages

Without constant returns to scale in matching, the job-finding rate, λ, varies over time when the market is not in steady state. We now examine the participation and search strategies of workers on a dynamic equilibrium path, which are determined by their expectation of the future path of the job-finding rate. We will assume here that λ is continuously differentiable with respect to time,[10] with a steady-state value λ^*.

The reservation wage of a worker with alternative income q satisfies equation (9). It follows from (10) that both his current market income $a(z, \lambda)$ and his search intensity s increase with λ and decrease with z. In the steady state, he participates if and only if $q \leq z^*$. Thus his reservation wage $z(q, t)$ is the solution of:

$$z = \max\{q, a(z, \lambda)\} + \frac{\dot{z}}{\rho + \delta} \tag{23}$$

$$\lim_{t\to\infty} z(q, t) = \max\{q, z^*\} \quad \text{where} \quad z^* = a(z^*, \lambda^*) \tag{24}$$

We can integrate (23) to obtain:

$$z(q, t) = (\rho + \delta) \int_t^\infty \exp\left[-(\rho + \delta)(\tau - t)\right] \max\{q, a(z, \lambda)\}\, d\tau \tag{25}$$

[9] If $\beta_1 + \beta_2 > 1 + \rho/\delta$, workers and firms have high incentives to search, and as activity rises search costs may increase faster than output.

[10] This will be so in equilibrium with a well-behaved matching function and continuous labour supply. Otherwise λ could jump; we could relax the assumption to piecewise differentiability to allow for this.

from which is it clear that $z(q,t)$ is continuously differentiable with respect to time, and can be interpreted as the worker's permanent income with a discount rate $\rho + \delta$. Note that it can never be optimal to accept a wage less than outside income: $z(q,t) \geq q$ for all q and t. Furthermore a worker who expects to participate for some future period, with $a(z(q),\lambda) > q$, has a reservation wage *strictly* above outside income. Workers enter or leave when current market income is equal to their outside income. From (23):

- *Entry and Exit.* At the time of entry (exit):

$$a(z(q),\lambda) = q, \quad \dot{a} \equiv a_z\dot{z} + a_\lambda\dot{\lambda} \geq 0\ (<0), \quad \text{and} \quad \dot{z} > 0\ (\geq 0)$$

Some may participate intermittently, but for those who participate continuously, and those who remain inactive forever, we can immediately describe the reservation wage:

- *Permanent Attachment.* A worker who is participating at time t and expects always to do so has a reservation wage independent of his outside income: $z(q,t) = \underline{z}(t)$, the solution to:

$$\underline{z} = a(\underline{z},\lambda) + \frac{\dot{\underline{z}}}{\rho+\delta} \qquad \underline{z} \to z^* \tag{26}$$

- *Permanent Inactivity.* For a worker who is not participating at time t and expects never to do so, $z(q,t) = q$.

As we would expect, workers with higher outside options participate less, have higher reservation wages, and search less intensively:

Proposition 4 *If $q_1 > q_2$:*

(i) if type q_1 participates at time t, then type q_2 also participates;

(ii) $z(q_1,t) \geq z(q_2,t)$ and $s(q_1,t) \leq s(q_2,t)$ with strict inequality unless they are both permanent participants.

Proof: See Appendix A. ■

Thus, workers' behaviour varies with their degree of attachment to the labour market, which is determined by their alternative income, relative to evolving market opportunities. Those with low outside incomes are permanently attached, and their outside income is irrelevant: they all set the same reservation wage and the same search intensity. Workers with higher levels of q expect to leave the market either permanently or temporarily in the future, and the higher their outside income the lower their attachment – that is, the less they expect to participate. Less attached workers have higher reservation wages and lower search intensity. Those with outside income above the steady-state level of market income, z^*, will eventually leave the market for permanent inactivity, with reservation wage equal to outside income.

4.1 Who Participates?

From Proposition 4, since workers with higher outside incomes are less likely to participate at any instant, it follows that there is a critical value $\hat{q}(t)$ such that all workers with $q \leq \hat{q}(t)$ participate at time t. The determination of the critical participant $\hat{q}(t)$ depends in general on the whole evolution of λ. For some precise results we focus on the case when λ evolves monotonically towards a steady state.

An Improving Market

Suppose that $\dot{\lambda} > 0$ for all future t, on a path towards a steady state. Differentiating (26):

$$\begin{aligned} \dot{\underline{z}} &= \dot{a} + \frac{\ddot{\underline{z}}}{\rho + \delta} \quad \text{and} \quad \dot{\underline{z}} \to 0 \\ \Rightarrow \dot{\underline{z}}\,(1 - a_z) &= a_\lambda \dot{\lambda} + \frac{\ddot{\underline{z}}}{\rho + \delta} \end{aligned}$$

from which it follows that the market reservation wage is also rising: $\dot{\underline{z}} > 0$.

One might think that in these circumstances any worker with $q < \underline{z}(t)$ would enter the market and stay forever, but this is not true, for two reasons. First, we know that the reservation wage lies strictly above outside income, unless the worker is forever indifferent to participation. A worker who will stay forever enters with reservation wage $\underline{z}$ when $a(\underline{z}(t), \lambda) = q$, and cannot be forever indifferent since $\underline{z}$ is changing. Hence $\underline{z}(t) > q$ at the moment of entry. He does not enter immediately when the market reservation wage reaches q because he would not want to commit himself to a job little better than his outside income; he prefers to wait until market opportunities improve. Second, although $\dot{\lambda} > 0$, it is possible that $a(\underline{z}(t), \lambda)$ will fall, causing some workers to leave the market temporarily.[11] However, sufficiently close to the steady-state it must be true that $\ddot{\underline{z}} < 0$, in which case (from the equation above) $\dot{a} > 0$, and any worker who is currently participating will never leave. Thus, we can say that:

Lemma 2 *If market opportunities are rising ($\dot{\lambda} > 0$ for all future t) then (i) $\dot{\underline{z}} > 0$ and*

[11] To see how this can happen, suppose that the market is expected to improve rapidly at some time in the future. While that time is distant, workers are happy to participate; as it gets nearer reservation wages rise, and those who have a relatively high outside income may withdraw, saving search costs, and return when the improvement has occurred.

(ii) sufficiently close to the steady state that $a(\underline{z}, \lambda)$ is increasing, all participants have reservation wage $\underline{z}$ and a worker of type q participates if and only if $a(\underline{z}, \lambda) \geq q$.

Proof: If type q leaves the market, the exit conditions with $\dot{\lambda} > 0$ imply $\dot{z}(q) > 0$, so $z > q$, and exit must be temporary. At the time of exit, $a(\underline{z}) > a(z(q)) = q$. When the worker re-enters permanently $a(\underline{z}) = q$, but this cannot happen since $a(\underline{z})$ is rising. Hence there is no exit from the market. Then, since all workers who participate must have reservation wage $\underline{z}$, they do so if and only if $a(\underline{z}, \lambda) \geq q$. ■

A Falling Market

Now consider the opposite case, in which $\dot{\lambda} < 0$ on a path to a steady state. Then, by the same argument as before, it follows that the market reservation wage is falling: $\dot{\underline{z}} < 0$. Clearly in this case workers with low outside incomes – below the steady-state income z^* – will remain in the market forever, with reservation wage $\underline{z}$. However, there will be other workers in the market who will leave at some time in the future, who will therefore have higher reservation wages than the permanent participants. Participation is described by the following result:

Lemma 3 *If market opportunities are falling ($\dot{\lambda} < 0$ for all future t) then (i) $\dot{\underline{z}} < 0$ and (ii) a worker of type q participates if and only if $a(q, \lambda) \geq q$.*

Proof: From the entry conditions, entry cannot occur when $\dot{\lambda} < 0$, so all workers outside the market are permanently inactive, with $z = q$ and hence $a(q, \lambda) < q$. And any worker for whom $a(q, \lambda) < q$ has $a(z(q), \lambda) \leq a(q, \lambda) < q$, so does not participate. ■

Summary

Drawing together the results in this section we can say that close to a steady-state, the marginal market participant $\hat{q}$ is determined by:

$$\hat{q} = \begin{cases} a(\hat{q}, \lambda) & \text{if} \quad \dot{\lambda} < 0 \\ a(z, \lambda) & \text{if} \quad \dot{\lambda} > 0 \end{cases} \tag{27}$$

where $z = \underline{z}(t)$, the reservation wage of a permanently attached worker.

5 Stability of Steady-State Equilibria

In section 3 it was shown that if the matching function has increasing returns at some levels of market activity there may be several steady-state equilibria, although for appropriate incentives for vacancy creation there is an upper bound $\bar{\eta}_0$ on the degree of increasing returns at equilibrium. In this section we will obtain conditions for dynamic stability in the neighbourhood of a steady state.

From section 4 we know that, close to a steady-state equilibrium when market conditions are improving (λ is rising), all participating workers have the same reservation wage and search intensity. So z, u, θ and s satisfy:

$$\begin{aligned} z &= a(z,\lambda) + \frac{\dot{z}}{\rho+\delta} && (28)\\ 0 &= b'(s) + \lambda\beta_1 S(z) && (29)\\ 0 &= -c + \frac{\lambda}{\theta}\beta_2 S(z) && (30)\\ \text{where} \quad \lambda &= \phi(m)\mu(\theta) \quad \text{and} \quad m = su\,\mu(\theta) && (31) \end{aligned}$$

When λ is falling, there are some workers who will leave before the steady state is reached and who therefore have different reservation wages and search intensity. But in the neighbourhood of the steady state, the differences between the average values $\bar{S}(z)$ and $\bar{s}$ and the corresponding values $S(z)$ and s for permanently attached workers are of second order. Ignoring these differences, the equations above are also satisfied in a falling market by u, θ, and the values of z and s for permanently attached workers.

Non-employed workers participate if and only if they have alternative income less than $\hat{q}$ determined by (27). Unemployment is the number of non-employed workers with outside income less than $\hat{q}$:

$$u = N(\hat{q}(t),t) \quad \Rightarrow \quad \dot{u} = n(\hat{q})\dot{\hat{q}} + \left.\frac{\partial N}{\partial t}\right|_{\hat{q}}$$

In the steady state the partial derivative n is discontinuous at z. But in any case it is bounded:

$$0 \leq n(\hat{q},t) \leq L'(\hat{q})$$

since the number of non-employed workers with outside income in any interval cannot exceed the total number of workers in this interval. Evaluating $\partial N/\partial t$ using (16) gives:

$$\dot{u} = n(\hat{q})\dot{\hat{q}} + \delta(L(\hat{q}) - u) - \Phi(m)\pi(z) \qquad (32)$$

In the neighbourhood of a steady state, market variables z, s, θ and u satisfy equations (28) to (30), and (32).

5.1 Local Dynamics

To analyse local dynamics, the system of equations (28) to (30) and (32) can be reduced to a 2-dimensional system in the reservation wage z of (almost all) participating workers, and market activity m. All other variables can be written in terms of z and m:

Lemma 4 *In the neighbourhood of a steady state, equations* (27) *and* (29) *to* (31) *determine s, θ, u and λ as implicit functions of z and m:*

$$\begin{aligned}
&s = \tilde{s}(\theta), && \tilde{s}_\theta > 0; \\
&\theta = \tilde{\theta}(z,m), && \tilde{\theta}_z < 0, \quad \tilde{\theta}_m \overset{\text{sgn}}{=} \eta - 1; \\
&u = \tilde{u}(z,m), && \tilde{u}_z > 0, \quad \tilde{u}_m \overset{\text{sgn}}{=} 1 + \epsilon - \eta(\alpha + \epsilon); \\
&\lambda = \tilde{\lambda}(z,m), && \tilde{\lambda}_z < 0, \quad \tilde{\lambda}_m \overset{\text{sgn}}{=} \eta - 1; \\
&\hat{q} = \tilde{q}(z,m), && \tilde{q}_z < 0, \quad \tilde{q}_m \overset{\text{sgn}}{=} \eta - 1.
\end{aligned}$$

The proof, including expressions for the elasticities, is given in Appendix A. Equations (28) and (32), for the dynamics of z and u, can now be written in terms of z and m:

$$\begin{aligned}
\dot{z} &= f_1(z,m) \\
(\tilde{u}_m - n(\hat{q})\tilde{q}_m)\,\dot{m} &= (n(\hat{q})\tilde{q}_z - \tilde{u}_z)\,\dot{z} + f_2(z,m) \\
\text{where} \quad f_1 &= (\rho + \delta)(z - a(z,\lambda)), \\
f_2 &= \delta(L(\hat{q}) - u) - \Phi(m)\pi(z), \\
\text{and} \quad & \hat{q} = \tilde{q}(z,m), \quad \lambda = \tilde{\lambda}(z,m), \quad u = \tilde{u}(z,m)
\end{aligned}$$

Linearising in the neighbourhood of the steady state, we can see that the stability of the system depends on the matrix:

$$\begin{pmatrix} f_{1z} & f_{1m} \\ \dfrac{f_{2z} + (n\tilde{q}_z - \tilde{u}_z)f_{1z}}{\tilde{u}_m - n\tilde{q}_m} & \dfrac{f_{2m} + (n\tilde{q}_z - \tilde{u}_z)f_{1m}}{\tilde{u}_m - n\tilde{q}_m} \end{pmatrix}$$

in which all functions are evaluated at the steady state (where $\hat{q} = z$). The derivatives of f_1 and f_2 can be evaluated using the elasticities in the proof of Lemma 4; in particular, note that $f_{1z} > 0$ and $f_{1m} \overset{\text{sgn}}{=} 1 - \eta$. The system is saddle-path stable if and only if the determinant:

$$\Delta = \frac{f_{1z}f_{2m} - f_{1m}f_{2z}}{\tilde{u}_m - n\tilde{q}_m}$$

is negative. It can be shown that if $\eta(m) < \bar{\eta}_0$ (the condition for stability with respect to vacancy creation) the denominator of Δ is positive, and the numerator is negative,[12] so the equilibrium is saddle-path stable. Moreover the direction of slope of the saddle path depends on the slope of $f_1 = 0$, and therefore on returns to scale. In summary:

Proposition 5 *A steady-state equilibrium that is stable with respect to vacancy creation is locally saddle-path stable. The saddle path is downward-sloping in m-z space if $\eta < 1$, flat if $\eta = 1$, and upward-sloping if $\eta > 1$.*

PROOF: See Appendix A. ■

5.1.1 Decreasing returns to scale

Figure 2a illustrates dynamic behaviour near an equilibrium where $\eta < 1$.[13] An equilibrium with decreasing returns to scale is always stable, with a downward-sloping saddle path: higher activity is associated with lower reservation wages because with decreasing returns matching is less efficient when there are more agents searching.

Following a shock, both z and m may jump. The expectational variable z can jump to any value, to ensure that the market moves onto the saddle path. Market activity m is constrained because employment adjusts slowly, and this affects the adjustment of unemployment. Some instantaneous adjustment of m takes place, however, since workers and firms can enter or leave the market.

Suppose, for example, that the market is in steady state, and an unexpected productivity shock destroys some jobs so that unemployment and hence market activity are too high. At the equilibrium reservation wage the job-finding rate λ falls. Workers lower their reservation wages instantaneously, and those who have the highest outside income leave the market, reducing u instantaneously. The more elastic is labour supply, the greater this initial jump. But (unless labour supply is perfectly elastic) unemployment is still above its equilibrium level, so firms create more vacancies and activity m is also high. Then market activity falls gradually as unemployed workers find jobs, raising the effectiveness of matching; workers gradually raise their reservation wages, and inactive workers re-enter, until equilibrium is restored. Using Lemma 4 it can be verified that as the market returns to equilibrium along the saddle path:

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[12]The numerator has the same sign as $\zeta_1' - \zeta_2'$; this is not surprising, since $f_1 = 0$ and $f_2 = 0$ are locally identical to $z = \zeta_1$ and $z = \zeta_2$.

[13]Note that the lines drawn are $\dot{z} = f_1 = 0$, and $f_2 = 0$ where $\dot{m} \overset{\text{sgn}}{=} -\dot{z}$, which together determine the slope of the saddle path. The line $\dot{m} = 0$ is not shown; the diagram is drawn for a case in which $\dot{m} = 0$ is positively sloped, but when labour supply is elastic (L' is high) it can slope downwards.

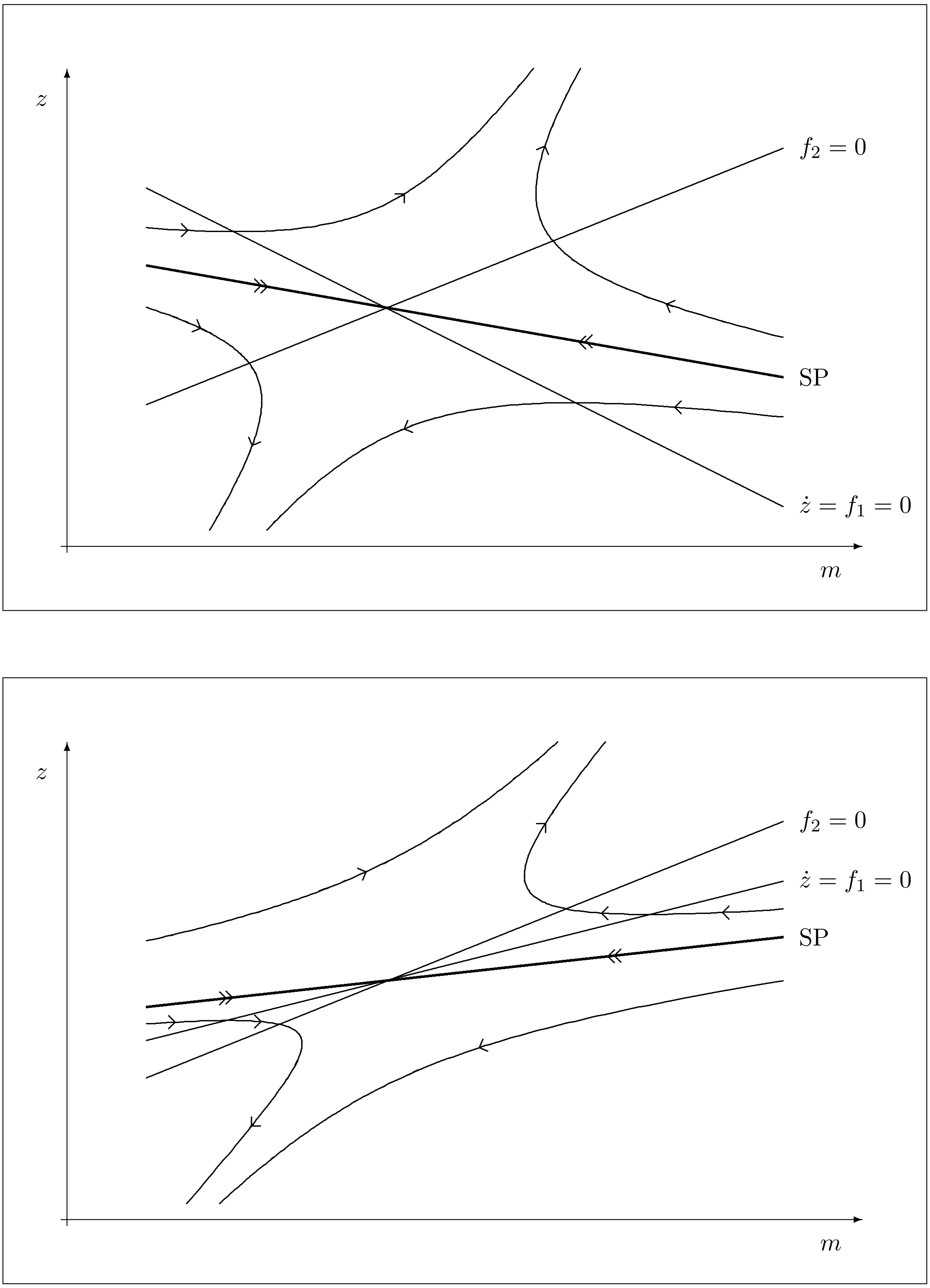

Figure 2: Phase diagrams for (a) $\eta < 1$, (b) $\eta > 1$

- Unemployment u falls; search intensity s rises, and total search su falls.
- Vacancies fall, but market tightness rises.
- Participation $L(\hat{q})$ rises, and the unemployment rate u/L falls.

5.1.2 Constant and increasing returns

If the equilibrium has locally constant returns to scale, the dynamics are much simpler. There is no need for the reservation wage to change while the market remains near the steady state, since the efficiency of matching is not affected by a change in activity level. As in the standard constant returns model, while unemployment and vacancies evolve to their steady-state values, the other variables – market tightness, reservation wage and search intensity – all remain constant at their equilibrium values; with no change in market opportunities, entry and exit of workers does not occur.

An equilibrium with increasing returns to scale is stable (both with respect to vacancies, and dynamically) provided that $\eta < \bar{\eta}_0$ – that is, if the responsiveness of the matching rate to a shock is not too high. The saddle path is upward-sloping. A shock that raises unemployment and market activity leads to a rise in the reservation wage, because matching is more effective; workers take advantage of temporarily more effective matching by raising their reservation wage to obtain more productive jobs, and new workers enter temporarily while the market is operating more effectively. Along the saddle path, z falls with m and u, as unemployed workers find jobs or leave the market.

5.2 Multiple Equilibria and Stability

Bringing together the results of sections 5.1 and 3, we can now describe the properties of the equilibria that can arise in a market where the matching function does not have everywhere constant returns to scale. Numbering them in order of decreasing market size, we can rule out even-numbered equilibria because they are not robust (unstable) with respect to incentives for vacancy creation. Odd-numbered equilibria are stable in this sense provided that returns to scale are not too high and labour supply is not too elastic ($\eta < \bar{\eta}_0$), in which case they are also dynamically saddle-path stable. The equilibrium with the largest market is welfare-superior to others, and is stable if $\eta < \bar{\eta}_0$. Figure 3 shows the same multiple equilibria that were illustrated in Figure 1. The welfare-superior equilibrium (1) has decreasing returns to scale, so it is stable, with a downward-sloping saddle path; (2) is unstable; (3) has increasing returns, so it is stable if and only if $\eta < \bar{\eta}_0$, in which case the saddle path slopes upwards.

5.3 Global Dynamics

With multiple rational expectations equilibria, agents face a severe coordination problem. We have established that steady-state equilibria are locally stable if returns to scale are not too high, but stability of a steady-state equilibrium also requires that all agents expect

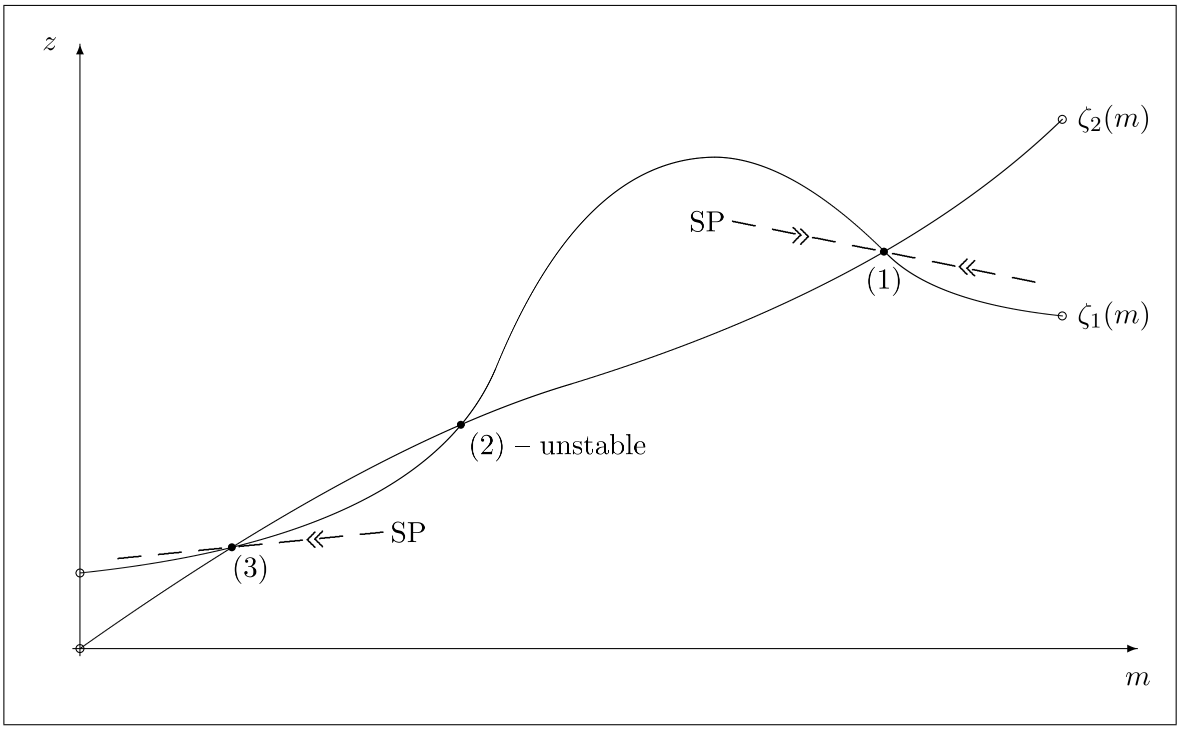

Figure 3: The functions ζ_1 and ζ_2 as in the existence diagram, with multiple equilibria and saddle paths for stable ones

the market to return to this steady state. Where there are several stable steady states, it is possible that any one of them can be reached from a given initial position, if agents believe that it will be. We can demonstrate this property by considering a simple special case of the model.

Suppose that every match has the same productivity $y > 0$, the utility of unemployment is zero, and search intensity is exogenous ($s = 1$). There is a mass normalised to one of workers with outside income zero, who always participate, and an equal mass with outside income $q_H \in (0, y)$. Thus labour supply to the market is:

$$L(q) = \begin{cases} 1 & \text{if} \quad 0 \leq q < q_H \\ 2 & \text{if} \quad q \geq q_H \end{cases}$$

Matches are always consummated and the expected surplus for a worker with reservation income z is $S(z) = y - z$. Suppose that the matching function is:

$$\Phi(m) = \begin{cases} \phi_\ell m & \text{if} \quad m \leq \tilde{m} \\ \phi_h m & \text{otherwise} \end{cases} \qquad \text{where} \quad \phi_\ell < \phi_h$$

Thus the average matching rate is either high or low depending on the level of market activity. It is straightforward to show that for a given value of the average matching rate ϕ, the steady-state equations (17) to (21) have a solution:

$$z = z(\overset{+}{\phi}),\ \theta = \theta(\overset{+}{\phi}),\ \lambda = \phi\mu(\theta),\ u = \frac{L(z)\delta}{\delta + \lambda},\ m = u\mu(\theta)$$

So we have two possible stable steady states (z_i, θ_i, u_i) corresponding to the two levels of the average matching rate ϕ_i; an equilibrium with a high (low) level of ϕ exists if the corresponding activity level m is above (below) $\tilde{m}$. The equilibrium with higher average matching rate has higher income, tightness, participation and activity, but a lower unemployment rate u/L. If:

$$0 < z_\ell < q_H < z_h < y \quad \text{and} \quad \frac{\delta\mu(\theta_\ell)}{\delta + \lambda_\ell} < \tilde{m} < \frac{2\delta\mu(\theta_h)}{\delta + \lambda_h}$$

we have multiple equilibria: a Pareto superior one in which all workers participate, and a second with low activity in which only low-types (with low outside income) participate.

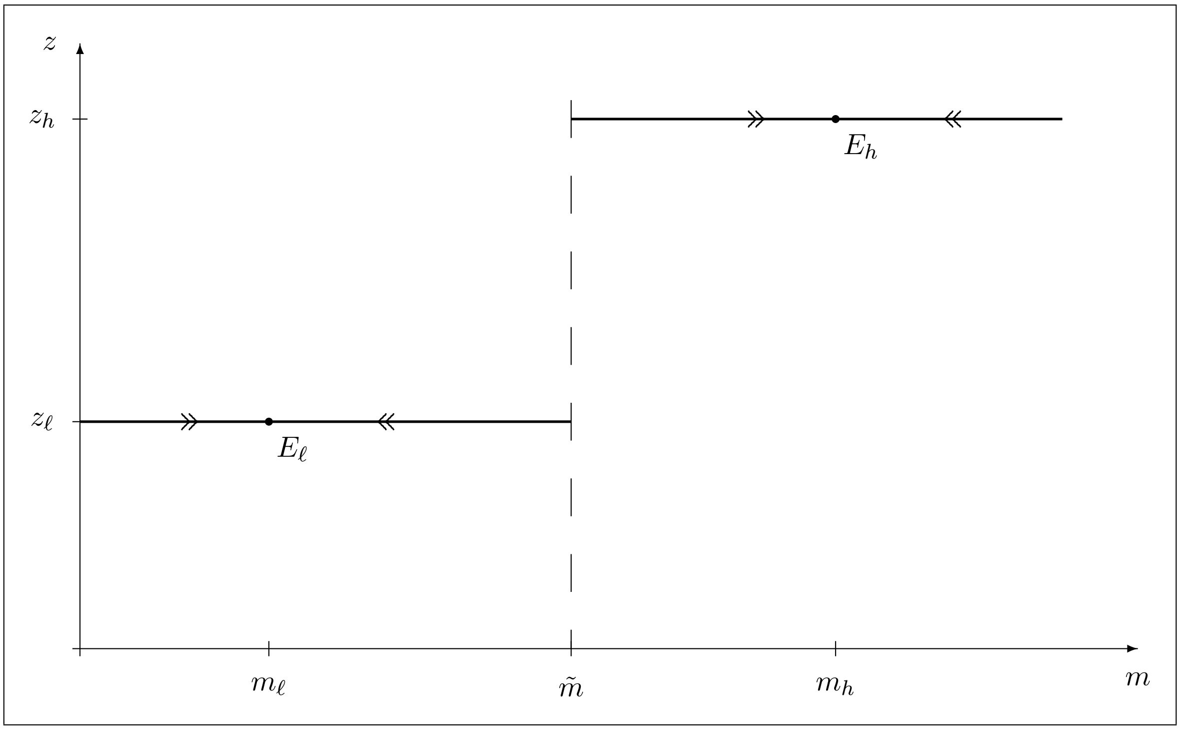

Figure 4: High and low activity equilibria and saddle paths

Since the matching function has constant returns to scale within the regions of high and low activity, the saddle paths for both equilibria are horizontal in m–z space (see Figure 4). The market may be able to jump straight onto either saddle path, with constant worker income and market tightness, and converge to the steady state as employment and hence market activity evolves. Whether this is possible from an arbitrary starting point depends on how many workers are initially employed – which determines whether market activity can instantaneously adjust to a level within the same region as the steady state.

Which equilibrium is reached from any initial position depends entirely on the expectations of workers about the future path of the market. It can converge to either of the two steady states (even if the horizontal saddle path is not reached immediately) and there are many equilibrium paths. And if the expectations of workers change, the market can shift instantaneously from an equilibrium path towards one steady state onto a path to the other. To illustrate the possibilities we describe two cases in more detail.

1. *If the market is at the high (low) steady state, it is possible for it to jump instantaneously to the low (high) saddle path.*
At the high steady state the number of unemployed low-types is lower than u_ℓ. If everyone believes that the market will move to the low equilibrium with $z = z_\ell$ for all future t, then the unemployed high-types immediately leave, firms close down vacancies and market tightness falls, and the market jumps to a point on the low horizontal saddle path with a level of market activity lower than m_ℓ, then converges to the low equilibrium as employment adjusts. Conversely, at the low steady state, all the high-types are inactive, and the number of unemployed low-types is higher than at the high steady state. So the market can instantaneously jump to the high saddle path with activity level above m_h.

2. *A "new" market, starting from zero employment, can converge to either equilibrium.*
The following are equilibrium paths:

a) All workers enter immediately, and the market converges to equilibrium on the high saddle path.

b) Only the low-types enter, and the market converges to the low equilibrium along the low saddle path. This can only happen if ϕ_ℓ is sufficiently low that $\mu(\theta_\ell) \leq \tilde{m}$, so that even when all the low-types are searching the activity level is below $\tilde{m}$.

c) All workers enter, and the average matching rate is initially high, but the market converges to the low equilibrium. Since workers know that the average matching rate will fall their reservation wages are lower than z_h (so market tightness is initially higher than θ_h). Unemployment falls, and eventually all the high-types leave the market together, permanently, and the low saddle path is reached. Exit of the high-types cannot occur until the number of employed low-types is sufficiently high (above $\tilde{e} \equiv 1 - \tilde{m}/\mu(\theta_\ell)$) that market activity below $\tilde{m}$ falls when the high-types leave.

Figure 5 shows the evolution of employment (e_L, e_H) of the two types of worker along these paths, for the case $\mu(\theta_\ell) \leq \tilde{m}$, when all three types of path are possible. Note, however, that the only paths shown are those on which the high types leave the market at most once; there are others on which the market alternates between high and low activity – possibly indefinitely.

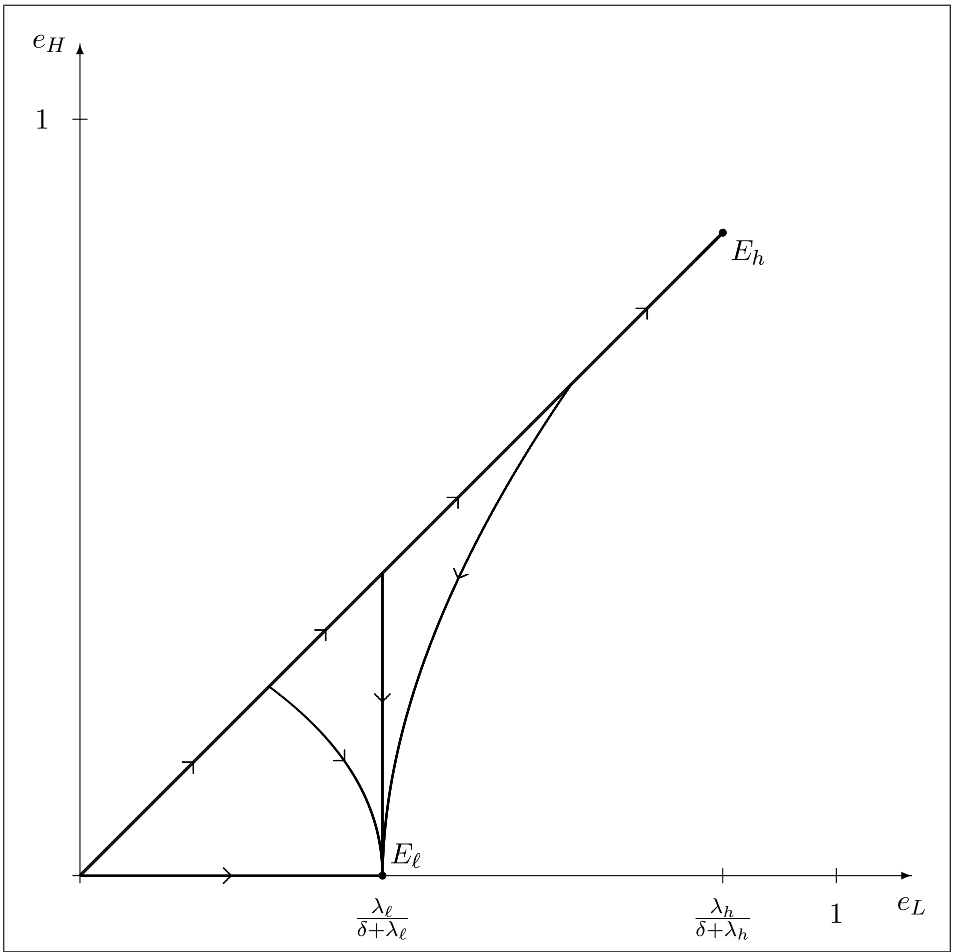

Figure 5: Equilibrium paths for a new market

6 Efficiency

It is well known that with constant returns to scale in matching, the decentralised matching market is efficient (subject to matching frictions) under the Hosios condition: when the surplus shares β_1 and β_2 are equal to the elasticities of matching with respect to unemployment and vacancies. This gives individual agents the right incentives to ensure that search externalities are internalised, and hence maximise total output net of search costs. Thus it is possible, in principle, for a market in which wages are determined by ex-post bargaining between worker and firm to achieve efficiency, although in practice there is no reason to expect the condition to be satisfied.

In this section we determine the social planner's optimum for the matching model with non-constant returns and derive a simple generalisation of the Hosios condition.

6.1 The Social Planner's Problem

Consider a social planner who wishes to maximise the present value of aggregate net income, continuously determining the number of vacancies (or equivalently market tightness), and for each type of worker the participation decision, search intensity, and reservation productivity level for match consummation. The planner is subject to the frictions imposed by the matching technology, which determine how the distribution of workers across the three labour market states evolves over time. The objective is to maximise:

$$\int_0^\infty \exp(-\rho t) \left[Y + \int pb(s)n \, dq - \theta c \int psn \, dq + \int (1-p)qn \, dq \right] dt$$

where Y is the total instantaneous output of current matches. The second and third terms are the utility of unemployed workers, and total vacancy costs, respectively, and the final term is the income of inactive workers. The current states of workers and total output cannot be directly controlled by the planner; from (16) $n(q)$ evolves according to:

$$\dot{n} = \delta(L' - n) - \lambda ps\pi(z)n$$

and the path of current output is given by:

$$\dot{Y} = \lambda \int ps\pi(z) \left[z + h(z) \right] n \, dq - \delta Y$$

6.2 First-order Necessary Conditions for Optimality

Proposition 6 *On a socially optimal dynamic path, participation $p(\cdot)$, reservation productivity $z(\cdot)$, search intensity $s(\cdot)$ and market tightness θ satisfy:*

$$p(q) = \begin{cases} 1 & \textit{if} \quad a^*(z(q), \lambda) \geq q \\ 0 & \textit{otherwise} \end{cases} \tag{33}$$

$$z = \max\{q, a^*(z, \lambda)\} + \frac{\dot{z}}{\rho + \delta} \tag{34}$$

$$\textit{where} \quad a^*(z, \lambda) = \max_s \left\{ b(s) + s\lambda \left(\eta_u S(z) + (1 - \eta_u)[S(z) - \bar{S}] \right) \right\} \tag{35}$$

$$0 = -c + \eta_v \frac{\lambda}{\theta} \bar{S} \tag{36}$$

PROOF: By the Principal of Optimality, the social planner's value function $\Omega(Y, n(\cdot))$ satisfies the Bellman equation:[14]

$$\rho\Omega(Y, n(\cdot)) = \max_{\theta,p(\cdot),z(\cdot),s(\cdot)} \Big\{ Y \;+\; \int pb(s)n\,dq - \theta c \int psn\,dq + \int (1-p)qn\,dq \\ +\; \Omega_Y \dot{Y} + \int \Omega_{n(q)} \dot{n}\,dq \Big\}$$

We can write down first-order conditions with respect to $\theta, p(\cdot), z(\cdot)$ and $s(\cdot)$, together with envelope conditions for Y and $n(\cdot)$. Starting with the first-order condition for $z(\cdot)$ and the envelope condition for Y we have:[15]

$$0 = p(q)\left(\Omega_Y z(q) - \Omega_{n(q)}\right) \tag{37}$$

$$\rho\Omega_Y = 1 - \delta\Omega_Y + \dot{\Omega}_Y \tag{38}$$

Since Ω_Y is bounded,[16] the solutions are:[17]

$$\Omega_Y = \frac{1}{\rho+\delta} \quad \text{and} \quad \Omega_{n(q)} = \frac{z(q)}{\rho+\delta} \tag{39}$$

These are the social marginal values of a unit of output, and an unemployed worker of type q, respectively. The first-order condition with respect to θ is:

$$0 = -\bar{s}uc + \eta_v \frac{\lambda}{\theta} \int ps\pi(z) \left[\Omega_Y[z + h(z)] - \Omega_{n(q)}\right] n\,dq$$

which, using (37) and (15), gives us condition (36).

The envelope condition for $n(\cdot)$ and the first-order condition for $s(\cdot)$ are:

$$\rho\Omega_{n(q)} = (1-p(q))q + p(q)\left\{b(s) - s\theta c + s\lambda[S(z) - \bar{S}] + \eta s\lambda\bar{S}\right\} - \delta\Omega_{n(q)} + \dot{\Omega}_{n(q)}$$

$$0 = b'(s) - \theta c + \lambda[S(z) - \bar{S}] + \eta\lambda\bar{S} \quad \text{when} \quad p(q) \neq 0$$

[14]Since Ω is a function of Y, Ω_Y denotes $\partial\Omega/\partial Y$ as usual; however, Ω is a *functional* of $n(\cdot)$, and we write $\Omega_{n(q)}$ for the partial derivative of Ω with respect to n at q *in the sense of Volterra*. Furthermore, since we are maximising the objective with respect to the *functions* $p(\cdot)$, $z(\cdot)$ and $s(\cdot)$, the first-order conditions for them will also be in terms of Volterra derivatives. (See Appendix B.)

[15]Observe that $\frac{\partial}{\partial Y}\Omega_Y \dot{Y} + \frac{\partial}{\partial Y}\int \Omega_{n(q)}\dot{n}\,dq = \Omega_{YY}\dot{Y} + \int \Omega_{Yn(q)}\dot{n}\,dq = \dot{\Omega}_Y$.

[16]The additional value, $d\Omega$, of an increase in output, dY, lies between 0 and dY/ρ, i.e. what the SP gets if the match that produced the extra output is destroyed immediately, and what she gets if it is never destroyed; so $0 \le \Omega_Y \le 1/\rho$.

[17]The reservation value $z(q)$ is indeterminate when $p(q) = 0$; however in that case we can simply set it equal to $(\rho+\delta)\Omega_{n(q)}$.

and eliminating $\Omega_{n(q)}$ and θ using (39) and (36), we obtain:

$$z = (1-p)\,q + p\,a^*(z,\lambda) + \frac{\dot{z}}{\rho+\delta} \tag{40}$$

where $a^*(z,\lambda)$ is given by (35).

Finally, consider the choice of $p(\cdot)$. The derivative of the maximand in the Bellman equation with respect to p at q, evaluated using the first-order conditions above, is:

$$n(q)\left(a^*(z(q),\lambda) - q\right)$$

Hence the social planner chooses $p(q)$ according to (33), and equation (40) for $z(q)$ can be written in the form (34). ■

6.3 The Steady State and Second-order Conditions

Inspection of the optimality conditions (33) to (36) shows, by the same argument as for the decentralised equilibrium, that in a steady state all participating workers have the same reservation value z and search intensity s. Thus in an optimal steady state z, s, θ and u satisfy:

$$\begin{aligned} z &= b(s) + s\lambda\eta_u S(z) && (41)\\ 0 &= b'(s) + \lambda\eta_u S(z) && (42)\\ 0 &= -c + \frac{\lambda}{\theta}\eta_v S(z) && (43)\\ \Phi(m)\pi(z) &= \delta(L(z) - u) && (44) \end{aligned}$$

These equations are clearly very similar to the equations for the decentralised steady state: they differ only in that the surplus shares are replaced by the elasticities of matching with respect to unemployment and vacancies. It appears that a generalisation of the Hosios condition holds. But we should first consider whether a solution to first-order conditions exists, and whether the first-order conditions are sufficient for a social optimum.

6.3.1 Existence

It can be shown in the same way as for the decentralised case that a solution to the social planner's first-order conditions exists provided that $M_v(1,0)$ is sufficiently large. The only substantive difference is that in the analogue of Lemma 1 z depends on market size through the marginal matching rate $\Phi'(m)$, rather than the average $\Phi(m)/m$. It is also useful to note that the crossing condition in this analysis implies that there can be at most one solution in any concave region of the matching function $\Phi(m)$.

6.3.2 Second-order Conditions

As well as satisfying the first-order conditions, an optimal choice of θ, $p(\cdot)$, $z(\cdot)$ and $s(\cdot)$ must correspond to a local maximum of instantaneous welfare.

With δf denoting a differential change in a function f, we write $\mathrm{E}[\delta f]$ for $\frac{1}{u}\int pn\, \delta f(q)\, dq$ and similarly for $\mathrm{Var}[\delta f]$. We decompose the relevant differential changes in θ, $p(\cdot)$ (whose changes are felt via changes in u), $z(\cdot)$ and $s(\cdot)$ into the vectors d_E and d_V, given by:[18]

$$d_E{}' = \begin{pmatrix} d\theta & \mathrm{E}[\delta s] & du & \mathrm{E}[\delta z] \end{pmatrix} \quad \text{and} \quad d_V{}' = \begin{pmatrix} \mathrm{Var}[\delta s] & \mathrm{Var}[\delta z] \end{pmatrix}$$

The second-order differential change in the objective Ω, at a steady-state solution of the first-order conditions, is then directly proportional to $d_E{}'\, H_E\, d_E - d_V{}'\, H_V\, d_V$, where the symmetric matrices H_E and H_V are given by:

$$\begin{pmatrix} \frac{\alpha}{\theta^2}\left(\alpha\frac{m\Phi''}{\Phi'} - \frac{1-\alpha}{\sigma}\right) & \frac{\alpha}{\theta s}\frac{m\Phi''}{\Phi'} & \frac{\alpha}{\theta u}\frac{m\Phi''}{\Phi'} & 0 \\ \cdot & \frac{1}{s^2}\left(\frac{m\Phi''}{\Phi'} - \frac{1-\alpha}{\epsilon}\right) & \frac{1}{su}\frac{m\Phi''}{\Phi'} & 0 \\ \cdot & \cdot & \frac{1}{u^2}\frac{m\Phi''}{\Phi'} & 0 \\ \cdot & \cdot & \cdot & \frac{1}{\eta}\frac{\pi'}{\pi h} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{1}{s^2}\left(\frac{1-\alpha}{\epsilon}\right) & 0 \\ \cdot & \frac{1}{\eta}\left|\frac{\pi'}{\pi h}\right| \end{pmatrix}$$

respectively, and σ is the elasticity of substitution between unemployment and vacancies. If H_E is negative definite, then this change is negative and we have a local maximum. On the other hand, if H_E is not negative definite, then there is a vector of changes that makes $d_E{}'\, H_E\, d_E$ non-negative and at the same time has $d_V = 0$, and so the second-order effect on Ω is non-negative.

If $L'(z) > 0$, it can be verified that H_E is negative definite if and only if $\Phi''(m) < 0$, giving necessary and sufficient conditions for a strict local maximum.[19] With an upward-sloping supply of workers to the market, it can never be optimal to be on a convex part of the matching function – since by bringing another worker into the market the social planner raises activity and the marginal matching rate, and hence increases income z for

[18] $du = dN(\hat{q})$

[19] If $\Phi'' = 0$, we have a maximum with respect to s, θ, z, but the second-order effect of changing u is zero.

all the workers in the market.

When labour supply is inelastic, $du = 0$. In this special case, the relevant matrix (the sub-matrix corresponding to changes in θ, s and z) is negative definite if and only if $m\Phi''/\Phi' < (1-\alpha)/(\epsilon+\alpha\sigma)$. This condition can be intepreted as saying that if, at an optimum, raising search activity would increase the matching rate, the effect must be small enough that the gain for workers is outweighed by the increased search costs.

6.3.3 Steady State Welfare

Evaluating total welfare at a steady-state solution of the first-order conditions gives:

$$\rho\Omega = \int \max\{z, q\}\, dL(q) + (L-u)h(z)\left(1 - \eta\frac{\delta}{\rho+\delta}\right)$$

Equivalently the *net social surplus* from the market – the difference between $\rho\Omega$ and total income when all workers are inactive – is:

$$\Delta(\rho\Omega) = \int^{z} (z-q)\, dL(q) + (L(z)-u)h(z)\left(1 - \eta\frac{\delta}{\rho+\delta}\right) \tag{45}$$

It follows that if there is a solution to the first-order conditions in a region of non-increasing returns, the social planner can achieve a positive social surplus. A positive surplus can occur at a solution with increasing returns, but if $\eta > 1 + \rho/\delta$ this requires the total rents to outweigh losses from employment. We discuss this case further below.

6.4 Necessary and Sufficient Conditions for Optimality

Putting together the results above we can now state our main efficiency result:

Proposition 7 *A steady-state allocation is a local welfare optimum if and only if*

(i) z, s, θ and u satisfy the social planner's first-order conditions (34) *to* (36)*, and*

(ii) $\Phi''(m) < 0$ (or $\frac{m\Phi''}{\Phi'} < \frac{1-\alpha}{\epsilon+\alpha\sigma}$ if labour supply is inelastic).

Corollary 1 *(Generalised Hosios Condition) A decentralised steady state equilibrium with variable labour supply is a local welfare optimum if and only if $\Phi''(m) < 0$ and $\beta_1 = \eta_u$, $\beta_2 = \eta_v$.*

We can also conclude that if the matching function $\Phi(m)$ has only one concave region, a local optimum is the unique global optimum provided that the net surplus (45) is positive.

Corollary 1 means that efficiency can in principle be achieved by a policy maker who sets the surplus shares of workers and firms, as in the standard constant returns case.

The difference is that with non-constant returns a tax or subsidy is required: the surplus shares do not sum to one. If the optimum has decreasing returns to scale, the surplus must be taxed in order to give agents appropriate search incentives. If the optimum has increasing returns the policy maker must use a subsidy. In particular, we can see from (45) that when $\eta > 1 + \rho/\delta$ employment is effectively loss-making, but the subsidy could nevertheless be financed by a lump-sum tax on workers, who earn rents in equilibrium.

This generalisation of the Hosios condition is quite intuitive. With constant returns to scale in matching, search externalities arise when the relative number of agents – market tightness – is suboptimal, and efficiency can be achieved by manipulating the relative return to search. With non-constant returns there are also market size effects: additional search externalities that arise when the absolute number of agents is not optimal. To achieve efficiency in this case, it is necessary to control both the relative and absolute return to search.

6.4.1 Efficiency away from the steady state

With everywhere-constant returns, the Hosios condition is necessary and sufficient for efficiency not only at the steady-state, but everywhere on the equilibrium path. With a more general matching function this is not the case. Comparing the social planner's first-order conditions in Proposition 6 with the dynamic equations for individuals in section 2.4, we can see that, while setting $\beta_2 = \eta_v$ allows firms to behave optimally, $\beta_1 = \eta_u$ is not sufficient for optimal worker behaviour when workers are heterogeneous. There is an additional external effect from a worker who has a different reservation wage from the average in the market. His reservation wage changes the average surplus $\bar{S}$; so it changes the incentives facing firms, affecting market tightness and hence the matching rate for other workers. For example, consider a worker who has high enough outside income that he will leave the market in future. His reservation wage is above average, so his expected surplus is below average. From (35) we can see that the social planner would lower his reservation wage, to mitigate his negative effect on market tightness and the aggregate matching rate. This result is similar to that of Shimer and Smith (2001) who show that the Hosios condition is not sufficient for efficiency with ex-ante heterogeneous agents who have different reservation productivities.

6.5 Interpretation: Search Externalities

Our efficiency analysis, as well as providing a neat generalisation of the Hosios condition, gives some insight into the different sources of inefficiency in random matching models.

Agents' decisions to search affect the returns to other agents by changing *market tightness* or *market size* (both of which can affect individual matching rates) or the *expected surplus* from a match. In a model with constant returns and homogeneous agents only the first of these has an external effect, and it can be offset by an appropriate sharing rule. Non-constant returns introduces a market size externality, which can be internalised using a uniform employment tax or subsidy. Heterogeneity of reservation values means that an agent's decision to search changes the expected surplus for agents on the other side of the market; this externality could be offset only with agent-specific taxes or subsidies.

Concluding Remarks

Empirical evidence suggests that labour markets are not always characterised by constant returns to scale in matching, and hence that market size matters. The generalisation of the standard labour market matching model developed here allows us to describe the implications of market size effects. Matching markets can have stable decentralised equilibria with either decreasing or increasing returns. When there are multiple equilibria, larger markets deliver higher welfare, but pessimistic expectations may nevertheless lead to a low activity equilibrium.

The key difference from the constant-returns case is that the private returns to search are determined by the average matching rate, which increases or decreases with market size depending on returns to scale. A deterministic model cannot provide a full account of the business cycle, but the non-constant average matching rate induces plausible dynamic variation in labour force participation and search intensity that is absent from the standard model, suggesting that introducing a market size effect into an empirical matching model would allow the incorporation of separate labour force and unemployment dynamics.

Appendix

A Various Proofs

Proof of Lemma 1: From (17) and (18), search intensity s satisfies $sb'(s) + z - b(s) = 0$ which determines s_1, with $s_1(0) = 0$ and $\frac{z}{s}\frac{ds_1}{dz} = \epsilon$. Then from (18) and (19): $\beta_1 c\theta + \beta_2 b'(s) = 0$. Since s is a function of z, this determines $\theta_1(z)$, with $\theta_1(0) = 0$ and $\frac{z}{\theta}\frac{d\theta_1}{dz} = 1$. Substituting $s = s_1(z)$ and $\theta = \theta_1(z)$, equation (19) can be written in terms of z and m only:

$$\phi(m) = \frac{c}{\beta_2 S(z)} \frac{\theta_1(z)}{\mu(\theta_1(z))} \tag{A.1}$$

The right-hand side is a continuous, differentiable and increasing function of z, with limits $\phi_0 \equiv c/\beta_2 S(0)\mu'(0)$ at 0 and ∞ at $\bar{y}$. Hence $\forall m$ such that $\phi(m) \geq \phi_0$ this equation has a solution $z = \zeta_1(m) \in [0, \bar{y})$ as required. Differentiating (A.1):

$$\frac{m}{z}\frac{d\zeta_1}{dm}\left(\frac{z}{h} + \frac{1-\alpha}{1+\epsilon}\right) = \eta(m) - 1 \tag{A.2}$$

Now consider the flow condition (20). Substituting $u = m/s\mu$ and using $s_1(z)$ and $\theta_1(z)$:

$$\Big(\delta L(z) - \Phi(m)\pi(z)\Big)\, s_1(z)\mu(\theta_1(z)) = \delta m \tag{A.3}$$

This determines $z = \zeta_2(m)$, with $\zeta_2(0) = 0$, and derivative:

$$\frac{m}{z}\frac{d\zeta_2}{dm}\left(\gamma L + u\frac{\alpha+\epsilon}{1+\epsilon} - (L-u)\frac{z\pi'}{\pi}\right) = u + (L-u)\eta \tag{A.4}$$

$\zeta_2(m)$ is strictly increasing, reaching a maximum value $\bar{y}$ at a finite value $\bar{m}$. ∎

Proof of Proposition 1: An equilibrium is a solution of $\zeta_2(m) = \zeta_1(m)$. If we define $\zeta_1(m) = 0$ when $\phi(m) < \phi_0$, then both ζ_2 and ζ_1 are continuous on $[0, \bar{m}]$, and $\zeta_1(\bar{m}) < \zeta_2(\bar{m}) = \bar{y}$, so a sufficient condition for a solution to exist is $\zeta_1(0) > \zeta_2(0) = 0$; that is, the average matching rate at 0 is greater than ϕ_0:

$$\begin{aligned} \lim_{m\to 0}\phi(m) &> \phi_0 \\ \Rightarrow M_v(1,0) = \Phi'(0)\mu'(0) &> \frac{c}{\beta_2 S(0)} \end{aligned}$$

If this condition holds, we have at least one solution (z, m) with $z > 0$ and $m > 0$; corresponding equilibrium values of the other variables are given by: $s = s_1(z)$, $\theta = \theta_1(z)$

and $u = m/s\mu(\theta)$. At any equilibrium, from (A.4) and (A.2):

$$\begin{aligned}\frac{d\zeta_2}{dm} - \frac{d\zeta_1}{dm} &\overset{\text{sgn}}{=} u\left(\frac{z}{h} + 1 + \gamma - \eta\left(\gamma + \frac{\alpha+\epsilon}{1+\epsilon}\right)\right) \\ &\quad + (L-u)\left(\eta\left(\frac{z}{h} + \frac{1-\alpha}{1+\epsilon}\right) + (1-\eta)\left(\gamma - \frac{z\pi'}{\pi}\right)\right) \\ &\overset{\text{sgn}}{=} L\left(\frac{z}{h} + 1 + \gamma - \eta\left(\gamma + \frac{\alpha+\epsilon}{1+\epsilon}\right)\right) + (L-u)(\eta-1)\left(1 - \frac{zh'}{h}\right)\end{aligned}$$

Since $zh'/h < 1$ by assumption, this is positive if $\eta(m) \leq \eta_0$. If this condition holds for all m, then ζ_1 and ζ_2 can only cross once, and the equilibrium is unique. ■

Proof of Proposition 3: Most of the argument is given in the text; it remains to prove that $\Phi(m)\pi(z)h(z)$ is increasing in m between equilibria. From the proof of Proposition 1, equilibrium values of z and m satisfy $z = \zeta_1(m)$ and:

$$\frac{d}{dm}(\Phi(m)\pi(z)h(z)) \overset{\text{sgn}}{=} \frac{m\Phi'}{\Phi} + m\frac{(\pi h)'}{\pi h}\frac{d\zeta_1}{dm} = \eta(m) - \frac{m}{h}\frac{d\zeta_1}{dm}$$

which is strictly positive from (A.2). ■

Proof of Proposition 4: From (23):

$$z(q_1) - z(q_2) = K(t) + \tfrac{1}{\rho+\delta}(\dot{z}(q_1) - \dot{z}(q_2)) \qquad \text{(A.5)}$$

where $K \equiv \max\{q_1, a(z(q_1))\} - \max\{q_2, a(z(q_2))\}$.

1. Suppose that $z(q_1) < z(q_2)$ for some t. Then $K > 0$, so $\dot{z}(q_1) - \dot{z}(q_2) < 0$, and $z(q_1) < z(q_2)$ for all future t. But this is impossible since from (24) $z(q_1) \geq z(q_2)$ in the limit. Hence we must have $z(q_1) \geq z(q_2)$ for all t.

2. If worker q_1 is participating at time t, $q_1 \leq a(z(q_1))$. Then from 1 (above), $a(z(q_2)) \geq a(z(q_1)) \geq q_1 > q_2$, so q_2 strictly prefers to participate.

3. Compare $z(q_1)$ and $z(q_2)$ at time t_0 in the following cases:

   (a) $a(z(q_1)) < q_1$ in an interval $[t_0, t_1)$ (so q_1 does not participate).
   If $z(q_1) = z(q_2)$ at t_0, $K(t_0) = q_1 - \max\{q_2, a(z(q_1))\} > 0$ so $\dot{z}(q_1) - \dot{z}(q_2) < 0$. Then $z(q_1) < z(q_2)$ immediately after t_0, which is impossible. Hence $z(q_1) > z(q_2)$ at t_0.

   (b) $a(z(q_1, t_0)) = q_1$, and $\dot{a} < 0$ (so q_1 exits at t_0).
   From 2 (above), q_2 strictly prefers to participate at t_0. Then $K(t) = q_1 - a(z(q_2))$ for some interval $t \in [t_0, t_1)$. If $z(q_1) = z(q_2)$ at t_0, $K(t_0) = 0$ and

$\dot{z}(q_1) = \dot{z}(q_2)$. Since $a(z(q_1))$ is strictly decreasing at t_0, so is $a(z(q_2))$, and $\dot{K}(t_0) > 0$. Differentiating (A.5), the second derivative of $z(q_1) - z(q_2)$ is strictly negative at t_0. Again this implies $z(q_1) < z(q_2)$ after t_0, which is impossible, so $z(q_1) > z(q_2)$ at t_0.

(c) $a(z(q_1)) \geq q_1$ in an interval $[t_0, t_1)$.
Then both q_1 and q_2 participate, and $z(q_1)$ and $z(q_2)$ satisfy the same equation $z = a(z,t) + \frac{1}{\rho+\delta}\dot{z}$ in this interval, which has a unique solution through any point (z,t). If q_1 will exit in future, $z(q_1) > z(q_2)$ at that time, and hence $z(q_1) > z(q_2)$, at t_0. Otherwise both participate permanently with $z(q_1) = z(q_2) = \underline{z}$ for all $t \geq t_0$.

Finally, differentiating the first-order condition for search intensity, $b'(s) + \lambda\beta_1 S(z) = 0$, shows that s is a strictly decreasing function of z. ■

Proof of Lemma 4: Eliminating λ and $S(z)$ from (29) and (30) $\Rightarrow$

$$\begin{aligned} s &= \tilde{s}(\theta), \quad \frac{s}{\theta}\frac{d\tilde{s}}{d\theta} = \epsilon; \\ \theta &= \tilde{\theta}(z,m), \quad \frac{h(z)\tilde{\theta}_z}{\theta} = \frac{-1}{1-\alpha}, \quad \frac{m\tilde{\theta}_m}{\theta} = \frac{\eta-1}{1-\alpha}. \end{aligned}$$

Then from $u = m/s\mu(\theta)$ and $\lambda = \mu(\theta)\phi(m)$:

$$\begin{aligned} u &= \tilde{u}(z,m), \quad \frac{h(z)\tilde{u}_z}{u} = \frac{\epsilon+\alpha}{1-\alpha}, \quad \frac{m\tilde{u}_m}{u} = \frac{1+\epsilon-\eta(\alpha+\epsilon)}{1-\alpha}; \\ \lambda &= \tilde{\lambda}(z,m), \quad \frac{h(z)\tilde{\lambda}_z}{\lambda} = \frac{-\alpha}{1-\alpha}, \quad \frac{m\tilde{\lambda}_m}{\lambda} = \frac{\eta-1}{1-\alpha} \end{aligned}$$

Finally, using (27): $\hat{q} = \tilde{q}(z,m)$

where in a rising market: $\tilde{q}_z = a_z + a_\lambda\tilde{\lambda}_z, \ \tilde{q}_m = a_\lambda\tilde{\lambda}_m$

and in a falling market: $(1-a_z)\,\tilde{q}_z = a_\lambda\tilde{\lambda}_z, \ (1-a_z)\,\tilde{q}_m = a_\lambda\tilde{\lambda}_m$

which can be evaluated using the results above together with, from (10):

$$\frac{\lambda a_\lambda}{a} = 1+\epsilon \quad \text{and} \quad \frac{h(z)a_z}{a} = -(1+\epsilon)$$

■

Proof of Proposition 5: If the equilibrium is stable with respect to vacancies, $\eta(m) < \bar{\eta}_0$. To determine the sign of Δ, the derivatives of f_1 and f_2 are:

$$f_{1z} = (\rho+\delta)\left(1 - a_z - a_\lambda\tilde{\lambda}_z\right) \qquad f_{1m} = -(\rho+\delta)a_\lambda\tilde{\lambda}_m$$

$$f_{2z} = \delta(L'(z)\tilde{q}_z - \tilde{u}_z) - \Phi(m)\pi'(z) \qquad\qquad f_{2m} = \delta(L'(z)\tilde{q}_m - \tilde{u}_m) - \Phi'(m)\pi(z)$$

which can be evaluated using the elasticities in the proof of Lemma 4. After a little rearrangement the numerator of Δ can be written:

$$f_{1z}f_{2m} - f_{1m}f_{2z} = f_{1m}\Phi\pi' - f_{1z}\Phi'\pi - \delta\left[f_{1m}L' + f_{1z}\tilde{u}_m - f_{1m}\tilde{u}_z\right]$$

(This expression takes the same form irrespective of whether λ rises or falls towards the steady state.) Substituting the expressions above gives, after some manipulation:

$$\delta\left[f_{1m}L' + f_{1z}\tilde{u}_m - f_{1m}\tilde{u}_z\right] = \frac{(\rho+\delta)(1+\epsilon)}{(1-\alpha)m}u\left(\frac{z}{h} + 1 + \gamma - \eta\left(\gamma + \frac{\alpha+\epsilon}{1+\epsilon}\right)\right)$$
$$f_{1m}\Phi\pi' - f_{1z}\Phi'\pi = -\frac{(\rho+\delta)(1+\epsilon)}{(1-\alpha)m}(L-u)\left(\eta\left(\frac{z}{h} + \frac{1-\alpha}{1+\epsilon}\right) + (1-\eta)\left(\gamma - \frac{z\pi'}{\pi}\right)\right)$$

so from the proof of Proposition 1 $f_{1z}f_{2m} - f_{1m}f_{2z} \stackrel{\text{sgn}}{=} \zeta_1' - \zeta_2'$, which is negative when $\eta(m) < \bar{\eta}_0$. In a rising market the denominator of Δ is:

$$\tilde{u}_m - n\tilde{q}_m = \frac{1+\epsilon}{1-\alpha}\frac{u}{m}\left(1 + \frac{n(z)z}{u} - \eta\left(\frac{\epsilon+\alpha}{1+\epsilon} + \frac{n(z)z}{u}\right)\right)$$

which is positive when $\eta < \bar{\eta}_0$. It is straightforward to show that this is also true in a falling market. Hence $\Delta < 0$ and the equilibrium is saddle-path stable. The slope of the saddle-path is determined by the sign of f_{1m},[20] which is equal to the sign of $1-\eta$. So it is downward-sloping, upward-sloping or flat when η is, respectively, less than, greater than, or equal to 1. ■

[20] In general the saddle-path of a system: $\begin{pmatrix} \dot{y} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}$ with $a > 0$ is downward-sloping if $b > 0$, flat if $b = 0$, and upward-sloping if $b < 0$.

B Notes on Volterra derivatives

Let f be a function and let I be a functional that takes f as its argument. We wish to determine the differential change in $I[f]$ resulting from a differential change in f. Following Volterra (1930), we fix a point q_1 in the domain of f and define a particular small change in f, denoted by Δf, as follows: (a) $\Delta f(q) \neq 0$ iff $q \in (q_1 - w/2, q_1 + w/2)$; (b) either $\Delta f \geq 0$ or $\Delta f \leq 0$; (c) $|\Delta f(q)| \leq h$. Consider the fraction

$$\frac{I[f + \Delta f] - I[f]}{\int \Delta f(q)\, dq}$$

as $w \downarrow 0, h \downarrow 0$. If the limit exists (uniformly in f and in q_1), then it is the first derivative of I w.r.t. f at q_1 in the sense of Volterra, and is variously denoted by $I'[f(\cdot); q_1]$, $I'[f; q_1]$, or $[\delta I/\delta f]_{q_1}$.

With δf denoting a general differential change in the function f, δI denotes the total differential of I (w.r.t. f), and is given by

$$\delta I = \int I'[f; q]\, \delta f(q)\, dq\,,$$

which is seen as the continuous analogue of $dI = \sum_{i=1}^{k} \frac{\partial I}{\partial f_i}\, df_i$ when I is a function of the k-vector $(f_1, \ldots, f_k)$.

Partial derivatives of a more general functional are defined similarly.[21] Also, since I' is again a functional, second- and higher order derivatives can be defined in the obvious way, and are denoted by $I''[f; q_1, q_2]$, etc., the total second-order differential of I (w.r.t. f) being given by

$$\delta^2 I = \iint I''[f; q_1, q_2]\, \delta f(q_1)\, \delta f(q_2)\, dq_1\, dq_2\,.$$

Of particular interest here, consider I being defined by an integral: $I[f] \equiv \int \psi(f(q))\, dq$ for some function ψ. In this case, $I'[f; q]$ is simply given by $\psi'(f(q))$, and so optimisation of I w.r.t. f is in effect pointwise. Further, when I' is *not* defined by an integral, the total second-order differential of I (w.r.t. f) is given by

$$\delta^2 I = \int \psi''(f(q))\, (\delta f(q))^2\, dq$$

which is the continuous analogue of cross-partial derivatives being zero.

[21] There seems to be no agreed notation for partial derivatives; when I is a functional of f and g we use $I_f[f, g; q]$ or, more simply, $I_{f(q)}$.

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