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UNBALANCED GROWTH

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Abstract

This paper considers a model of endogenous growth with two sectors. It shows that it may be desirable to concentrate research in one sector, and so have unbalanced growth, but that the pressure of competition in research may cause the private sector to spread research too widely, and so have growth that is too balanced. It is also shown that even if private agents concentrate research in one sector, they may do so in the wrong one.

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1. Introduction

Models of economic growth often have only one sector or focus on growth paths which are balanced in the sense that all sectors grow at equal rates. In practice, growth is far from balanced. For example in the Industrial Revolution cotton and to a lesser extent iron production grew at a rate far outstripping the growth of the rest of the economy.[1] One question of interest is whether such unbalanced growth is desirable. In the 1950's it was argued that it was not on the grounds that unbalanced growth would lead to supply shortages that would slow down overall economic growth (see Scitovsky (1987) for a survey). At a more formal level, support for this notion comes from the so-called turnpike theorems which provide conditions under which optimal paths converge to balanced expansion paths (see McKenzie (1986)).

On the other hand, Hirschman (1957) argued that unbalanced growth might actually be desirable as concentrating on certain sectors might actually increase the rate of technical change.[2] Such a notion is hard to assess in the usual neo-classical model of economic growth since the rate of technical change is taken to be exogenous. More recent models of growth which treat technical change as endogenous do, however, offer the prospect of examining such an argument. This paper aims to do so in a simple model.

Existing models of endogenous growth, such as Romer (1990), are one-sector in the sense that there is only one final output.[3] They focus on whether overall technical progress, achieved via deliberate research or learning-by-doing, is likely to be socially optimal. With more than one final output, there is the additional question of whether the private sector will achieve the correct balance of technical progress between sectors.

The paper formulates a simple two-sector version of the kind of endogenous growth model studied by Romer (1990) and Grossman and Helpman (1991a). It shows that even if the sectors are completely symmetric it may be desirable to have unbalanced growth on account of the increasing returns created by the research process. The private equilibrium may, in contrast, tend to achieve balanced growth. Unfettered private competition may therefore lead to growth that is insufficiently unbalanced in contrast to the usual supposition.

To understand this, suppose that the government wishes to have research concentrated in one sector. The private incentive to do research in a sector depends on the reward to making an invention in that sector. This in turn depends on the flow of profits an inventor receives while his patent lasts and the expected duration of his patent. If a large number of other researchers are working in a sector, then that sector may tend to expand rapidly and so increase the incentive to work in that sector. For example, rapid technical innovation in microchips may reduce the price of hardware significantly and so increase sales of computers and expand the market for new software. On the other hand, a large number of other researchers in the same sector may also mean that there is a higher likelihood of new inventions being rapidly displaced. If the first factor dominates, then private equilibrium will tend to be concentrated. If the second does, then private researchers will prefer to to work in sectors where there is less competitive pressure. The paper shows that even if from a social point of view it would be desirable to have research and development concentrated, the second effect may mean that the private sector will fail to achieve this.

Whether this is of practical importance is of course hard to assess. It does however provide a caution against uncritical support for government intervention to ensure balance in the process of growth. Indeed, it might suggest that concentration would be desirable. For example, if, before they make specific investments in human capital, researchers could be allocated to either computer research or biotechnology, then there might be a case for subsidising one sector in order to achieve greater concentration. Similarly, governments which have pursued policies of specialisation in development may be on the right lines rather than those attempting to achieve balance via planning.

This argument contrasts with the argument for a 'big push'of simultaneous industrialisation across many sectors advanced by Rosenstein-Rodan (1943) and formalised by Murphy, Shleifer and Vishny (1989). In the latter paper, there are multiple equilibria on account of increasing returns and to escape from the low-level equilibrium requires co-ordinated investment across firms. Which policy is appropriate depends on whether such demand externalities are most important or whether technological externalities are the key to development. In the latter case, concentration of effort may be a virtue.

Section 2 lays out the basic model and section 3 analyses the public optimum. The model is a simple two-sector version of the basic structure found in recent models such as Romer (1990) or Grossman and Helpman (1991a). Technical progress in each sector depends on the stock of knowledge in that sector and the amount of human capital devoted to research. It is shown that even if consumer preferences are symmetric between the goods, unbalanced growth may be desirable. Section 4 describes the underlying private economy and section 5 analyses the private equilibrium. Private sector research and development is modelled following Grossman and Helpman (1991a) (which in turn builds on Aghion and Howitt (1992)). It shows, for the reasons outlined above, that the private equilibrium may be insufficiently balanced. It also shows that even if the private sector achieves unbalanced growth it may specialise on the wrong sector: if it is desirable to be in a sector with high research intensity then researchers may choose to be in a sector simply because everyone else has made the same choice. Section 6 discusses the implications of alternative specifications of the relationship between human capital and technical progress and shows that if there are increasing returns to human capital, the public optimum may involve alternating between sectors in research. Section 7 concludes briefly.

2. The Model

The basic structure of the model is that found in Romer (1990) or Grossman and Helpman (1991a). There are two sectors 1 and 2. The stock of knowledge in the sectors will be denoted A and B respectively. These evolve according to the equations

$$\dot{A} = \delta H_A A \tag{1a}$$

$$\dot{B} = \delta H_B B \tag{1b}$$

where H_A and H_B are the amounts of skilled labour, or human capital, devoted to research in the respective sectors. Time is assumed continuous and dots denote derivatives. This specification of technical progress is common to many recent models of growth. The precise mechanism generating this reduced form is unimportant for characterising the public optimum but when solving for the private equilibrium will need to be specified. Note that given A technical progress is linear in H_A, so that there are in a sense constant returns in the research sector. The implications of relaxing this assumption will be discussed briefly in Section 6. As is well known, linearity in A is important if steady-state growth is to be possible. If there are diminishing returns in A, then growth will eventually peter out unless human capital grows without bound. If there are increasing returns, then the growth rate will be increasing over time.[4]

The formulation in (1) assumes that there are no spillovers between sectors in the accumulation of knowledge. One should therefore think of the industries as being fairly distinct: say cotton and iron rather than cars and motor cycles. It also assumes that the constant of proportionality, δ, is the same in both sectors. This is because the aim is to show that it may be desirable to have asymmetric growth even when the model is completely symmetric. Generalising the analysis to allow for asymmetric technical progress functions presents no difficulties and will be discussed briefly below.

Final output in each sector is produced by knowledge and labour alone with the following production functions:

$$X_t = A_t L_t^X \tag{2a}$$

$$Y_t = B_t L_t^Y \tag{2b}$$

where X_t and Y_t are the final outputs of goods produced in sectors 1 and 2 at time t respectively and L_t^X and L_t^Y are the corresponding inputs of labour. In other words, there are constant returns to scale and technical progress enhances the productivity of labour. For simplicity it is assumed that there is no capital.

In what follows the expressions D_t and $D(t)$ will be used to refer to the value of the variable D at time t and will be used interchangeably.

It is assumed that skilled labour can move freely between sectors and so the social planner faces a constraint of the form

$$H_A(t) + H_B(t) \leq H(t) \tag{3}$$

where $H(t)$ is the amount of skilled labour available at time t. Similarly labour can be freely allocated between sectors and so there is a constraint of the form

$$L_t^X + L_t^Y \leq L(t) \tag{4}$$

So far it has not been specified whether skilled and unskilled labour are freely interchangeable. The results for the public optimum are valid whether they are or not. If they are not then the overall constraint has the form

$$H_A(t) + H_B(t) \leq \bar{H} \tag{5a}$$

$$L_t^X + L_t^Y \leq \bar{L} \tag{5b}$$

where $\bar{H}$ and $\bar{L}$ are the supplies of skilled and unskilled labour respectively. If they are freely interchangeable, then the constraint is

$$H_A(t) + H_B(t) + L_t^X + L_t^Y \leq M \tag{5$'$}$$

where M is the total supply of labour.

Formulation $(5)'$ is the one adopted in one-sector models as the question of interest is whether too much or too little Research and Development takes place. Here the question focussed on is the allocation between sectors and the results for the public optimum vary the allocation of skilled labour holding unskilled labour constant, so either formulation will do. In the case of the private optimum the qualitative results also hold irrespective of the formulation used but explicit description of the growth paths is easier with assumptions $(5a)$ and $(5b)$, so these will be used for concreteness. Under $(5)'$ it is possible that there will be no research if the supply of labour, M, is too low (see Grossman and Helpman (1991a)). The analysis applies when M is large enough, otherwise the question of balance is trivial.

There is assumed to be a single representative consumer who supplies skilled and unskilled labour inelastically. He has a utility function which is additively separable over time and has constant discount rate, r, so his overall utility function has the form

$$\int_0^\infty e^{-rt} V(X, Y)\, dt \tag{6}$$

The within period utility function, V, is assumed to be homothetic so that

$$V(X, Y) = F(U(X, Y)) \tag{7}$$

where F is increasing and concave and U is homogeneous of degree 1. Concavity of F is not necessary for homotheticity but ensures overall preferences are concave. The shape of F does not affect within period preferences but it does affect substitutability between periods. The force of the homotheticity assumption is that if relative prices between the goods remain fixed, as they will in balanced growth, then the goods are consumed in a constant ratio. If this were not so then preferences would change as the economy grew and there is no reason to think that balanced growth is desirable.

As interest will focus on the possibility of unbalanced growth when the model is symmetric it will be assumed that U is symmetric, that is

$$U(X,Y) = U(Y,X) \tag{8}$$

although the results generalise easily without this assumption.

3. The Public Optimum

The **planner's problem** is to choose $\{H_A(t), H_B(t), L_t^X, L_t^Y\}$ to maximise

$$\int_0^\infty e^{-rt} V(X,Y)\, dt$$

subject to the constraints given by (1), (2), (3), (4) and (5). $H_A(t)$ and $H_B(t)$ will be assumed piecewise continuous to ensure a well-defined solution to (1).

The assumption of homotheticity allows considerable simplification of the problem. Consider the problem of allocating labour to production of final goods in period t. Suppose that the planner has decided to allocate a total amount of labour L_t for this purpose. Either (5) or (5)$'$ may be assumed for the purpose of this argument. In order to maximise overall utility, L_t must be allocated between goods 1 and 2 to maximise utility. Taking into account (2) the problem becomes,

$$\begin{array}{rl} \text{Max} & U(X,Y) \\ \text{subject to} & \dfrac{X}{A} + \dfrac{Y}{B} \leq L_t \end{array}$$

In other words, one can regard the inverse of the productivity A as the price of units of good 1 in terms of labour. Now, because U is homogeneous of degree one, the optimal ratio of X to Y is independent of L_t and depends only on the ratio of A to B. L_t simply determines the absolute levels of X and Y. One can therefore write the maximised value of U as $\psi(A,B)L_t$ where ψ is increasing in its arguments and homogeneous of degree one. To interpret ψ note that standard consumer theory (see for example Varian (1992) p. 147) shows U has an expenditure function of form $c(1/A, 1/B)u$ and associated indirect utility function $\frac{y}{c(1/A,1/B)}$, where y denotes income. One can think of c as a unit cost function and so ψ is one over the unit cost function. L_t plays the role of income here and so $\psi(A,B)L_t$ is the indirect utility function.

One can therefore simply analyse the **modified planner's problem**

$$\text{Max} \int_0^\infty e^{-rt} V(\psi(A_t, B_t)L_t)\, dt$$

subject to (1), (3), (5) or (5)$'$.

The aim in what follows will be to find conditions on the optimal allocation of skilled labour to research and the allocation of labour to production will be held constant, so either (5) or (5)' may be assumed.

One could derive necessary conditions by setting up a Hamiltonian but it is more instructive to give a direct argument. The presence of terms of the form $H_A A$ in (1) in any case means that the problem is non-convex and so one cannot apply the usual Hamiltonian sufficiency conditions, whereas the approach used here is well-suited to showing sufficiency.

Note that one can integrate (1) to obtain

$$A(t) = A_0 \exp\{\int_0^t \delta H_A(s)\, ds\} \tag{9a}$$

$$B(t) = B_0 \exp\{\int_0^t \delta H_B(s)\, ds\} \tag{9b}$$

Note that the level of productivity at time t in a sector depends on the amount (or integral) of skilled labour devoted to research up to time t but not on the time path followed. If one denotes $H_A(t) + H_B(t)$ by $H(t)$, the total amount of skilled labour devoted to research at time t, one can multiply these equations together to obtain

$$A(t)B(t) = A_0 B_0 \exp\{\int_0^t \delta H(s)\, ds\} \tag{10}$$

In other words, the levels of $A(t)$ and $B(t)$ achievable at time t are constrained to lie on a certain rectangular hyperbola. This is illustrated in Figure 1. Not all values on the hyperbola are achievable because $H_A(t)$ and $H_B(t)$ cannot be negative.

Looking at the modified planner's problem an intuitive necessary condition for an optimum is that $\psi(A_t, B_t)$ be locally maximised subject to the constraint that A_t and B_t lie on the feasible region defined by (10) (and the non-negativity constraints). For, if say increasing A_t would be desirable one could increase $H_A(s)$ for $t - \epsilon < s < t$ and decrease H_B by a corresponding amount. If one decreases $H_A(s)$ and increases $H_B(s)$ for $t < s < t + \epsilon$ one obtains a path for $\{A(t), B(t)\}$ which coincides with the original one for $s \notin [t - \epsilon, t + \epsilon]$. ψ attains higher values in this interval, for sufficiently small ϵ, since ψ was not maximised at time t. It follows that ψ must be maximised subject to (10) and the non-negativity constraints, This argument can be made rigorous by an explicit variational argument.

Suppose that the optimum involves positive levels of research in both sectors at time t, so that one can increase or decrease $H_A(t)$ and $H_B(t)$ by small amounts freely. In Figure 1, one can therefore attain points either side of (A_t, B_t). If ψ is maximised subject to the constraint that (A, B) lies on the hyperbola, it follows that the level curve of ψ through (A, B) must be tangent to the hyperbola, as is shown in Figure 1. This will be so if and only if the marginal rates of substitution are equal, that is

$$\frac{A_t}{B_t} = \frac{\partial\psi/\partial B_t}{\partial\psi/\partial A_t}$$

or equivalently

$$A_t \frac{\partial \psi}{\partial A_t} = B_t \frac{\partial \psi}{\partial B_t} \tag{11}$$

Dividing through by ψ, it follows that the elasticity of ψ with respect to A and B must be equal.

$$\frac{A_t}{\psi} \frac{\partial \psi}{\partial A_t} = \frac{B_t}{\psi} \frac{\partial \psi}{\partial B_t} \tag{11$'$}$$

To interpret this condition, note that from equation (9), an increase in the amount of human capital allocated to sector 1 increases the level of A proportionately and so the condition for an optimum is simply that the proportionate increase in ψ be offset by the corresponding reduction caused by the reduction of human capital allocated to sector 2. Now as noted above ψ is in fact one over the unit cost function associated with U. Shephard's Lemma shows that the derivatives of c are the (compensated) demand and it follows that the elasticity of c, and so of minus ψ, equals the budget shares.[5] It follows that (11) is equivalent to

$$w_A = w_B \tag{12}$$

where w_A denotes the budget share of A and similarly for w_B. Now since there are only two goods the budget shares must of course be 1/2. Summarising this discussion

Proposition 1 *If research is undertaken in both sectors at time t then condition (11) or equivalently (12) must be satisfied and so both goods have the same budget share.*

So far no use has been made of (8), that is U or equivalently ψ is symmetric. If ψ is symmetric, then if one starts from an initial position when $A = B$, setting $H_A(t) = H_B(t)$ so that A and B grow at the same rate ensures that (1) is always satisfied. In other words,

Corollary *If U is symmetric and the productivity levels in the two sectors are initially equal then balanced growth always satisfies the necessary condition (12) for optimality.*

Condition (12) is, however, only a necessary condition for optimality. It merely states that ψ is tangent to (10). In order for this to imply that ψ is maximised one requires that ψ be more concave than (10). This is so in Figure 1 but is violated in Figure 2. If the situation of Figure 2 holds therefore balanced growth will not be optimal.

To investigate this further, attention will focus on the case when U has the CES form

$$U(X, Y) = (X^\rho + Y^\rho)^{1/\rho} \qquad \rho \leq 1 \tag{13}$$

where as usual $\rho = 0$ corresponds to Cobb-Douglas preferences and $\rho = -\infty$ corresponds to Leontief preferences. ρ less than or equal to 1 guarantees that preferences are concave. In this case, it is easy to show that ψ has the form

$$\psi(A, B) = (A^\sigma + B^\sigma)^{1/\sigma} \tag{14}$$

where σ satisfies

$$\frac{1}{\rho} - \frac{1}{\sigma} = 1 \tag{15}$$

In other words ψ is also has the CES form. This follows easily from Varian (1992) p. 56, noting that ψ is one over the unit cost function.

If U is Cobb-Douglas then ψ is also Cobb-Douglas and so has the form

$$\psi(A, B) = AB$$

Comparing this with (10), one sees that the indifference curve through any point coincides with the constraint set. In this case any combination of A and B with the same product has the same utility and so it is irrelevant how research is distributed among the two sectors. So any kind of growth pattern, balanced or not, with the same total level of research is equally good.

If ρ is greater than zero, however, then σ exceeds zero and ψ is less concave than the constraint curve. In this case, balanced growth is never optimal, even if the sectors have initially equal productivity levels. This can easily be seen from Figure 2. The dotted lines indicate higher level indifference curves and it can be seen that the further away from the point $A = B$ one is on the constraint curve, the higher the level of utility. This is easy to check formally. If one starts from a point where the productivity levels are equal, therefore, then the optimal policy is to devote all human capital to one sector for ever onwards. This ensures that at any point the economy is as far as possible from the 45° line and so utility is as high as it can be. (Note that the hyperbola the economy lies on at any time depends only on the total amount of labour devoted to research and not its allocation between sectors.) If the productivity levels are initially unequal, then it is easy to see that one should solely develop that sector that is leading. The development paths are therefore those shown in Figure 2, where arrows denote directions of research.

This asymmetry of solution is perhaps surprising given the symmetry of the underlying model. One can perhaps understand it more clearly by considering the special case $\rho = 1$. In this case U is given by

$$U(X, Y) = X + Y$$

so the goods are perfect substitutes and ψ is given by

$$\psi(A, B) = \max\{A, B\}$$

Since the goods are perfect substitutes one may as well devote all labour to improving the level of productivity of one good.

If ρ is less than one then the consumer displays a love of variety, since U is concave. This therefore provides an incentive to develop both goods. If, however, ρ is greater than zero then this offset by the consideration that devoting all research resources to one sector enables that sector to grow much more quickly.

Since the model is symmetric it is irrelevant which sector one chooses to develop but once one has chosen one, only that sector should be developed. In other words, it does not matter which direction one chooses but once one has started in a given direction one should continue. One might think that one should develop sector 1 for a while, then switch to sector 2 and so on. In Section 6 a model will be outlined where this is the case but here it is not. Switching to developing sector 2 would bring the economy back closer to the 45° line and it is preferable to keep on developing sector 1.

If ρ is less than zero then ψ is more concave than the constraint and so balanced growth is optimal. In this case, ψ decreases as one moves away form the 45° line along the hyperbola, as shown in (10). If the economy starts on the 45° line, therefore, it is optimal to continue on it. If one sector is initially ahead, then all resources should be devoted to raising productivity in the other sector and when the productivities are equal, balanced growth should follow. This is illustrated in Figure 1.

To summarise:

Proposition 2 *If U has the CES form, (14), and initial productivity levels in the two sectors are equal then all research should take place in one sector if ρ exceeds zero, whilst growth should be balanced if ρ is less than zero. If initial productivity levels differ then only the leading sector should be developed in the first case, while in the second only the lagging sector should be developed until the productivity levels are equal. The pattern of development is irrelevant when ρ equals zero.*

It is easy to show that when ρ exceeds zero and all research takes place in one sector, then the budget share of the leading sector goes to one. In this sense, one sector is inessential. When ρ is less than zero, the leading sector's budget share would go to zero if development were concentrated in that sector. One might argue that in this case both sectors are essential, which provides another perspective on the two cases.

These features of the CES utility function are often considered rather undesirable. Either the sectors are both completely inessential, in the first case, or completely essential, in the second. An intermediate position, where a product's market share rises initially as its price falls but may eventually fall because demand for the other product becomes inelastic might be more reasonable. In this model one could imagine a combination of the behaviour shown in Figures 1 and 2: initially the level curves of ψ are less concave than the constraint set but eventually become more so, as is shown in Figure 3. In this case, ψ would initially increase as one moved along the constraint away from the 45° line but then fall. The optimal policy in this case would be to have initially unbalanced growth, concentrating on just one sector, until reaching the point where the level curves of ψ are tangent to the constraint, points D or E in Figure 3. From that point on growth would be balanced, with research allocated equally between the two sectors to maintain the optimal ratio of A to B.

The discussion above assumes that that the potential for technical progress, as measured by δ, is the same in both sectors. Equations (11) and (12) in fact generalise easily to the case when the productivity of skilled labour in research in sector A, δ_A, differs from that in sector 2, δ_B. (9a) and (9b) are as before except that δ_A and δ_B replace δ in the respective equations. (10) therefore becomes

$$A(t)^{1/\delta_A} B(t)^{1/\delta_B} = A_0^{1/\delta_A} B_0^{1/\delta_B} \exp\{\int_0^t H(t)\, dt\} \qquad (10)'$$

The constraint curve is therefore no longer a rectangular hyperbola but a similar argument to that before yields that if research takes place in both sectors then it must be true that

$$\delta_A A_t \frac{\partial \psi}{\partial A_t} = \delta_B B_t \frac{\partial \psi}{\partial B_t} \qquad (11)''$$

or equivalently in budget share form

$$w_A / w_B = \delta_B / \delta_A \qquad (12)'$$

In other words if research occurs in both sectors, their budget shares are inversely related to their potential for progress. This is because if sector 1 is capable of growing more quickly than sector 2 then the only reason doing research in sector 2 is if it is larger and so improvements in productivity are more valuable. Of course unbalanced growth is more likely with asymmetric sectors. For example with symmetric Cobb-Douglas preferences, one simply does research in the sector with higher δ. On the other hand if preferences are sufficiently concave one will in the long run have balanced growth in accordance with $(12)'$.

The discussion above applies to the case of two sectors but the conditions above generalise immediately to the case of n sectors. Applying the argument to any pair of sectors shows that $(11)''$ or equivalently $(12)'$ must hold between any pair of sectors which both enjoy positive levels of research.

In summary, this section has derived necessary conditions for the public optimum and used them to show that even if the model is completely symmetric it may be optimal to have unbalanced growth. Essentially this arises from the fact that concentrating research sector allows that sector to grow very rapidly and this more than offsets any desire to have balanced consumption of both goods.

4. The Private Economy

The following two sections investigate the extent to which the private sector will achieve the optimum described in the previous section. It shows that the desire to avoid competition from other researchers may lead private research to spread between sectors when the public optimum would be to have it concentrated. Growth as a consequence may be insufficiently unbalanced.

In order to do this one needs to specify a model of the research process. The model adopted here is a simple generalisation to two sectors of the model of quality ladders due to Grossman and Helpman (1991a), whose work in turn builds on that of Aghion and Howitt (1992) and Segerstrom, Anant and Dinopoulos (1990). Since the generalisation is fairly obvious, the reader is referred to Grossman and Helpman (1991a) for omitted details.

It is assumed that final output in each sector is in fact a composite of many goods. One might for example think of one sector being food and the composite goods being different kinds of foods. Each sector consists of a continuum of such sub-industries indexed by $j \in [0,1]$. Innovation in these industries takes the form of discrete increases in quality. The quality of the mth generation of goods in sub-industry j in sector i, $q^i_m(j)$, is λ times that of the $m-1$th, so $q^i_m(j) = \lambda q^i_{m-1}(j)$, where $\lambda > 1$ is a constant independent of generation, sub-industry and sector. Once an innovation is made it becomes available for all subsequent researchers to improve on, so there are unappropriated gains from research.

It is assumed that preferences are separable between sectors, so that one can think of there being sub-utility functions for each sector. If a consumer consumes $x^i_{mj}(j)$ units of quality m of good j in sector i at time t, then his utility from consuming sector i's products, X^i, is given by

$$\log X^i(t) = \int_0^1 \log(\sum_m q^i_m(j) x^i_{mj}(j))\, dj \tag{16}$$

In other words, goods of a different quality in the same industry are perfect substitutes for one another once one makes an appropriate quality adjustment. Units are chosen so that $q^i_m(0) = 1$ and so $q^i_m(j) = \lambda^{m_j}$. Furthermore, according to (16) the output of the different sub-industries is combined according to a continuous version of the Cobb-Douglas utility function.

These sub-utilities are combined into an overall instantaneous utility function $V(X^1, X^2)$, which is in turn added up over time to give overall utility

$$\int_0^\infty e^{-rt} V(X,Y)\, dt$$

exactly as in the last section. X and Y will be used rather than X_1 and X_2 for consistency with the last section. One can therefore think of the consumer deriving utility from different kinds of food and X as summarising his total utility from food. This is combined with utility from consuming goods from the non-food sector to give overall utility. One could equivalently think of a final good X being assembled from various intermediate goods and then sold to consumers. Either interpretation will do equally well.

From (16) it follows that consumers will only buy those qualities with the lowest quality-adjusted price $p^i_{mt}(j)/q^i_m(j)$. Furthermore, the Cobb-Douglas form of (16) implies that spending in a given sector will be equally spread across all products. So if the consumer spends E^i on sector i the demand for industry j's products is

$$x^i_t(j) = \frac{E^i}{p^i_{st}(j)} \tag{17}$$

where s is the quality offering the lowest quality-adjusted price (in equilibrium the highest). All other qualities have zero demand.

As in Section 2 it will be assumed that the overall utility function V is homothetic and so has the form $F(U(X,Y))$ where U is homogeneous of degree one. In order to be able to solve explicitly for the equilibrium it will be assumed that U is of the CES form

$$U(X,Y) = (X^{\rho} + Y^{\rho})^{1/\rho} \qquad \rho \leq 1 \tag{18}$$

It will also be assumed that F is the logarithm function, that is

$$V(X,Y) = \log U(X,Y)$$

As noted in section 2, the form of F does not affect the allocation of expenditure within a period but it does affect the allocation of expenditure between periods. The current formulation is convenient as it allows one to assume that interest rates are constant. It is straightforward to show that under this assumption the optimal allocation of expenditure between periods obeys

$$\frac{\dot{E}}{E} = R - r$$

where R is the rate of interest. If one normalises the price level so that

$$E(t) = 1 \tag{19}$$

for all t, it therefore follows that interest rates are constant with

$$R = r$$

and r will denote interest rates throughout.[6]

This completes the description of the consumer side of the economy. As far as producers are concerned, it is assumed that innovators produce goods of higher quality. This know-how is then licensed to a single firm to manufacture the new goods or equivalently the patent holder sets up his own company. In any given industry, the good of highest quality competes with those of lower quality. Competition is by prices. The only input to produce each good is labour and one unit of any good requires one unit of labour. So unit production costs are ω, where ω is the wage rate.

Consider the pricing strategy of the producer with currently highest quality. The producer with next highest will cut prices down until he is charging marginal cost, that is ω. The producer with highest quality can charge $\lambda\omega$ and still win the entire market as consumers value his product λ times as much as the next best product. From (17), it follows that demand is unit elastic and so there is no point in charging a price below $\lambda\omega$. This discussion assumes that the highest quality product is exactly one step ahead of its nearest rival. It is easy to show as in Grossman and Helpman (1991a) that the currently leading firm undertakes no research, essentially because it has less incentive to do so because it is earning positive current profits, and so this is in fact true.

It follows that all products sell at price

$$p = \lambda\omega \tag{20}$$

and that a firm in sector 1 and 2 makes respective profits

$$\pi_A = (1-\alpha)w_A \tag{21a}$$
$$\pi_B = (1-\alpha)w_B \tag{21b}$$

where $\alpha = 1/\lambda$ and w_A is the budget share of sector 1. The latter holds since the normalisation (19) implies that each sector has w_A spent on it and from (17) a price of p yields sales of w_A/p.

Finally, the research technology will be specified. It is assumed that once an improvement in quality has been made, all inventors are able to use this as the basis for future improvements. It is assumed that in each sub-industry, success arrives according to a Poisson process. If a firm employs an amount of skilled labour h then in the small interval of time dt it has probability γhdt of succeeding in discovering how to produce the next highest quality. Once this discovery has been made, it enables other inventors to seek to improve upon it and thus there an unappropriated spillover from R&D.

From (21) it follows that it is equally profitable to be an industry leader in any industry within a given sector and so attention will be restricted to equilibria in which all industries within a given sector have equal levels of research devoted to them.

To see that this description of the private economy is consistent with the specifications for output and research given by (1) and (2) in section 2, note that for sector 1 (16) is equivalent to

$$X(t) = A_t \exp \int_0^1 \ln x_t(j)\, dj \tag{22}$$

where

$$A_t = \exp \int_0^1 \log(\lambda^{m_t(j)})\, dj = \lambda^{\int_0^1 m_t(j)\, dj} \tag{23}$$

$x_t(j)$ denotes the consumption of j at time t, assuming only the highest quality is produced and $m_t(j)$ denotes the generation of the current highest quality of sub-industry j. A_t corresponds to the level of knowledge in Section 2 and measures the average quality level in sector 1. From the discussion above, it follows that if L_t^X units of labour are devoted to sector 1 it will be spread equally across industries. Since it takes one unit of labour to produce one unit of each good, (22) reduces to

$$X(t) = A_t L_t^X$$

which is precisely (2a). If H_A units of skilled labour are devoted to sector 1, it follows from the above discussion that it will be devoted equally to industries and so each has probability of $\gamma H_A dt$ of making a quality improvement of λ. From (23) and the properties of the Poisson process it follows (see Grossman and Helpman (1991a) for details) that

$$\dot{A} = (\ln \lambda)\gamma H_A A$$

which is precisely ($1a$) with $\delta = (\ln \lambda)\gamma$.

It remains to determine the equilibrium conditions for research and development. It is assumed that an inventor receives an infinitely long patent on his innovation. By licensing the invention, or using it to produce output himself, he receives the full flow of profits, given in (21), from it until he is displaced by the next improvement. The values of holding an invention in sectors 1 and 2, V_A and V_B respectively, are determined by the condition

$$\pi_A + \dot{V}_A - \gamma H_A V_A = rV_A \qquad (24a)$$

$$\pi_B + \dot{V}_B - \gamma H_B V_B = rV_B \qquad (24b)$$

π_A gives the flow of profits received by the inventor. $\dot{V}_A$ is the capital gain on holding a patent in sector 1. The final term on the left-hand side represents the risk of being displaced: if H_A units of skilled labour are being devoted to research in every industry in sector 1 (recall the equilibrium is symmetric) then there is probability $\gamma H_A dt$ in length of time dt that an inventor will succeed and so V_A be lost. The value of the patent given by the left-hand side must in equilibrium be equal to what could be obtained by selling it and investing in a risk free bond with return r, which is the right-hand side.

If an investor considers employing labour in research in sector 1 for a short interval dt then if h_A units of labour are employed there is probability γh_A that he makes a breakthrough and wins prize V_A. Since this probability is linear in h_A it follows that if there is to be research in both sectors in must be the case that the rewards are equal in the two sectors, that is

$$V_A = V_B \qquad (25)$$

Under assumption (5), that is skilled labour can only be used in research, this is the only condition needed, since the supply of labour is inelastic. If one were to assume $(5)'$, then the productivity of labour in a sector where research takes place would have to equal that in production, so $\alpha V_A = w$ would be required in addition. Explicit determination of asymmetric growth paths is slightly easier with (5) but the other results only depend on the truth of (25) independently of (5).

Attention will be restricted to equilibria in which $H_A(t)$ and $H_B(t)$ are piecewise continuous in order to ensure well-defined solutions to (1).

5. The Private Equilibrium

With the description of the private economy completed one can now apply the equilibrium conditions to determine its equilibrium behaviour. In section 3 it was shown that completely unbalanced growth is optimal when ρ exceeds zero. If all research is to be concentrated in, say, sector 1 it must be the case that the value of a patent in sector 1, V_A, exceeds that of one in sector 2, V_B. Now from (24) one can see that concentrating research in sector 1 has two effects: firstly it changes the market share and so profitability of sector 1 and secondly it increases the likelihood of losing a patent in that sector. From (21), if concentration makes the share of sector 1 grow then this increases its attractiveness and this is indeed the case when ρ exceeds zero, as discussed in section 3. This effect works in favour of concentration. On the other hand, the fact that one is more likely to lose a patent in sector 1 because of the higher level of research makes switching to sector 2 more attractive. The overall effect is ambiguous.

In the case of Cobb-Douglas preferences it is easy to see that one cannot have unbalanced growth. Here the market share of both sectors is always 1/2, independently of the productivity levels in the two sectors. Therefore the profits levels, π_A and π_B, are always equal. The only respect in which the sectors differ is therefore the research intensities and so they must be equal, otherwise all researchers would wish to switch to the sector with lower probability of losing their patent if they win it. In the Cobb-Douglas case the allocation of research effort between sectors is irrelevant and so the private equilibrium is efficient. If ρ is close to zero, however, then it is easy to see that the same force will work in favour of inefficient dispersion of research efforts if the productivity levels of the sectors and so profit levels are initially close. The private equilibrium will therefore be insufficiently unbalanced.

In this section, it is shown that if ρ is positive but less than a critical value, then the private equilibrium is always insufficiently unbalanced. Indeed, from a position of equal initial productivity levels there will always be equal research in both sectors. If productivity levels are unequal then eventually all research will be concentrated in the leading sector but not immediately as it should be. If ρ exceeds this value, then there will always be specialisation, as there ought to be, but there are multiple equilibria and there may be specialisation in the wrong sector. The effect of expectations is so strong that there may be specialisation in the lagging sector if this is expected to be the leading sector in the long run. Finally if ρ is less than zero, the private economy will converge to balanced growth, as it ought to, but research will begin in the leading sector before equal productivity levels are achieved, in contrast to the public optimum.

To proceed more formally, one needs to determine V_A and V_B. Integrating (24) yields,

$$V_A(0) = \int_0^\infty e^{-rt-\gamma \int_0^t H_A(s)\,ds} \pi_A(t)\,dt \tag{26a}$$

$$V_B(0) = \int_0^\infty e^{-rt-\gamma \int_0^t H_B(s)\,ds} \pi_B(t)\,dt \tag{26b}$$

In other words, the value of the patent is simply the discounted value of the profits derived from it, where the discount rate is adjusted for the probability of losing the patent. These equations assume that $e^{-rt}V_A(t)$ goes to zero as t goes to infinity, and similarly for V_B, in other words that there are no bubbles in the asset values. This the case here as there is a single infinitely long-lived consumer.

Now π_A and π_B are proportional to the respective budget shares of the two products. Since the relative price of the two goods is B/A it follows from Varian (1992) (p. 112) that the budget share of sector 1 is

$$w_A = \frac{A^\sigma}{A^\sigma + B^\sigma} \tag{27}$$

where σ equals $\rho/1-\rho$. Using (9) to determine A and B and (21) one therefore obtains

$$V_A(0) = (1-\alpha)\int_0^\infty e^{-rt}\frac{A_0^\sigma \exp\{\frac{\rho(\delta+\gamma)-\gamma}{1-\rho}\int_0^t H_A(s)\,ds\}}{\Delta} \tag{28a}$$

$$V_B(0) = (1-\alpha)\int_0^\infty e^{-rt}\frac{B_0^\sigma \exp\{\frac{\rho(\gamma+\delta)-\gamma}{1-\rho}\int_0^t H_B(s)\,ds\}}{\Delta} \tag{28b}$$

where

$$\Delta = A_0^\sigma \exp\{\sigma(\delta\int_0^t H_A(s)\,ds)\} + B_0^\sigma \exp\{\sigma(\delta\int_0^t H_B(s)\,ds)\}$$

Suppose that initially the productivity levels of the two sectors are equal so that $A_0 = B_0$. If one sets $H_A(t)$ equal to $H_B(t)$ for all t then clearly $V_A(0) = V_B(0)$ so that it is equally attractive to pursue research in either sector. The same holds for all subsequent times. One therefore obtains

Proposition 3 *If the productivity levels of the two sectors are equal initially then balanced growth of both sectors is always an equilibrium.*

Intuitively, if both sectors are growing at an equal rate and research is taking place at an equal rate in both sectors, then researchers are indifferent between them. Thus balanced growth is always an equilibrium. If ρ exceeds zero then this is socially inefficient, as seen in section 3.

One might however ask if there is another equilibrium in which there is concentration on one sector. This cannot be the case if ρ lies between zero and θ, where $\theta = \frac{\gamma}{\gamma+\delta} = \frac{1}{1+\ln\lambda}$. Suppose that one is looking for an equilibrium in which all research takes place in sector 1. Then $H_A(t) > H_B(t)$ for all t and in particular

$$\int_0^t H_A(s)\,ds > \int_0^t H_B(s)\,ds \tag{29}$$

for all positive t. Yet examining (28) one sees that if ρ is less than θ the integrand in (28a) is less than that in (28b) for all t. (Note that the denominators are the same and that $A_0 = B_0$ here.) It follows that $V_A(0) < V_B(0)$. If this is the case, however, then it is more valuable to own a patent in sector 2 than sector 1 initially and so all research would take place in sector 2, contrary to supposition.

It follows that one cannot have completely unbalanced growth, that is specialisation in one sector. In fact balanced growth is the unique equilibrium if initial productivity levels are equal. Note first of all, one cannot have one sector always leading another: if one were to have $A(t) > B(t)$ for all $t > 0$ then, from (9), (29) would have to be true. Yet this was just shown to yield a contradiction. It follows that if one starts from a position of equal productivity one cannot escape from it permanently and must always return to it.

On the other hand, one can show

Lemma 1 *If $\theta > \rho \geq 0$ then $H_A \geq H_B$ if $A > B$ and $H_B \geq H_A$ if $A < B$.*

A formal proof can be found in the Appendix but the intuitive argument is that if A exceeds B, then sector 1 has a larger budget share than sector 2 and so research will take place in sector 2 only if this is compensated for by a lower probability of losing a patent, that is $H_B \leq H_A$. One has to argue slightly more carefully because the future value of holding the patent in sector 2 might be large even if current flow profits are low.

Lemma 1 implies that if ever A is strictly greater than B then this will be true for ever after. It follows that if one ever leaves the line $A = B$ one will never return to it again. It was shown above, however, that if one starts on that line one must return to it. It therefore follows that if initially $A = B$, this must be true for ever after. To summarise

Proposition 4 *If $0 < \rho < \theta$ and the productivity levels of the two sectors are initially equal, then the productivity levels of the two sectors will always be equal. The private equilibrium is therefore socially inefficient.*

Not merely will the private sector fail to achieve complete specialisation, as the public optimum requires, but it cannot even have one sector's productivity always growing faster than the other's.

The value of θ depends on λ. The larger is λ then the more rapidly sector 1 grows and so the more valuable it is to hold a patent in that sector for a given rate of research in that sector.

In terms of Figure 4, therefore, the public optimum has the system always moving vertically or horizontally, with research in only one sector. In the private equilibrium it is impossible to escape permanently from the 45° line. It is natural to ask what happens if one starts close to but not on the 45° line, that is the two sectors have similar levels of productivity initially. The same argument as above shows that one cannot have complete specialisation if one starts close to the 45° line. On the other hand, Lemma 1 shows that one cannot have convergence to the 45° line: if initially $A > B$, then $H_A \geq H_B$.

In the Appendix an explicit equilibrium is constructed for this case. Initially, research takes place in both sectors but H_A exceeds H_B only by a small amount, to ensure $V_A = V_B$. As the difference between A and B grows so does the difference between H_A and H_B until the line OD is reached (see Figure 4). Along this line, an inventor is indifferent between holding a patent in sector 1 and sector 2 given that all future research will take place in sector 1: that is the benefit of being in a large market exactly offsets the greater threat of losing the patent. When OD is reached, research in sector 2 falls discontinuously to zero: V_A and V_B are continuous but their derivatives are not. Beyond this point, investors strictly prefer to hold a patent in sector 1 and $V_A > V_B$. A symmetric story holds if initially $B > A$. It is also shown that this is the unique equilibrium of the model. Assumption (5) is only used in the argument of this paragraph. It makes explicit determination of the asymmetric equilibrium paths easier, as there is a known maximum level of research, but it could probably be relaxed.

In the case when ρ lies between 0 and θ, therefore, the fear of competition from other inventors prevents efficient specialisation. If one starts with initially equal levels of productivity one cannot escape from that position permanently. If one starts with nearly equal levels of productivity, one can achieve A/B tending to infinity, as one ought to, but the competitive pressure prevents this happening as rapidly as it ought to.

If ρ exceeds θ then the benefits of owning a patent in the larger market outweigh the competitive pressure from other researchers and it is possible to achieve complete specialisation from initially equal levels of productivity, although the inefficient balanced growth equilibrium still remains. This follows straightforwardly from (28).

The advantages of being in the faster growing sector, are however, so great that it is possible for the private economy to specialise in the wrong sector. If $A > B$ then the public optimum requires that the economy specialise completely in sector 1. Examining (28) shows that if A is close to B then it is an equilibrium to have $H_A(t) = 0$ and $H_B(t) = \bar{H}$ for all t. Although, the flow profits from being in Sector 2 are lower and the likelihood of losing the patent higher than in Sector 1, the fact that 2 is expected to become the leading sector outweighs this effect.

Lemma 1 becomes

Lemma 2 *If $\rho > \theta$ then if $A > B$ either $H_A \geq H_B$ or $H_A = 0$ and $H_B > 0$. H_B can only be positive if eventually the system crosses the 45° line and if $H_A = 0$, then H_A must remain zero until the 45° line is crossed.*

In other words, it is only possible for there to be positive research in sector 2 when $A > B$ if it is believed that at some point in the future sector 2 will have a larger market share than sector 1. It is shown in the Appendix that if $A \neq B$, then there must eventually be complete specialisation in one sector. There are, however, equilibria in which this need not happen immediately if initial productivity levels are close. In the Appendix a rather curious equilibrium in which research reverses direction is constructed. If $A < B$ then initially research in sector 2 exceeds that in sector 1. At some point research in sector 2 drops to zero and all research takes place in sector 1 subsequently. The force of expectations is very strong in this parameter range and there can therefore be multiple equilibria, although the equilibria with research reversing directions seem rather perverse.[7]

In terms of Figure 5, therefore the natural equilibria involve complete specialisation in one sector, though possibly the wrong one, as shown. As noted, however, there are equilibria in which initially the system moves away from the 45° in one direction but then switches to complete specialisation in the other sector.

To summarise:

Proposition 5 *If $\rho > \theta$ then completely unbalanced growth can be achieved from equal levels of productivity growth and the private economy is socially efficient, although there is also an inefficient balanced growth equilibrium if productivity levels are initially equal. If productivity levels are initially unequal there will eventually be complete specialisation but it may be in the wrong sector if initial productivity levels are similar.*

If ρ equals θ it follows from ($28a$) and ($28b$) that if $A > B$ then $V_A(0) > V_B(0)$ no matter what the allocation of research effort, and so the unique equilibrium has complete specialisation in whichever sector has initially higher productivity levels.

If ρ is less than zero then the market with lower productivity has a higher market share and one converges to balanced growth from unequal productivity levels. This is fairly clear since Lemma 1 is reversed in this case but formal details are in the Appendix. Note, however, that in the social optimum one ought to have no investment in the leading sector until productivity levels are equal. If, however, the economy is close to the 45° line, then the market shares of the two sectors will be nearly equal and so one cannot have research in the two sectors very different. It follows that research will start in the leading sector before the productivity levels are equalised. Again, formal details can be found in the Appendix. The convergence pattern is therefore as shown in Figure 6.

To summarise:

Proposition 6 *If $\rho < 0$ then the economy converges to balanced growth but the convergence is too balanced.*

These results suggest that the private sector may pursue research on too wide a frontier and that as a result growth may be too balanced. The reason is that competition from other inventors gives an incentive for researchers to seek less competitive areas. In doing so they do not take into account the social benefit that greater concentration leads to higher growth because of the spillovers from research.

It is easy to see that the model is fairly robust. Suppose for example that the sectors have unequal rates of potential productivity growth, that is unequal γ's. If preferences are Cobb-Douglas then, as shown in section 3, one should have all research concentrated in the sector with higher γ. This is not achievable in the private economy if the level of human capital is high. Note that with with these preferences the flow of profits in each sector is always equal. If $\gamma_A > \gamma_B$ then if $V_A = V_B$ research in sector 1 is more attractive since making a discovery is more likely there. If however the level of research is high enough in sector 1 compared to sector 2, this will be offset by the greater competitiveness of sector 1.

Similarly suppose that γ is the same in both sectors but preferences are asymmetric. If they have the Cobb-Douglas form then it is optimal to have all research concentrated in the sector with higher weight in the utility function. The budget shares of the two sectors, and therefore the flow of profits from a patent, are independent of the levels of A and B. Therefore even if sector 1 has a higher budget share, if the level of research in sector 1 is high compared to that in sector 2 this will make it less attractive. This therefore again limits the degree of concentration of research that can be achieved. In both the examples of the last two paragraphs the results will also hold for ρ near zero.

The specific results of the model were derived for particular functional forms but the intuition clearly applies more widely. The threat of competition from other researchers tends to make research more widely dispersed than is desirable from the social standpoint.

6. Oscillating Growth

In the previous sections it was assumed that there are constant returns to research at any time in as much as the basic equation

$$\dot{A} = \delta H_A A$$

is linear in H_A. Doubling the input of researchers double the incremental number of ideas over a small interval.

It is arguable that there may be increasing returns to research. For example, academics often prefer to write joint papers. If this is so one would have equations of the form

$$\dot{A} = \phi(H_A)A \qquad (1a)'$$

$$\dot{B} = \phi(H_B)B \qquad (1b)'$$

where $\phi'' > 0$. This creates a further incentive for specialisation.

In fact there is no optimum when ρ is less than zero. The source of the difficulty is that the presence of increasing returns means that the government has an incentive to pursue a randomised policy. Suppose that it wishes both sectors to grow at equal rates. It could simply allocate equal amounts of skilled labour to each sector. The fact that that ϕ is convex, however, implies that

$$\frac{1}{2}\phi(\bar{H}) + \frac{1}{2}\phi(0) > \phi(\frac{1}{2}\bar{H})$$

In other words expected growth in ideas would be higher if the government gave each sector a fifty-fifty chance of having all the available skilled labour rather than sharing it equally for sure. The same preference for randomisation holds in any allocation that involves both sectors using skilled labour.

The precise interpretation of a randomised strategy is open to debate, since it involves choosing randomly at each instant, but it seems reasonable to suppose that if there is probability $c_A(t)$ that all skilled labour is allocated to sector 1 and probability $c_B(t)$ that it is allocated to sector 2 at time t then A and B evolve according to

$$\dot{A} = c_A(t)\phi\big(H(t)\big)\, A \tag{1a)$''$}$$

$$\dot{B} = c_B(t)\phi\big(H(t)\big)\, B \tag{1b)$''$}$$

where $H(t)$ is the total amount of labour devoted to research at time t. This seems reasonable as, making an informal appeal to the law of large numbers, if there is probability $c_A(t)$ that sector 1 is chosen at time t, then sector 1 will be chosen approximately a proportion $c_A(t)$ of the time in a small interval of time dt.[8] This interpretation is standard in control theory — see for example Young (1969).

If ρ is exceeds zero this causes no difficulty since the government always wishes to specialise anyway. If however ρ is less than zero then there is a difficulty. If one allows the government to randomise then it would like to have equal shares allocated to the two sectors. This follows since if one writes $c_A(t)$ as $H_A(t)/H(t)$ and similarly for $c_B(t)$ then $(1)''$ becomes

$$\dot{A} = H_A(t)\chi_t\, A$$

$$\dot{B} = H_B(t)\chi_t\, B$$

where χ_t equals $\phi(H(t))/H(t)$. This has the same form as (1) and arguing as in section 4 shows that it is optimal to have $H_A(t) = H_B(t)$ if $A = B$.

If one does not allow the government to randomise it will attempt to approximate this by rapidly switching skilled labour between sectors: that is allocate all labour to sector 1 for a while, then to sector 2 and so on. It can however always improve the approximation and so its payoff by switching more rapidly. There is therefore no optimum.

In the control theory literature this phenomenon is known as chattering (see Young (1969) for a more detailed discussion.) In practical terms, this solution seems rather curious because it seems rather hard to imagine infinitely rapid switching between sectors. On the other hand, if one added adjustment costs, the switching would not be so rapid and one can interpret the answer here as saying that one ought to have periods of rapid growth in a given sector followed by slow growth if switching costs are low. Constructing an explicit model is beyond the scope of the paper but it is clear one could construct one with oscillatory growth.

In practice, the degree of increasing returns may be limited. After all, one sees fewer three author than two author papers, at least in Economics. Kortum (1993) suggests that in the aggregate there may be decreasing returns to Research and Development.[9] If ϕ were globally concave, then this would reduce the range of ρ for which unbalanced growth is optimal but unless ϕ is very concave would not eliminate it entirely. The point, however, remains that some degree of initial increasing returns may be a reason for fluctuations being optimal.

7. Conclusion

The paper has examined a simple model of endogenous growth and has shown that the private sector may produce growth that is too balanced. The reason being that private agents have an incentive to develop sectors in which competition, and so the risk of being displaced, is less. As a result, research effort may be more dispersed than is socially desirable. The government may therefore wish to intervene, by for example subsidising research in certain sectors, in order to encourage concentration.

The model is clearly somewhat special in its choice of functional forms. It is also a closed economy model. If increasing returns in research are limited, and knowledge flows imperfectly across borders, then the desirability of specialisation will clearly be greater for small countries than large ones. The point that unbalanced growth may sometimes be in the public interest but not achievable without intervention should, however, still continue to hold in more general models.

Footnotes

1. See Jackson (1992) p. 20 Table 11 for some recent estimates of sectoral growth rates and Crafts and Harley (1992) for further discussion of overall growth rates.

2. See also Scitovsky (1959).

3. Some models, such as Lucas (1988), allow accumulation of both human and physical capital, and so have two sectors, but there is only one kind of final output.

4. See for example Solow (1994). Jones (1995) considers a one-sector model in which there are decreasing returns to A but population growth allows continuing technical progress.

5. See Deaton and Muellbauer (1990) exercise 2.13, p. 43.

6. This normalisation is employed by Grossman and Helpman (1991b). It of course does not affect the real allocations.

7. These equilibria involve taking different actions at the same ratio of A/B. Since π_A and π_B depend only on A/B this seems unnatural and restricting attention to Markov strategies which depend only on A/B would rule them out.

8. There are of course well known difficulties with formalising the idea of a law of large numbers with a continuum of random variables — see for example Judd(1985).

9. Jones (1995) and Stokey (1995) consider one-sector growth models with this feature.

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Appendix

Proof of Lemma 1

Suppose first of all that H_A and H_B are both strictly positive for some interval of time during which $A > B$ and $H_A < H_B$. (One may assume this since H_A and H_B are by assumption piecewise continuous and hence A and B and also V_A and V_B are continuous.) During that interval V_A and V_B must be equal. Now from (24)

$$V_i(0) = \int_0^\tau e^{-\Gamma_i(t)}\pi_i(t)\,dt + e^{-\Gamma_i(\tau)}V_i(\tau) \tag{A1}$$

where τ is an arbitrary time and $i = A$ or B and $\Gamma_i(t)$ is defined by

$$\Gamma_i(t) = rt + \gamma\int_0^t H_i(s)\,ds \tag{A2}$$

Now suppose that 0 and τ are in the given interval. Then one must have $V_A(\tau) = V_B(\tau)$. On the other hand, if $\rho \geq 0$ and $A > B$ then $\pi_A(t) \geq \pi_B(t)$ for all t in $[0,\tau]$. It follows from $(A1)$ that $V_A(0) > V_B(0)$ if $H_A(t) < H_B(t)$ for all t in $[0,\tau]$, a contradiction.

Suppose therefore that H_B is strictly positive and H_A is zero during some interval where $A > B$. Let 0 be a point in this interval and let $T = \inf\{t : H_A(t) > 0\}$ and $S = \inf\{t : A(t) < B(t)\}$. If $T = \infty$ and $S = \infty$ then $A > B$ for all t and using (24), or equivalently $(A1)$ with $\tau = \infty$, and arguing as in the first paragraph yields a contradiction.

If $T < \infty$ then $V_A(T) = V_B(T)$ since $H_A(t) > 0$ implies $V_A(t) \geq V_B(t)$ and so this follows from the definition of T and continuity of V_A and V_B. If $A(T) \geq B(T)$, using $(A1)$ yields a contradiction. If $A(T) < B(T)$, then $S < T$ and $A(t) > B(t)$ for $t < S$ and $A(t) < B(t)$ for $T > t > S$. Now

$$V_i(0) = \int_0^S e^{-\Gamma_i(t)}\pi_i(t)\,dt + \int_S^T e^{-\Gamma_i(t)}\pi_i(t)\,dt + e^{-\Gamma_i(T)}V_i(T) \tag{A3}$$

Now in $[0,S]$, $\pi_A(t) \geq \pi_B(t)$, so the first term is larger when $i = A$. Also when $t \in [S,T]$ the second term is larger when $i = A$ by equation (28) (with limits of integration S and T rather than 0 and ∞) as $\rho < \theta$. Also $\Gamma_A(T) < \Gamma_B(T)$ since $H_A(t) < H_B(t)$ for t in $[0,T]$. Hence $V_A(0) > V_B(0)$, contradicting the fact that $H_A(0) < H_B(0)$.

If $T = \infty$ and $S < \infty$ then applying the argument of the last paragraph to $(A3)$ with the last term equal to zero yields a contradiction.

Equilibrium off $45°$ *line when* $0 < \rho < \theta$

It is enough to consider the case when $A > B$. From (28) it follows that there is a unique value K of A/B such that it is an equilibrium to have $H_A = \bar{H}$ and $H_B = 0$ for all time if one starts at (A,B). The equilibrium described in the text will be constructed by looking for strategies which depend only on $x = A/B$. Note that π_A and π_B depend only on x. Abusing notation by writing $H_A(x)$, $V_A(x)$ and so on, the equation for V_A becomes

$$V_A'\delta x(H_A - H_B) + \pi_A - (r + \gamma H_A)V_A = 0 \tag{A4}$$

Note that $\dot{x} = \delta(H_A - H_B)x$. If H_A and H_B are both positive for some interval then $V_A = V_B$ and using ($A4$) and the corresponding equation for V_B and the fact that $H_A + H_B = \bar{H}$ and $\pi_A + \pi_B \equiv 1 - \alpha$ yields

$$H_A = \frac{\bar{H}}{2} + \frac{\pi_A - \pi_B}{2\gamma V_A} \tag{A5}$$

and

$$V_A'\delta x(\pi_A - \pi_B) - (r + \gamma\frac{\bar{H}}{2})\gamma V_A^2 + \frac{1-\alpha}{2}\gamma V_A = 0 \tag{A6}$$

Provided $\pi_A \neq \pi_B$ this equation is separable in variables and can be integrated to obtain V_A. Imposing the condition that $V_A(K)$ equals the value of a patent when $H_A = \bar{H}$ and $H_B = 0$ starting from $A/B = K$ determines the value function off the 45° degree line uniquely. $\pi_A = \pi_B$ corresponds to starting on the 45° line and yields the value of staying there forever.

It needs to be shown that ($A4$) prescribes values for H_A that do not exceed $\bar{H}$. Note that at K the amount of research in the future on good A will never exceed $\bar{H}$ and its profit level will be at least $\pi_A(K)$ and so using ($26a$),

$$V_A(K) > \frac{\pi_A(K)}{(r + \gamma\bar{H})} \tag{A7}$$

Similarly the research level on good B will never be less than zero and its profitability will be at most $\pi_B(K)$, so

$$V_B(K) < \frac{\pi_B(K)}{r} \tag{A8}$$

Using the fact that $V_A(K) = V_B(K)$ and ($A7$) and ($A8$) yields

$$\gamma\bar{H}V_A(K) > \pi_A(K) - \pi_B(K) \tag{A9}$$

Using ($A9$) in ($A5$) shows that at $x = K$, $H_A < \bar{H}$. That is there is discontinuous jump in H_A at $x = K$. V_A is continuous but V_A' is not.

To show that $H_A < \bar{H}$ for all $x < K$ it will be shown that V_A is decreasing and hence from ($A5$) that H_A is increasing. Reverting to time-dependent form, if both H_A and H_B are positive on an interval then $V_A = V_B$ and $\dot{V}_A = \dot{V}_B$ and so adding together ($24a$) and ($24b$) yields

$$2\dot{V}_A + (1-\alpha) - (2r + \gamma\bar{H})V_A = 0 \tag{A10}$$

It follows that V_A is decreasing if $V_A(K) < \frac{1-\alpha}{2r+\gamma\bar{H}}$.

Now for $x > K$, $H_A = \bar{H}$ and $H_B = 0$, so adding together ($24a$) and ($24b$) yields

$$\dot{V}_A + \dot{V}_B = \gamma\bar{H}V_A + r(V_A + V_B) - (1-\alpha) \tag{A11}$$

Note that $(A10)$ and $(A11)$ imply that $\dot{V}_A + \dot{V}_B$ is continuous across $x = K$. Now as $x \to \infty$, $\pi_A \to (1-\alpha)$ and $\pi_B \to 0$ and so from $(26a)$ and $(26b)$, $V_A \to (1-\alpha)/(r+\gamma\bar{H})$ and $V_B \to 0$. Hence $V_A + V_B \to (1-\alpha)/(r+\gamma\bar{H}) < 2(1-\alpha)/(2r+\gamma\bar{H})$. Now either $V_A + V_B < 2(1-\alpha)/(2r+\gamma\bar{H})$ for all $x \geq K$ or $V_A + V_B$ has a maximum in the range $K \leq x < \infty$ at least equal to $2(1-\alpha)/(2r+\gamma\bar{H})$. In the former case, since $V_A(K) = V_B(K)$, then certainly $V_A(K) < (1-\alpha)/(2r+\gamma\bar{H})$, so consider the latter case.

At the maximum of $V_A + V_B$ for $x \geq K$, one must have $\dot{V}_A + \dot{V}_B \leq 0$. Since $V_A \geq V_B$ for $x \geq K$, with inequality strict if $x > K$, it follows from $(A11)$ that at the maximum

$$(r + \gamma\frac{\bar{H}}{2})(V_A + V_B) \leq (1-\alpha) \tag{A12}$$

with strict inequality if the maximum occurs for $x > K$. This yields a contradiction unless the maximum occurs at $x = K$ and $V_A(K) + V_B(K) = 2(1-\alpha)/(2r+\gamma\bar{H})$. In this case, however, $(A11)$ implies that $\dot{V}_A + \dot{V}_B = 0$ at $x = K$ and differentiating $(A11)$ yields that at $x = K$ (since $\dot{V}_A$ is discontinuous, interpret the expressions as right-hand derivatives)

$$\ddot{V}_A + \ddot{V}_B = \gamma\bar{H}\dot{V}_A \tag{A13}$$

Since $\dot{V}_A > 0$ to the right of $x = K$ this implies that $\dot{V}_A + \dot{V}_B > 0$ just to the right of $x = K$ and so $V_A + V_B$ is increasing. This contradicts the assumption that the maximum of $V_A + V_B$ for $x \geq K$ occurs at $x = K$.

The only possibility therefore is that $V_A + V_B < 2(1-\alpha)/(2r+\gamma\bar{H})$ for $x \geq K$. It follows that

$$V_A(K) = V_B(K) < \frac{1-\alpha}{2r+\gamma\bar{H}} \tag{A14}$$

Hence V_A is decreasing and H_A is increasing and so is strictly less than $\bar{H}$ for $x < K$.

Uniqueness of Equilibrium when $0 < \rho < \theta$.

Uniqueness of equilibrium when the system starts on the 45° line was established in the text. It therefore suffices to consider the case when $A > B$. The following lemma is useful:

Lemma A1 *If* $\theta > \rho > 0$ *and* $A > B$ *then if* $H_A(t) = \bar{H}$ *and* $H_B(t) = 0$ *then* $\dot{V}_A(t) > 0$ *and* $\dot{V}_B(t) < 0$. *Moreover if* $H_A(t) = \bar{H}$ *then* $H_A(s) = \bar{H}$ *for all* $s \geq t$.

This follows since from $(24a)$

$$\dot{V}_A = -\pi_A(t) + (r+\gamma\bar{H})V_A(t) \tag{A15}$$

From Lemma 1, A/B is increasing and research in A will never be higher than $\bar{H}$, so the same argument that established $(A7)$ shows that $V_A(t) > \pi_A(t)/(r+\gamma\bar{H})$ and so $\dot{V}_A(t) > 0$. Simililarly $V_B(t) < \pi_B(t)/r$ and so $\dot{V}_B(t) < 0$. Now suppose that at some subsequent times $H_B > 0$ and let T be the infimum of such times. By continuity of V_A and V_B, $V_A(T) = V_B(T)$. But $\dot{V}_A(t) > 0$ and $\dot{V}_B(t) < 0$ for $t < T$ contradicts the fact that for $t < T$ $V_A \geq V_B$ as $H_A > 0$.

From Lemma 1 it follows that A/B increases without bound and since $\pi_A(x) \to (1-\alpha)$ and $\pi_B(x) \to 0$ as $x \to \infty$, it follows that for sufficiently large x, $H_A = \bar{H}$. Applying Lemma A1 it follows that from any point x there exists a unique time τ, possibly zero, such that $H_A = \bar{H}$ for $t > \tau$ and, if $\tau > 0$, $H_A < \bar{H}$ for $t < \tau$. Using continuity of V_A and V_B it follows that if $\tau > 0$, $V_A(\tau) = V_B(\tau)$ and hence that at this point $x(\tau) = K$. It follows that if $x \geq K$, $H_A = \bar{H}$ for all subsequent times and if $x < K$, both H_A and H_B are positive until the system hits K and so evolves according to the equilibrium constructed above. This establishes uniqueness.

Proof of Lemma 2

Note that the argument of the first paragraph of Lemma 1,which rules out both H_A and H_B being positive when $A > B$ with $H_A \leq H_B$ applies when $\rho \geq \theta$. On the other hand if H_A and H_B are both positive when $A > B$ and the system never crosses the 45° line, then (28) yields a contradiction. Note finally, that if $H_A = 0$ when $A > B$, then the argument of the first two sentences of the third paragraph of Lemma 1 applies when $\rho \geq \theta$ and shows that the 45° line must be crossed before H_A becomes positive.

Proof of Proposition 5

Construction of reversing equilibrium

As in the construction of equilibrium for the case $0 < \rho < \theta$, let K be the value of $x = A/B$ such that the value of a patent in sector 1 equals that in sector 2 if $H_A = \bar{H}$ at all subsequent dates. Since $\rho > \theta$, $K < 1$. Now the argument leading to ($A14$) applies even if ρ exceeds θ. Hence provided that

$$\pi_B(K) - \pi_A(K) \leq \gamma \bar{H} V_A(K) \tag{$A16$}$$

the previous construction applies: for $1 > x \geq K$, H_B and H_A are both positive with H_B rising. At $x = K$, the system switches to $H_A = \bar{H}$.

($A16$) may but need not hold. If it does not, there are two possibilities. Consider setting $H_B = \bar{H}$ for all $1 > x > K$ and solve for V_A and V_B using ($A4$) with the boundary condition that $V_A(K)$ and $V_B(K)$ assume the above value. Denote the derived value functions $\underline{V}_A(x)$ and $\underline{V}_B(x)$. If ($A16$) fails then it follows from ($A4$) that $\underline{V}_B(x) > \underline{V}_A(x)$ in the neighbourhood of $x = K$ (for $x > K$). If $\underline{V}_B(x) \geq \underline{V}_A(x)$ for all $1 > x > K$ then setting $H_B = \bar{H}$ is an equilibrium. If not then let $\underline{x}$ be the value of x such that $\underline{V}_A(x) \leq \underline{V}_B(x)$ for $K \leq x \leq \underline{x}$ and $\underline{V}_A(x) > \underline{V}_B(x)$ for all $x > \underline{x}$ sufficiently close to $\underline{x}$. Now in time-dependent form the system evolves for $x < \underline{x}$ according to (note that $\dot{x} < 0$)

$$\dot{\underline{V}}_B = (r + \gamma\bar{H})\underline{V}_B - \pi_B \tag{$A17a$}$$

$$\dot{\underline{V}}_A = r\underline{V}_A - \pi_A \tag{$A17b$}$$

Now in that range $\underline{V}_A \leq \underline{V}_B$ and so it follows that

$$\dot{\underline{V}}_A + \dot{\underline{V}}_B \leq (2r + \gamma\bar{H})\underline{V}_B - (1-\alpha) \tag{$A18$}$$

Let $\hat{V} = (1-\alpha)/(2r + \gamma\bar{H})$. Now $\underline{V}_B(K) \leq \hat{V}$ from $(A14)$ and so $\dot{\underline{V}}_A + \dot{\underline{V}}_B \leq 0$ at $x = K$. Now this implies $V_A + V_B \leq 2\hat{V}$ for $x \leq \underline{x}$. For

$$\dot{\underline{V}}_A + \dot{\underline{V}}_B = r(\underline{V}_A + \underline{V}_B) + \gamma\bar{H}\underline{V}_B - (1-\alpha) \tag{A19}$$

Hence if $\underline{V}_A + \underline{V}_B > 2\hat{V}$, one must have $\dot{\underline{V}}_A + \dot{\underline{V}}_B > 0$ as $\underline{V}_B \geq \underline{V}_A$. This implies that that $\underline{V}_A + \underline{V}_B$ is increasing and $(A19)$ and the fact that $\underline{V}_B \geq \underline{V}_A$ imply that $\underline{V}_A + \underline{V}_B$ strictly exceeds $2\hat{V}$ for subsquent times and hence that $\underline{V}_B$ strictly exceeds $\hat{V}$. This contradicts the fact that $\underline{V}_B(K) \leq \hat{V}$.

It follows that $\underline{V}_B(\underline{x}) \leq \hat{V}$. Moreover the fact that $\underline{V}_A(x) > \underline{V}_B(x)$ for $x > \underline{x}$ implies that $\underline{V}'_B(\underline{x}) < \underline{V}'_A(\underline{x})$ and it follows from $(A4)$ and the equivalent equation for $\underline{V}_B$ that $\pi_B(\underline{x}) - \pi_A(\underline{x}) < \gamma\bar{H}\underline{V}(\underline{x})$. Hence it is an equilibrium for the system to evolve according to $(A5)$ and $(A6)$, with H_A and H_B both positive until it hits $\underline{x}$ and then set $H_B = \bar{H}$ until it hits $x = K$. Exactly one of the equilibria constructed must exist for all values of $\rho > \theta$. Numerical examples show one can find examples of all three.

Proof that there must be eventual specialisation when $\rho > \theta$

It follows from Lemma 2 that if the system does not cross the 45° line there must be complete specialisation. It will be shown that the system can only cross the 45° line a finite number of times and so eventually there must be complete specialisation. Now it follows from Lemma 2 that one cannot have $H_B \geq H_A > 0$ and $A > B$ and so if the system crosses the 45° line from $A > B$ one can find an interval of time $[0, T]$ for which $H_B = \bar{H}$ and $A(t) > B(t)$ and $A(T) = B(T)$. If the system is to cross the 45° line again then there must exist finite S such that $H_A(t) > 0$ for some interval beginning at $S + T$. Now

$$V_A(0) = \int_0^T e^{-rt}\pi_A(t)\,dt + \int_T^{S+T} e^{-rt}\pi_A(t)\,dt + e^{-r(S+T)}V_A(S+T) \tag{A20a}$$

$$V_B(0) = \int_0^T e^{-rt-\gamma\bar{H}t}\pi_B(t)\,dt + \int_T^{S+T} e^{-rt-\gamma\bar{H}}\pi_B(t)\,dt + e^{-(r+\gamma\bar{H})(S+T)}V_B(S+T) \tag{A20b}$$

Now since H_A becomes positive at $S+T$, $V_A(S+T) = V_B(S+T)$. Also, $\pi_A(T+\epsilon) = \pi_B(T-\epsilon)$ if $|\ \epsilon\ | < \max\{S, T\}$ and $\pi_A > \pi_B$ if $A > B$. Since γ is strictly positive it follows that for $V_B(0) \geq V_A(0)$ one must have $S > T$.

Suppose that starting from $A > B$ the system crosses the 45° line infinitely often. Denote by T_1 and S_1 the times derived above for the first crossing. Now after the first crossing, one has $H_B \geq H_A$, until $H_A = \bar{H}$ and the system crosses the 45° line again (by Lemma 2). If S'_1 denotes the time to reach the 45° line again and T'_1 the time until H_A becomes positive, then $S'_1 \geq S_1$ and a symmetric argument to that given above shows that $T'_1 > S'_1$. If T_2 denotes the time to hit the 45° line on the second crossing from $A > B$, then $T_2 \geq T'_2$. Arguing by induction, one has $T_1 < S_1 < T_2 < S_2 < \ldots < T_n < S_n < \ldots$, using obvious notation. Suppose that the sequence T_n is bounded above. Then S_n and T_n must tend to a common limit, $\underline{T}$ say.

Now from $(A20a)$ and $(20b)$, since $V_A(S+T) = V_B(S+T)$, for H_B to be positive on the nth crossing one must have

$$\int_0^{T_n} e^{-rt}\pi_A(t)\,dt + \int_{T_n}^{S_n+T_n} e^{-rt}\pi_A(t)\,dt \le \int_0^{T_n} e^{-rt-\gamma\bar{H}t}\pi_B(t)\,dt + \int_{T_n}^{S_n+T_n} e^{-rt-\gamma\bar{H}t}\pi_B(t)\,dt$$

But

$$\int_0^{\underline{T}} e^{-rt}\pi_A(t)\,dt + \int_{\underline{T}}^{\underline{T}+\underline{T}} e^{-rt}\pi_A(t)\,dt > \int_0^{\underline{T}} e^{-rt-\gamma\bar{H}t}\pi_B(t)\,dt + \int_{\underline{T}}^{\underline{T}+\underline{T}} e^{-rt-\gamma\bar{H}t}\pi_B(t)\,dt$$

and this contradicts the previous inequality for sufficiently large n since S_n and T_n tend to $\underline{T}$. (Note that if T_n converges to $\underline{T}$ then the initial starting points for each crossing must converge to x such that $xe^{-\delta\bar{H}\underline{T}} = 1$.) Hence T_n and hence S_n must be unbounded. This implies that the system must attain arbitrarily large and small values of x.

On the other hand, if x is sufficiently large then one must have $H_A = \bar{H}$ for all subsequent times since $\pi_A \to 1-\alpha$ and $\pi_B \to 0$ as $x \to \infty$. Similarly one must have $H_B = \bar{H}$ for all subsequent times if x is sufficiently small. It follows that the system cannot cross the 45° line infinitely often and so there must eventually be complete specialisation.

Proof of Proposition 6

First a preliminary lemma:

Lemma A2 *If $\rho < 0$ and $A > B$, then $H_A < H_B$ and if $A > B$ initially then the system converges to the 45° line but does not cross it.*

Reversing the argument of Lemma 1, if $H_A(t) \ge H_B(t)$ for all t then $V_A(0) < V_B(0)$ as profits are higher in sector 2 and competition is lower. If $H_B(t) > H_A(t)$ at some point let T be the infimum of such points. By continuity $V_A(T) = V_B(T)$. Since $H_A \ge H_B$ for all $t < T$, one cannot cross the 45° line. Again profits are lower and competition is greater in sector 1 and so $V_A(0) < V_B(0)$, a contradiction.

Starting from any point $H_B \ge H_A$. The system cannot be bounded away from the 45° line as then $H_B - H_A$ must become arbitrarily small for all but a arbitrarily small length of time. This is not possible as $\pi_B - \pi_A$ is bounded away from zero since $x = A/B$ is bounded away from unity, and if this were the case V_A would become strictly less than V_B and so H_B would have to equal $\bar{H}$.

The 45° line cannot be crossed, since starting from that point if the system moves to the region $A < B$ one must have $H_A < H_B$ initially. This contradicts the first part of the Lemma.

Construction of Equilibrium

As above, let $\hat{V} = (1-\alpha)/(2r+\gamma\bar{H})$. Define L to be the unique value of x such that

$$r\hat{V} = \pi_A(x) \qquad (A21)$$

$$(r+\gamma\bar{H})\hat{V} = \pi_B(x) \qquad (A22)$$

It is straightforward to check that setting $H_A = 0$ and $H_B = \bar{H}$ for $x > L$ and letting the system evolve for $1 \leq x \leq L$ according to ($A5$) with $V_A = V_B = \hat{V}$ is an equilibrium. Note that for $x > L$, $\dot{V}_A > 0$ as

$$\dot{V}_A = rV_A - \pi_A \tag{$A23$}$$

At $x = L$, $\dot{V}_A = 0$. If $\dot{V}_A < 0$ for x just greater than L, one would have $\dot{V}_A < 0$ at $x = L$ since π_A is increasing as $\rho < 0$. Hence $\dot{V}_A > 0$ for x just greater than L and using ($A23$), for all $x > L$. Similarly $\dot{V}_B < 0$ for $x > L$. Hence since $V_A(L) = V_B(L)$, $V_A(x) < V_B(x)$ for $x > L$ and so $H_B = \bar{H}$ is an equilibrium.

Uniqueness when $\rho < 0$

It is enough to consider the case $A > B$. Note that $H_A \leq H_B$. It will be shown that at a point where $H_A = 0$ just before that point and $H_A > 0$ just after (a 'switch point'), then $\dot{V}_A \geq 0$ and $\dot{V}_B \leq 0$.

Lemma A3 *H_A and H_B are continuous at a switch point and hence so are $\dot{V}_A$ and $\dot{V}_B$.*

From (24a) and (24b)

$$\dot{V}_A = (r + \gamma H_A)V_A - \pi_A \tag{$A24a$}$$
$$\dot{V}_B = (r + \gamma H_B)V_B - \pi_B \tag{$A24b$}$$

After the switch point $\dot{V}_A = \dot{V}_B$, so if there is a discontinuous rise in H_A at the switch point $\dot{V}_A < \dot{V}_B$ just before the switch point since V_A, V_B, π_A and π_B are continuous. This contradicts the fact that $V_A \leq V_B$ before the switch point and $V_A = V_B$ at the switch point. The continuity of H_B follows immediately and continuity of $\dot{V}_A$ and $\dot{V}_B$ follows from ($A24a$) and ($A24b$).

Suppose that the claim is false. That is at some switch point either $\dot{V}_A < 0$ or $\dot{V}_B > 0$. Consider the former case. Since $\dot{V}_A$ and $\dot{V}_B$ are continuous, it follows that $\dot{V}_A = \dot{V}_B < 0$ in a neighbourhood just after the switch. Adding (24a) and (24b) yields

$$\dot{V}_A = (r + \gamma\frac{\bar{H}}{2})V_A - \frac{(1-\alpha)}{2} \tag{$A25$}$$

It follows that in that region $V_A = V_B < \hat{V}$ and that V_A is decreasing so long as H_A is positive. Note that H_A is assumed piecewise continuous. If H_A were to become zero,

$$\dot{V}_A = rV_A - \pi_A \tag{$A26$}$$

If $\dot{V}_A$ is strictly negative before H_A becomes zero it must be negative after the drop since V_A and π_A are continuous and H_A has decreased (see ($A24$)). Since π_A is increasing it follows that $\dot{V}_A < 0$ while $H_A = 0$. At any subsequent switch point it follows that again $\dot{V}_A < 0$ and the same argument applies. It follows that if ever $\dot{V}_A < 0$ at a switch point, $\dot{V}_A < 0$ for all subsequent times and so, since V_A is bounded below it decreases to a limit, which must be less than $\hat{V}$. A similar argument shows that if $\dot{V}_B > 0$ at a switch point, $\dot{V}_B > 0$ for all subsequent times and so V_B increases to a limit, exceeding $\hat{V}$, since it is bounded above.

Either case yields a contradiction. By Lemma A2, the system converges to the 45° line but does not hit it. It follows that close enough to the 45° line H_A becomes arbitrarily close to H_B except for a proportion of time that tends to zero. Now when H_A is positive the system obeys ($A24a$) and ($A24b$), so since then H_A becomes arbitrarily close to H_B and π_B approaches π_A and the proportion of time for which $H_A = 0$ goes to zero, it follows that V_A and V_B must approach the common limit $\hat{V}$. This gives a contradiction.

It follows that $\dot{V}_A \geq 0$ and $\dot{V}_B \leq 0$ at a switch point. Since they are continuous there (Lemma A3), it follows that $\dot{V}_A = \dot{V}_B = 0$. From ($A24a$) and ($A24b$) it follows that this can only be true if $V_A = V_B = \hat{V}$. Since $H_B = \bar{H}$, it follows that the only possible switch point is $x = L$. On the other hand, there must be a switch point since otherwise the system would hit the 45° line in finite time. It follows that the unique equilibrium is the one constructed above.

FIGURES

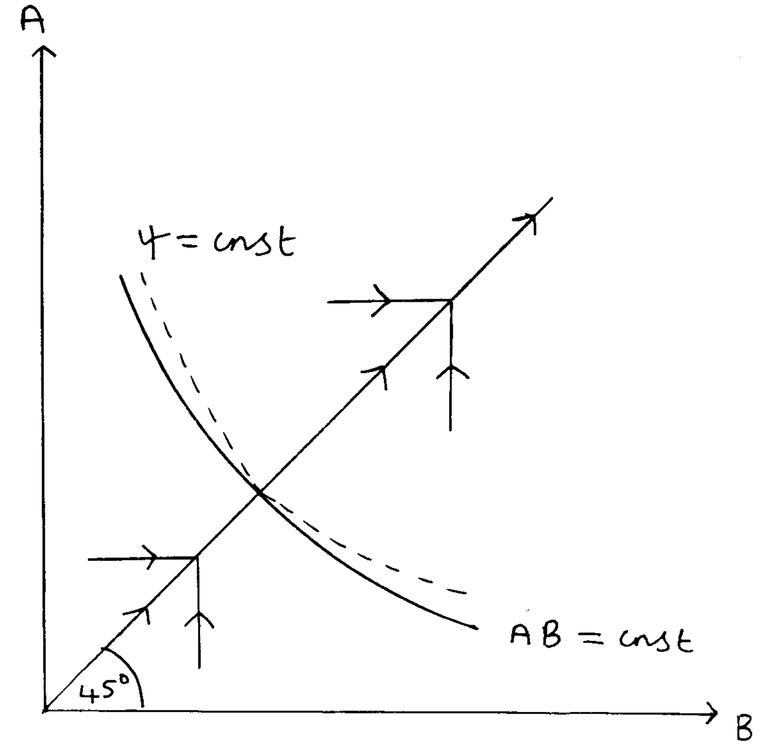

Figure 1

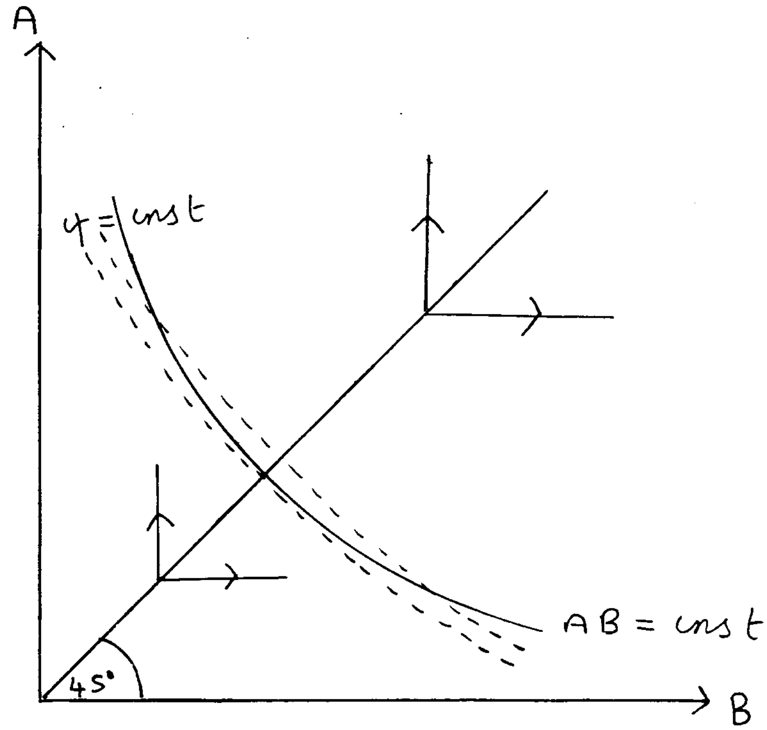

Figure 2

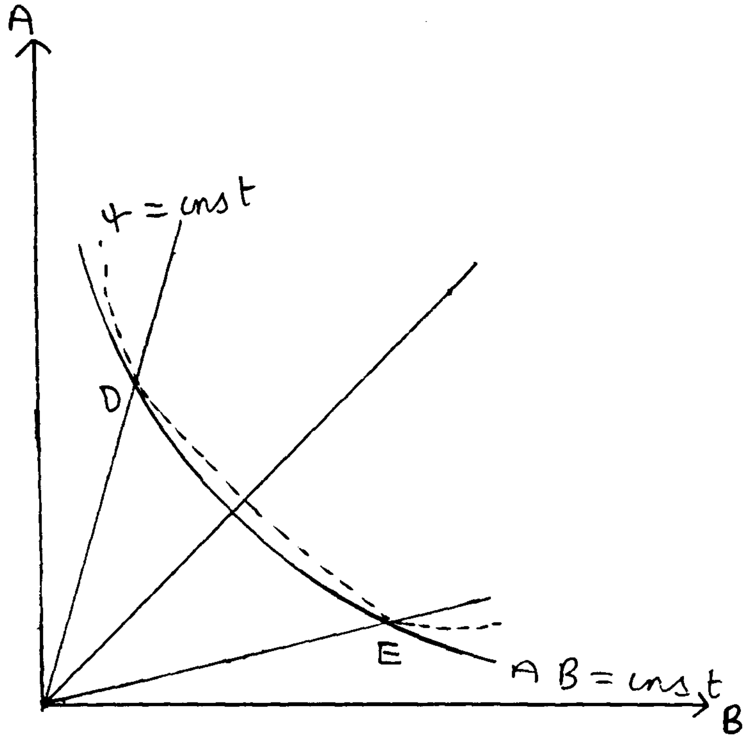

Figure 3

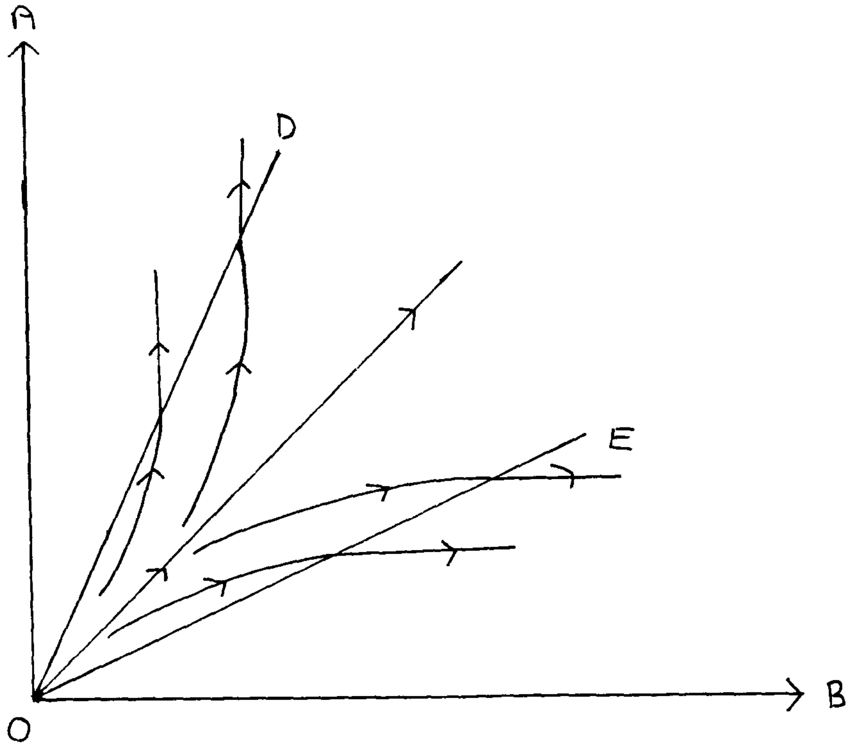

Figure 4

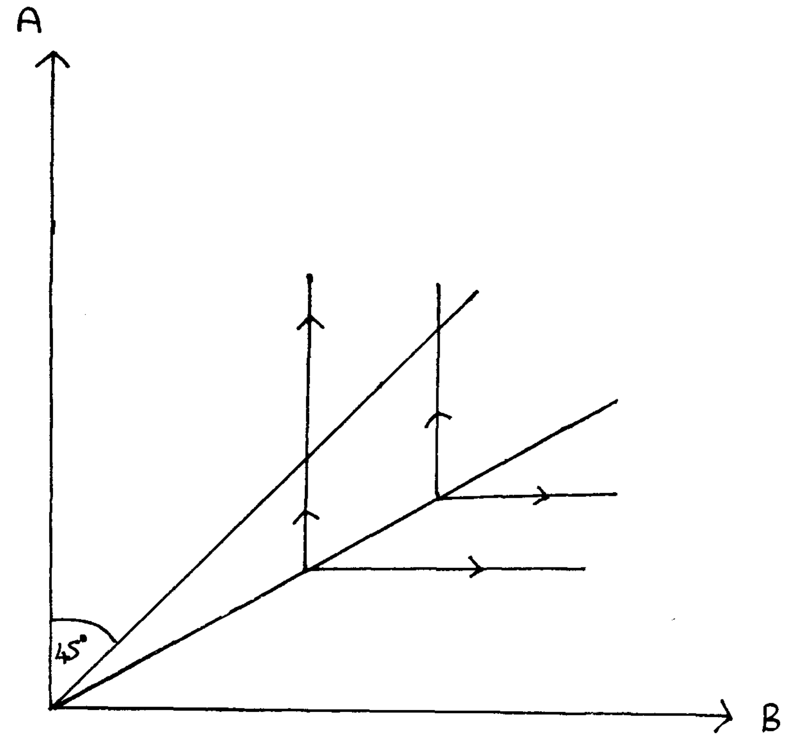

Figure 5

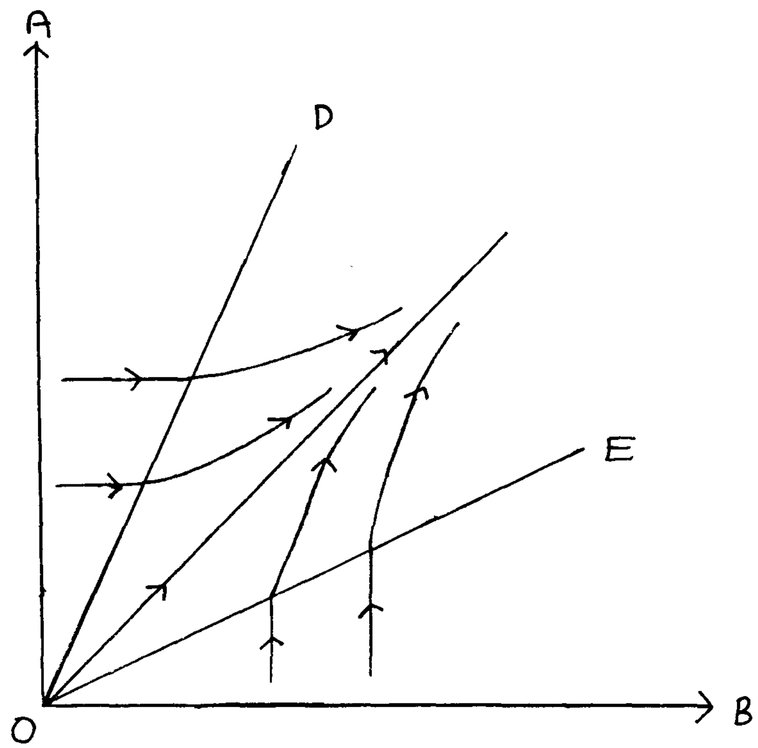

Figure 6