Advanced Macro: Growth Theory Lecture 2

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1 Human Capital and AK Models

There has been a lot of talk about the role of human capital in the growth process, so much so that human capital is now strongly part of the current credo about the origin of growth. But there is little agreement as to what is meant by human capital. For some, it is schooling and nutrition, which are clearly embodied in people; for others it is knowledge, which is disembodied.

The first version of the concept – especially schooling – has struck a chord with neo-classical growth economists when Barro and Mankiw, Romer and Weil demonstrated that adding education to a standard growth regression solves many of the problems. Since then, all growth regressions include schooling and justify the inclusion by the need to control for human capital. The rest of the theory is safe. MKW is a good illustration of this approach. The policy implication is straightforward: to get rich, not only do you need to work hard and be thrifty, you also need to study hard in school. Weberian ethics re-revisited. The second concept of human capital – knowledge – is quite different as knowledge is not embodied in people.

1.1 Human capital as schooling (MRW again)

The starting point is a straightforward extension to the Solow model:

$$\begin{aligned} Y(t) &= K(t)^{\alpha} H(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta} \\ \dot{k}(t) &= s_k y(t) - (n+g+\delta)k(t) \\ \dot{h}(t) &= s_h y(t) - (n+g+\delta)h(t) \end{aligned}$$

where $y = Y/AL$, $k = K/AL$ and $h = H/AL$. We assume that $\alpha + \beta < 1$, i.e., that there are decreasing returns to capital. Savings rate in physical capital s_k and human capital s_h are taken as exogenous. Note that physical and human capital are assumed to depreciate at the same rate. This assumption is not essential, however. The steady state is:

$$\begin{aligned} k^* &= \left(\frac{s_k^{1-\beta} s_h^{\beta}}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}} \\ h^* &= \left(\frac{s_k^{\alpha} s_h^{1-\alpha}}{n+g+\delta}\right)^{\frac{1}{1-\alpha-\beta}} \\ y^* &= \left(\frac{1}{n+g+\delta}\right)^{\frac{\alpha+\beta}{1-\alpha-\beta}} s_k^{\frac{\alpha}{1-\alpha-\beta}} s_h^{\frac{\beta}{1-\alpha-\beta}} \end{aligned}$$

from which we can get a gdp level equation:

$$\begin{aligned} \ln \frac{Y}{L} &= \ln(y^* A(0) e^{gt}) \\ &= \ln A(0) + gt + \frac{\alpha}{1-\alpha-\beta} \ln s_k + \frac{\beta}{1-\alpha-\beta} \ln s_h - \frac{\alpha+\beta}{1-\alpha-\beta} \ln(n+g+\delta) \end{aligned}$$

The advantage of this formulation is that, if human capital is treated just like another type of capital, the neo-classical model comes out essentially unscathed. The only difference is that the steady state level of output now depends on schooling as well as saving. Transition dynamics are unaffected: without technological change, the economy converges to its steady state; with technological change, the economy converges towards its balanced growth path, which is driven solely by technological change.

1.2 Human capital as knowledge (Romer 1986)

Romer speaks of human capital in his seminal 1986 paper (popularized by Lucas 1988), but he obviously thinks of something else than what MSW or Barro have in mind. The starting point of his model is a production function of the form

$$Q(k(t), K(t), x(t))$$

where $k(t)$ is the knowledge of the individual firm, $K(t)$ is the knowledge of the economy as a whole, and $x(t)$ stands for intermediate inputs (that in this context include equipment and labor). All agents are price takers and take $K(t)$ as given; it is an externality. The accumulation of knowledge follows

$$\frac{\dot{k}}{k} = g\left(\frac{I}{k}\right)$$

To close the model, we have a representative consumer maximizing discounted utility

$$\max \int_0^\infty e^{-\delta t} U(c(t)) dt$$

The first step in solving the model is to 'optimize out' the choice of $x(t)$. Since the choice of intermediate inputs is made within each period (this is obviously a long term model), it can be optimized out. If x^* is the optimal choice of x for given values of k and K, then define:

$$f(k, K) \equiv Q(k, K, x^*)$$

The second step is to assume symmetry, i.e., that all S firms are the same size. (This is not an obvious assumption since, later, Romer assumes that production is constant returns to scale.) This means that $K = Sk$. The third step is to define a function $F(k) \equiv f(k, Sk)$. Next, Romer defines two optimization frameworks. The first framework is a simple social planner framework in which the externality is internalized. The second framework is a competitive equilibrium framework.

1.2.1 Social planner framework (externality internalized)

$$\max \int_0^\infty e^{-\delta t} U(c(t))dt \quad \text{subject to}$$

$$\frac{\dot{k}(t)}{k(t)} = g\left(\frac{F(k(t)) - c(t)}{k(t)}\right)$$

$$\dot{k}(t) \geq 0 \text{ for all } t \text{ and } k(0) = k_0$$

Romer first has a theorem to say that the solution to the above optimization problem exists, i.e., is bounded/does not explode. To characterize the solution, he uses the Hamiltonian approach. Romer writes the Hamiltonian as:

$$H(k, \lambda) \equiv \max_c U(c) + \lambda \left[kg(\frac{F(k) - c}{k}) \right]$$

The solution path is defined by a system of first order conditions:

$$\begin{aligned} \dot{k}. &= H_\lambda(k, \lambda) \\ \dot{\lambda} &= \delta\lambda - H_k(k, \lambda) \\ U'(c) &= \lambda g'\left(\frac{F(k) - c}{k}\right) \end{aligned}$$

Using these three equations, Romer is able to characterize the bounds and the shape of the phase diagram [see paper]. Using the transversality condition,

$$\lim_{t\to\infty} \lambda(t)k(t)e^{-\delta t} = 0$$

he eliminates paths that diverge, that is, paths that cross $\dot{\lambda} = 0$. This demonstrates that the economy can growth without bounds by accumulating knowledge.

1.2.2 Competitive equilibrium framework

The trick here is to split the derivation of the competitive equilibrium in two steps. First, Romer treats the entire path of aggregate knowledge as given. For a given path of knowledge, the competitive equilibrium corresponds to a social planner problem with the path of K given. This optimization problem is:

$$\max \int_0^\infty e^{-\delta t} U(c(t))dt \quad \text{subject to}$$

$$\frac{\dot{k}(t)}{k(t)} = g\left(\frac{f(k(t), K(t)) - c(t)}{k(t)}\right)$$

$$\dot{k}(t) \geq 0 \text{ for all } t \text{ and } k(0) = k_0$$

The above can be solved for any path of $K(t)$. The second step is then to look for the path of $K(t)$ such that the individual accumulation of knowledge generates exactly $K(t)$, i.e., the path for which $K(t) = Sk(t)$.

Formally, this means forming the Hamiltonian

$$\widetilde{H}(k, \lambda, K) \equiv \max_c U(c) + \lambda \left[kg(\frac{f(k, K) - c}{k})\right]$$

where $K(t)$ is regarded as exogenously given. The solution path is defined as before by a system of first order conditions:

$$\begin{aligned} \dot{k}. &= \widetilde{H}_\lambda(k, \lambda, K) \\ \dot{\lambda} &= \delta\lambda - \widetilde{H}_k(k, \lambda, K) \\ U'(c) &= \lambda g' \left(\frac{F(k) - c}{k}\right) \end{aligned}$$

The second step is then to replace K with Sk in all the above first order conditions. These conditions can similarly be manipulated to find $\dot{k} = 0$ and $\dot{\lambda} = 0$ and to construct a phase diagram very similar to the other one, i.e., with unbounded growth.

Of course, this growth could slow down or accelerate over time. To get a sense of this, Romer considers a special case with a Cobb-Douglas formulation

$$f(k, K) = k^\nu K^\gamma$$

He illustrates permanent growth with this example and also uses it to compare competitive growth and socially optimal growth, which is higher since the social marginal return to knowledge is higher than the private return. See the paper for details.

1.3 The AK model/Romer revisited

The AK model can be seen as a special case of Romer's model. The attraction of the AK model is that, with suitable functional form assumptions, it is amenable to a rigorous treatment. We start from Romer's model but ignore the issue of externalities which are unessential for the permanent growth. We consider the following optimal control problem:

$$\max \int_0^{\infty} e^{-\delta t} U(c(t))dt \quad \text{subject to}$$

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= g\left(\frac{F(k(t)) - c(t)}{k(t)}\right) \\ k(0) &= k_0 \\ \lim_{t\to\infty} \lambda(t)k(t)e^{-\delta t} &= 0 \end{aligned}$$

$U(c)$ is a utility function, k stands for knowledge, c is consumption, $F(k)$ is a production function, and $g(.)$ relates investment in knowledge to growth in the stock of knowledge. The solution is obtained by forming the Hamiltonian

$$H(c,k,\lambda) \equiv U(c) + \lambda k g\left(\frac{F(k) - c}{k}\right)$$

and deriving the first order conditions:

$$\begin{aligned} \dot{k}. &= kg\left(\frac{F(k(t)) - c(t)}{k(t)}\right) \\ \dot{\lambda} &= \lambda\left(\delta - G(k,c)\right) \\ U'(c) &= \lambda g'\left(\frac{F(k) - c}{k}\right) \end{aligned}$$

where:

$$G(k,c) \equiv \frac{\partial}{\partial k}\left[kg\left(\frac{F(k) - c}{k}\right)\right]$$

If $G(k,c)$ is decreasing in k such that, for any (reasonable) value of c, there exist a value of k^* with $G(k^*,c) = \delta$, then the above system is identical to the standard neo-classical growth model and it can be analyzed in the same manner.[1]

By analogy with Jones and Manuelli, if $G(k,c) > \delta$ for all k, permanent unbounded growth obtains. Along an unbounded growth path, we note that any path that crosses the $\dot{k} = 0$ locus from above is ultimately infeasible. Consequently, the optimal path must remain above the $\dot{k} = 0$ locus. This implies permanent unbounded growth. Paths along which k grows faster than λ falls

[1] Romer focuses instead on the case where the economy exhibits global increasing returns such that $G(\widehat{k},c) > \delta$ for all $k > \widehat{k}$. Romer claims that, in this case, the economy experiences permanent unbounded growth. To demonstrate this, Romer draws a phase diagram where the $\dot{\lambda} = 0$ locus is downward sloping and never intersects the $\dot{k} = 0$ locus. From the fact that the $\dot{\lambda} = 0$ locus is everywhere above the $\dot{k} = 0$ locus, Romer deducts that permanent growth is possible.

ultimately violate the transversality condition. Along the optimal path, λ must fall fast enough so that $\lim_{t\to\infty} \lambda(t)k(t)e^{-\delta t} = 0$.

To illustrate permanent unbounded growth, we consider the AK model as a special case of Romer's model. Let $g(x) = x$, $F(k) = ak$, and $U(c) = \log(c)$. We assume that $a > \delta$. The Hamiltonian is:

$$H(c, k, \lambda) \equiv \log(c) + \lambda(ak - c)$$

and the canonical equations become:

$$\begin{aligned} \dot{k} &= ak - \frac{1}{\lambda} \\ \dot{\lambda} &= \lambda(\delta - a) \end{aligned}$$

where we have made use of the first order condition $\frac{1}{c} = \lambda$ to replace c in the first equation. The general solution to this system of differential equations is of the form:

$$\begin{aligned} \lambda(t) &= \lambda(0)e^{(\delta-a)t} \\ k(t) &= d(0)e^{at} + \frac{e^{(a-\delta)t}}{\delta\lambda(0)} \end{aligned}$$

where $\lambda(0)$ and $\delta(0)$ are initial conditions yet to be determined. We first eliminate paths that violate the transversality condition. After some straightforward algebra, we obtain:

$$\lambda(t)k(t)e^{-\delta t} = d(0)\lambda(0) + \frac{e^{-\delta t}}{\delta}$$

From the above, it clear that $d(0) = 0$ is required for the transversality condition to be satisfied. Using the initial condition $k(0) = k_0$, we then solve for $\lambda(0)$ using:

$$k_0 = k(0) = \frac{1}{\delta\lambda(0)}$$

The optimum path is thus:

$$\begin{aligned} k(t) &= k_0 e^{(a-\delta)t} \\ \lambda(t) &= \frac{e^{(\delta-a)t}}{\delta k_0} \end{aligned}$$

from which we see that knowledge is growing at constant rate $a - \delta$. Illustrate the optimum path. As anticipated, the optimum path is everywhere above the $\dot{k} = 0$ locus: $\lambda = \frac{1}{\delta k}$ along the optimal path; $\lambda = \frac{1}{ak}$ along the $\dot{k} = 0$ locus.

1.4 Human capital as skill (Lucas 1988)

Lucas also proposed a model of growth based on human capital accumulation. His model is closer to the standard neo-classical model and is easier to integrate to the neo-classical framework. Lucas envisions an economy of infinitely lived workers. Their skill level index h can be any positive number in $[0, \infty)$. The notation $N(h)$ denotes the number of workers with skill level h. In equilibrium, all workers will be the same so that this notation becomes superfluous.

From these two assumptions, it is already doubtful that the model will be a reasonable representation of the accumulation of skills in workers. For one thing, actual workers are not infinitely lived; whatever skill they learn is lost when they die. This means that whatever skills the previous generation gained has to be relearned by the new generation. Also, making the skill level take any value in $[0, \infty)$ is not a realistic approximation of the skills actual workers can gain in the market place. At any point in time, achievable skill levels are likely to bounded. The bound may go up as one adds to the stock of knowledge, though. It is like saying that workers (students) can learn more skills when there is more knowledge to be learned. In this sense, Lucas' model is not very different from that of Romer 1986. But let's continue with the model.

Lucas assumes that at any point in time a fraction $u(h)$ of workers of skill level h participate to production; the others are in school acquiring new skills. At any point in time the workforce measured in skill efficiency units is denoted:

$$N^e = \int_0^\infty u(h)N(h)h \, dh$$

Production is written $Y = F(K, N^e)$. Workers are paid their efficiency wage. The hourly wage is thus

$$F_N(K, N^e)h$$

The average level of skills in the economy is

$$h_a = \frac{\int_0^\infty hN(h)dh}{\int_0^\infty N(h)dh}$$

it is regarded as an external effect (similar to Romer's model).

Now, assume that all workers are identical. The notation simplifies to $N^e = uhN$ and $h_a = h$. The budget constraint of the economy is

$$N(t)c(t) + \dot{K}(t) = A(t)K(t)^\beta [u(t)h(t)N(t)]^{1-\beta} h_a^\gamma$$

The skill variable under the $1 - \beta$ exponent is the direct effect, which is fully internalized by workers (i.e., it is reflected in their wage). The skill variable h_a^γ denotes a external effect. The structure of the model is thus very close to that of Romer, except that it includes capital and workers. The law of motion for human capital is:

$$\dot{h}(t) = h(t)^\zeta G(1 - u(t)) \text{ with } G' > 0 \text{ and } G(0) = 0$$

1.4.1 Case 1: $\zeta < 1$

In this case we get

$$\frac{\dot{h}(t)}{h(t)} \leq h(t)^{\zeta - 1} G(1)$$

so that $\dot{h}/h$ must eventually tend to 0. This means that in this case the accumulation of skills cannot serve as permanent engine of growth. We still need technological change in $A(t)$ to explain growth in the long run.

1.4.2 Case 2: $\zeta = 1$ and $G(x) = \delta x$

With these assumptions, the accumulation of human capital is linear in human capital itself, i.e., we have:

$$\dot{h} = h\delta(1-u)$$

The analysis of this case can then proceed as in Romer 1986. Lucas sets up a social planner problem (in which externalities are internalized) and a perfect competition problem (in which they are not). The utility function of the representative consumer is unchanged, i.e.:

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} N(t) dt$$

where ρ is the discount rate and $N(t)$ is (exogenously determined) population. Maximizing the above subject to the budget constraint and the law of motion of human capital is accomplished by constructing the Hamiltonian:

$$H(K, h, \theta_1, \theta_2, c, u, t) = \frac{N}{1-\sigma}(c^{1-\sigma} - 1) +$$

$$\theta_1[AK^\beta [uhN]^{1-\beta} h^\gamma - Nc] + \theta_2[\delta h(1-u)]$$

where θ_1 and θ_2 are the costate variables associated with the two constraints. This is a more complicated Hamiltonian because there are more variables floating around, but the logic is the same. To solve it, you must differentiate with respect to c and u (the instantaneous decisions) and use optimal control theory. This yields a bunch of first order-like equation. Combined with two transversality conditions (one for each constraint) and with initial value for capital and skill level, they characterize the solution to the problem. Similar work can be done on the competitive economy, using the same trick as in Romer 1986.

As it turns out, this economy has a balanced growth path where the time spent learning $1-u$ is constant across periods. [Lucas does not, however, formally proves that the economy would converge to this balanced path from any starting point.] Along the balanced growth path, we get:

$$\begin{aligned} \frac{\dot{c}}{c} &= \kappa \\ \frac{\dot{h}}{h} &\equiv \nu = \delta(1-u) \end{aligned}$$

After some (relatively straightforward) algebra, Lucas can derive the balanced growth rate of the economy as a function of initial parameters for the social planner case (first best) and the competitive equilibrium case (externalities not internalized). He gets:

$$\nu^* = \frac{1}{\sigma}[\delta - \frac{1-\beta}{1-\beta+\gamma}(\rho - \lambda)] \quad \text{(first best)}$$

$$\nu^* = \frac{1}{\sigma(1-\beta+\gamma)-\gamma}[(1-\beta)(\delta - \rho + \lambda)] \quad \text{(competitive equilibrium)}$$

where, as before, ρ stands for discount rate and λ stands for population growth. Here we assume that technological change $\mu = 0$. The above calculations thus demonstrate that indefinite growth can be achieved without technological change. The accumulation of 'skills', which is endogenous to the model, is by itself capable of generating never ending growth.

What is remarkable from the above equations is that balanced growth is possible even if $\gamma = 0$, that is, even if there are no externalities. In this case the balanced growth rate is

$$\nu = \frac{\delta - \rho + \lambda}{\sigma}$$

This means that externalities are not essential to the growth process with skill/human capital accumulation. If we combine this observation with the earlier result that permanent growth requires that $\zeta = 1$, we get what is usually known as the AK model idea: if an economy has an infinitely accumulable factor whose productivity does not drop over time, then growth can continue indefinitely. Here, the AK idea is a little obscured because, at first glance, it appears that output is less than linear in human capital h. This is forgetting that the rate at which human capital is accumulated increases proportionally with the stock of human capital, i.e., $\dot{h} = h\delta(1-u)$. As the economy grows, the human capital stock grows exponentially, and the rate of growth of skills remains constant. In this sense it is an AK model: output can be forever increased by adding to the stock of human capital.

1.5 Conclusion on human capital

Thanks to the work of Romer 1986 and Lucas 1988, the process of long term growth has been endogenized. Their work is based on the abstract but intuitive idea that we live better than 200 years ago not so much because we have more equipment but because we know more about the world. Growth is the result of scientific discovery and the accumulation of technological knowledge.

The way knowledge affects output remains, however, very abstract. Also this early literature confuses infinitely accumulable knowledge and skills or schooling embodied in workers, which by definition cannot be infinitely accumulated. This confusion may come from the fact that knowledge is spread across the population through education.

We have essentially two types of mechanisms in front of us. The first one is the accumulation of private skills/knowledge (Lucas with $\gamma = 0$). This process alone can generate permanent growth provided that there are increasing returns in the accumulation of knowledge (captured in $\dot{h} = h\delta(1-u)$). The second mechanism is knowledge spillovers. These externalities can speed up growth or make it 'easier' from a purely mathematical point of view. But they are not necessary to construct a model of endogenous growth. This issue has attracted a lot of attention because the presence of externalities has serious policy implications. Those who dislike government intervention have been prompt to latch onto Lucas' model and claim that policy intervention is not required for permanent growth. As is often the case in this literature, people choose their models not based on what is reasonable but based on the policy prescriptions they generate.

Our next job is to examine the process of knowledge accumulation more in detail.

2 Learning by doing: Young's model

Several authors have proposed a view of growth as fueled by learning by doing. These authors think of growth as a process by which new goods are invented and where the invention process benefits from advances made in learning how to make other goods. We review one such model by Young, which builds on an earlier model by Stokey (1988). In this model the learning process is country specific, without spillovers across countries.

Young proposes a simple model in which permanent growth is fueled by learning by doing. The attraction of his model is that it has two economies trading with each other. The model is thus about the interaction between growth and trade. It is the first such model we examine in this module. Young assumes that growth is based on learning by doing which has spillovers on other (neighboring) industries/goods. He also assumes that learning is bounded for each good, i.e., once we have learned how to produce a hammer, there is no more learning. He considers two countries, each with its own learning. There are no knowledge/learning spillovers across the two countries. The effect of trade on growth thus depends on whether the static comparative advantage of the countries induces specialization in goods for which learning by doing is mostly exhausted or not.

The model has the following features: continuum of goods; labor sole factor; preferences symmetric and separable; firms behave in a perfectly competitive manner; growth occurs through a changing basket of goods, with additional welfare gains due to increasing variety. Increase in product diversity is a welfare gain from which both countries benefit, whether or not they are learning and see their production grow. In this limited sense, trade is welfare increasing always.

2.1 General autarky case

We begin with the autarky case. The notation of Young's model is like this. Goods s are indexed along the continuum $[0, \infty)$; only one dimension is considered. They are ordered according to the sophistication of their production process. Labor is the sole factor of production and output is linear in labor.

Firms thus behave in a perfectly competitive manner. The labor requirement for a good s has a lower bound $\bar{a}(s)$ which is assumed to be continuous and non-increasing in s. In other words, the cost of production of more sophisticated goods is lower than simpler goods once all the gains from learning have been achieved. All goods are normalized by the effect they have on utility, i.e., are measured in util units. The actual labor requirements at any point in time t is denoted $a(s,t)$ and is assumed to be such that $\lim_{s\to\infty} a(s,t) = \infty$ [in finite time, there are always more sophisticated goods that cannot be produced].

Learning by doing is modeled as follow. First of all, since labor requirements are bounded from below, if $a(s,t) = \bar{a}(s)$ then $\frac{\partial a(s,t)}{\partial t} = 0$. If $a(s,t) > \bar{a}(s)$, however, there is learning in the sense that

$$\frac{\dot{a}(s,t)}{a(s,t)} = -\int_0^\infty B(s,v,\frac{a(v,t)}{\bar{a}(v)})L(v,t)dv$$

where $L(v,t)$ is labor allocated to the production of good v at time t and function $B(.)$ is the learning by doing function. The following assumptions are made:

1. $B(s,v,\frac{a(v,t)}{\bar{a}(v)}) \geq 0$ for all v [output of other industries cannot increase my production costs].

2. $B(s,v,1) = 0$ [industries for which learning by doing is exhausted do not contribute to reduce production costs in other industries]

3. $B(s,v,\frac{a(v,t)}{\bar{a}(v)}) > 0$ for all $v \in (s-\alpha_s, s+\alpha_s)$ such that $a(v,t) > \bar{a}(v)$. [learning by doing is local, i.e., my industry gains mostly from learning in neighboring/similar industries]

4. $\sup_v \sup_s B(s,v,\frac{a(v,t)}{\bar{a}(v)}) < \infty$ [learning is not instantaneous]

5. $B(s,v,\frac{a(v,t)}{\bar{a}(v)})$ is continuous in s [technical assumption needed for the math]

Consumers maximize $\int_0^\infty u(c(s,t))ds$ with $u'(0) < \infty$ and $u(.)$ concave. Consumers thus have a preference for diversity but it is possible for them to consume a zero quantity of certain goods. Finally, there is no storage and labor is in fixed supply. The wage rate is taken as numeraire, which means that the price of goods is $a(s,t)$. There is no accumulation. The optimization problem facing a representative consumer at each point in time is:

$$\max_{\{c(s,t)\}} V(t) \equiv \int_0^\infty u(c(s,t))ds \text{ subject to}$$

$$wl = 1 \geq \int_0^\infty a(s,t)c(s,t)ds$$

since $l = 1$ by normalization of units and $w = 1$ because it is the numeraire. For simplicity, the intertemporal discount factor is assumed to be one – an innocuous assumption since there is no accumulation in the model. Since there is no accumulation, we can solve each period separately. The first order condition takes the usual form:

$$u'(c(s,t)) = \lambda a(s,t) \text{ for all } s,t$$

where λ is the Lagrange multiplier (and marginal utility of income at the optimum). In equilibrium, we must therefore have for all consumed goods:

$$\frac{u'(c(s,t))}{a(s,t)} = \frac{u'(c(v,t))}{a(v,t)} = \lambda$$

Young then tries to characterize the optimum. He notes that at any point in time there is a good for which $a(s,t)$ is lowest. This good must be in the consumption basket. Other goods at slightly higher prices are also consumed, albeit in smaller quantities. Then there is a level of price at which all income has been spent. Goods with production cost above that price are not. [show graph]. [For instance, if $U(c) = c^{\beta}$, then the consumer would spend the same amount on each good. The maximum price would be for the good that exhausts the budget constraint.]

Having characterized the instantaneous equilibrium, Young then moves to the traditional measure of growth in GDP per head $g(t)$. Of course this measure underestimates the growth in welfare since it does not take into account the increase in welfare due to the fall of prices over time. We have:

$$g(t) = \frac{\int_0^\infty a(s,t)\frac{\partial X(s,t)}{\partial t}ds}{\int_0^\infty a(s,t)X(s,t)ds} - \frac{\frac{dL}{dt}}{L(t)}$$

where the first term is instantaneous growth in total output $X(s,t) = x(s,t)L$, weighted by initial prices $a(s,t)$, minus growth in the labor force/ population. Note that the index basis is constantly updated. Now, since the value of all output is equal to the wage bill, the denominator of the first term is simply $L(t)$. After some manipulation, Young gets:

$$g(t) = \frac{-\int_0^\infty \frac{\partial a(s,t)}{\partial t}X(s,t)ds}{L(t)}$$

which shows that growth takes place as long as production costs are decreasing, that is, as long as there is learning by doing. Moreover, provided that $\lim_{s\to\infty} \bar{a}(s) = 0$, then $\lim_{t\to\infty} V(t) = \infty$: there is unbounded growth.

2.2 Special autarky case

Before we use the model to examine international trade, we need to simplify it a bit as follows:

1. $V(t) = \int_0^\infty \log(c(s,t)+1)ds$.
2. $\bar{a}(s) = \bar{a}e^{-s}$.

3. $B(s, v, \frac{a(v,t)}{\bar{a}(v)}) = 2$ for all s, v such that $\frac{a(v,t)}{\bar{a}(v)} > 1$.

With these additional assumptions, the economy becomes 'symmetric' in the sense that *if*, at an arbitrary time 0, $a(s,0) = \bar{a}e^{-s}$ for all $s \leq T(0)$ and $a(s,0) = \bar{a}e^{-T(0)}e^{s-T(0)}$ for all $s \geq T(0)$, then for all t we have that $a(s,t) = \bar{a}e^{-s}$ for all $s \leq T(t)$ and $a(s,t) = \bar{a}e^{-T(t)}e^{s-T(t)}$ for all $s \geq T(t)$ [show graph] and

$$\frac{dT(t)}{dt} = \int_{T(t)}^{\infty} L(s,t)ds$$

that is, growth in $T(t)$ is proportional to labor allocation to the production of the goods for which there is still learning.

With the new utility function, it is also possible to show that as $T(t)$ increases, so does the range of goods being produced. In addition, the range of produced goods moves to the right, that is, low end goods are abandoned in favor of more complex goods for which production costs have dropped enough to make them affordable. Welfare thus increases because both the range and quantity of goods improve.

Finally, if we dub $T(t)$ 'technical progress', it can be shown that

$$g(t) = \frac{dT(t)}{dt} = \frac{L(t)}{2}$$

2.3 Equilibrium path with trade

Now we consider two economies trading with each other under free trade and perfect competition. One is supposed to be an LDC, the other is a DC*, hence we assume $T^* > T$. Population is constant in both economies and they have identical preferences. Most importantly, there are no spillovers of knowledge/learning by doing from one economy to the other. Since $T^* > T$, we have that $a^*(s) \leq a(s)$ with strict inequality for all $s > T$: the DC has lower labor requirements for all goods of index greater than T [see graph].

We are interested in the technology gap between the two economies, $X \equiv T^* - T$ [beware of the change in notation: here X is not output but technology gap]. Let the wage rate in the LDC be the numeraire, that is, define the wage gap as $\omega \equiv \frac{w^*}{w}$ and define the price as $p(s) \equiv \frac{P(s)}{w}$. We have $p(s) = a(s)$ for goods that are produced in the LDC and $p(s) = \omega a^*(s)$ for goods produced in the DC. From this, Young computes budget constraints and the trade balance. He gets five types of production configurations under free trade, coded A, B, C, D, and E. [see graphs]. Configuration A occurs when $\omega = 1$ [the two countries have the same real wage; this is also the lowest possible wage for the DC]; configuration B occurs when $\omega = e^{2X}$ [highest possible relative wage for the DC]. The other three categories of configurations occur when the wage ratio is in between these two bounds. Young then examines which configuration arise along the equilibrium path depending upon initial conditions.

1. When $L^* > L$ [large DC], the first configuration is with $\omega = 1$ [same wages]; the configuration is thus A. Then the large DC progressively transfers the production of goods to the LDC until all goods from lowest M to middle T are produced in the LDC. Then the wage in the DC rises above that in the LDC; the configuration is C. Since H [the intersection point, see graph] is above T, there is still technical progress/learning in the LDC, but at a rate slower than in the DC. As a result the two economies progressively shift to configurations D and E [see graph]

2. When $L^* = L$, for small technological gap X, we initially get configuration C. Then moves to D [see graph]

3. When $L^* = L$ [small DC], the first configuration is B for small enough X. In other words, the relative wage of the DC is at its highest. Then progressively the production of goods in the overlap [see graph] are transferred to the LDC until all goods in the upper learning curve are produced in the DC. At that point, the relative wage of the DC ω begins to fall, moving the configuration to C. The equilibrium then stays in configuration C [see graph].

4. There also exists a special case whereby the technology gap remains constant over time; see paper.

Except for the special case, in all equilibria we have (1) $\frac{dX}{dt} > 0$: a small initial technological advantage gets reinforced over time; the advanced country gets 'dynamic gains from trade'; (2) $g^* > g$: the high-tech economy grows faster because there is more learning taking place. Welfare implications are ambiguous, though. Immediate welfare gains from trade are always positive; this is because of comparative advantage. The DC country always benefit from trade [except in the special case when the initial technology gap is small and the DC country has a small population/production, in which case catching up is possible]. Depending on the rate at which the future is discounted, the LDC gains from trade in 'most' cases but there exist cases in which it loses, meaning that the immediate gains from trade are more than compensated by slower growth in the future. Gains are more likely if the LDC country is small, losses are more likely if the LDC country is large. In the latter case, the LDC would benefit from imposing autarky in the hope of capturing technical progress in the future. In this case welfare will eventually rise above what it would have been without autarky, but the immediate welfare cost is high [consumers pay more for less diversified and less sophisticated products; think of China or former USSR].

The aggressive policy conclusions that come out of this model have a Schumpeterian flavor: try to gain an edge by subsidizing industries where you wish to catch up [e.g., European airplane industry]. Extending this model to two LDC's would trigger trade wars as both LDC's try to gain an edge on the other. This is a world dominated by survival of the fittest, not nice optimization at the margin. Conclusions would be very different, however, if there were technological spillovers from trade, i.e., if the LDC could learn to produce sophisticated goods by trading. This is a hotly debated empirical issue.

2.4 Conclusion on learning by doing

Stokey and Young's models are about learning by doing. They also are about economies in which growth takes place through the creation of new goods. Long term growth takes place because the cost of producing old goods and of starting the production of new goods falls with experience in the production of old goods. In that sense, the models combine learning by doing with the accumulation of knowledge which makes the production of new goods possible. Both models also generate specialization and complexification of the economy over time, features that were noted as early as by Adam Smith himself. Trade patterns (in Young's model) are also more convincing, albeit in a fairly demoralizing way.

What the models do not do, however, is to depart from the perfect competition assumption at the firm level. Learning takes the form of perfect externalities. There is no deliberate investment in the invention of new goods or in the reduction of production costs. These are models of technological progress without research and development. The accumulation of knowledge just happens, nobody plans or controls it. In that sense, these models (as those of Romer 1986 and Lucas 1988) are still not truly endogenous growth models: the engine of growth itself is still ultimately out of one's control. Incorporating deliberate research and development investment is the object of the next section.

Young's models is also disturbing for another reason. He shows that the size of an economy matters in the sense that the long term prospects are better for large countries than for small ones [an American bias?]. Empirical evidence is nevertheless replete with examples in small size was not an impediment to fast growth. If we examine the model closer, however, we realize that what defines a 'country' in Young's model is simply the boundary of learning by doing: a country is the unit that shares the learning by doing externalities. It is very doubtful that externality spillovers perfectly fit country boundaries. There may be situations in which they spill over across boundaries (e.g., from England to Belgium in the early 1800's, for instance), and other situations where they remain concentrated within the same country (e.g., the eastern seaboard in the US). This brings to light the possible role of geographical and social (communication) proximity in the spreading of learning by doing externalities.

There is also an empirical literature on learning by doing, which uses are more precise, narrow definition of learning by doing as embedded in workers or organizations. This work usually shows that for a firm learning how to produce a product only takes a few months, maximum a few years if the product is very complex (e.g., a ship). The productivity gains can be very large but they are finite, i.e., they converge to a lower limit on costs ($\bar{a}(s)$ in Young's notation). Evidence on learning by doing externalities is more confused, although some evidence suggests that workers take away some of the skills they learn to other firms, either as workers or as new entrepreneurs.