Part II: Comparative Statics for decisions under uncertainty and Comparative Information

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Decisions Under Uncertainty

Recall that a family $\{f(\cdot,s)\}_{s\in S}$ is an SCP family (for $S\subseteq\mathbb{R}$) if for any $x''>x'$, the function $\Delta:S\to\mathbb{R}$ defined by

$$\Delta(s)=f(x'',s)-f(x',s)$$

crosses the horizontal axis at most once.

Definition: A function $\Delta:S\to\mathbb{R}$ is said to have property ($\star$) if, whenever $s''>s'$,

$$\Delta(s')\geq\ (>)\ 0\Longrightarrow\Delta(s'')\geq\ (>)\ 0.$$

Definition: Consider the family $h(\cdot,t):S\to\mathbb{R}$ where $S\subseteq\mathbb{R}$. This family obeys log increasing differences if $\{\ln h(\cdot,t)\}_{t\in T}$ obeys increasing differences. Equivalently, whenever $s''>s'$,

$$\frac{h(s'',t)}{h(s',t)}\ \text{increases with } t.$$

Decisions Under Uncertainty

The following result is very useful in applications.

Theorem 6: Suppose $S \subseteq \mathbb{R}$, the function $\Delta : S \to \mathbb{R}$ has property ($\star$), and $\{h(\cdot, t)\}_{t \in T}$ obeys log increasing differences. Then

$$\phi(t) = \int_S \Delta(s) h(s,t) ds \text{ has property } (\star).$$

Indeed, if Δ crosses the horizontal axis at $s = s_0$, then, for $t_2 > t_1$,

$$\phi(t_2) \geq \frac{h(s_0, t_2)}{h(s_0, t_1)} \phi(t_1).$$

Decisions Under Uncertainty

Proof: We split the integral $\phi(t_2) = \int_S \Delta(s)h(s,t_2)ds$ into two:

$$\phi(t_2) = \int_{-\infty}^{s_0} \Delta(s)h(s,t_1)\,\frac{h(s,t_2)}{h(s,t_1)}ds + \int_{s_0}^{\infty} \Delta(s)h(s,t_1)\,\frac{h(s,t_2)}{h(s,t_1)}ds.$$

Between $[-\infty, s_0]$, $\Delta(s) \leq 0$, so the first term on the right is greater than

$$\frac{h(s_0,t_2)}{h(s_0,t_1)} \int_{-\infty}^{s_0} \Delta(s)h(s,t_1)\,ds.$$

Between $[s_0.\infty]$, $\Delta(s) \geq 0$, so the second term is greater than

$$\frac{h(s_0,t_2)}{h(s_0,t_1)} \int_{s_0}^{\infty} \Delta(s)h(s,t_1)\,ds.$$

Adding up the two lower bounds gives us

$$\phi(t_2) \geq \frac{h(s_0,t_2)}{h(s_0,t_1)} \int_S \Delta(s)h(s,t_1)\,ds = \frac{h(s_0,t_2)}{h(s_0,t_1)}\phi(t_1).$$

. QED

Decisions Under Uncertainty

Application: consider Bernoulli utility functions $v(\cdot, t)$ parameterized by t. Assume that the family $\{v'(\cdot, t)\}_{t \in T}$ obeys log increasing differences. This is equivalent to

$$-\frac{v''(z, t_2)}{v'(z, t_2)} \leq -\frac{v''(z, t_1)}{v'(z, t_1)}$$

whenever $t_2 > t_1$; i.e., type t_2 has a lower coefficient of risk aversion than type t_1.

Agent has wealth w and chooses between a safe and risky asset to maximize expected utility

$$V(x, t) = \int v((w - x)r + xs, t)\, \lambda(s)\, ds$$

– r and s are payoffs of safe and risky assets, latter with density λ; $x \in [0, w]$ is the investment in the risky asset.

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We want to show that investment in the risky asset increases with t, i.e., investment in the risky asset increases as risk aversion decreases.

This is a well-known result but the standard proof uses a first order argument that assumes that the agent is risk averse (so that the objective function is concave in the choice variable). Notice that our argument in the next slide does not rely on that assumption, i.e., it does not require u to be concave.

Decisions Under Uncertainty

Recall $V(x,t) = \int v((w-x)r + xs, t)\,\lambda(s)\,ds$. Differentiating w.r.t x,

$$V'(x,t) = \int_{s \in S} (s-r)\lambda(s)\,v'((w-x)r + xs, t)\,ds.$$

By Theorem 6 and the fact that $\{v'(\cdot,t)\}_{t \in T}$ obeys log increasing differences, for any $t_2 > t_1$,

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So whenever $x'' > x'$,

$$V(x'',t_2) - V(x',t_2) \geq \frac{v'(wr,t_2)}{v'(wr,t_1)} \left[V(x'',t_1) - V(x',t_1)\right].$$

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Clearly, the family $\{V(\cdot,t)\}_{t\in T}$ obeys SCP.

By Theorem 1, $\text{argmax}_{x\in[0,w]} V(x,t)$ is increasing in t,

i.e., the less risk averse agent invests more in the risky asset.

Decisions Under Uncertainty

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Definition: A family of density functions $\{\lambda(\cdot, t)\}_{t \in T}$ (defined on $S \subseteq \mathbb{R}$) that obeys log increasing differences is said to respect the monotone likelihood ratio (MLR) order.

Loosely speaking, an agent with beliefs represented by the density function $\lambda(\cdot, t'')$ thinks that higher states are more likely than an agent with beliefs represented by $\lambda(\cdot, t')$, if $t'' > t'$.

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Fact: If $\{\lambda(\cdot, t)\}_{t \in T}$ is MLR-ordered then it is ordered by first order stochastic dominance (FOSD), i.e., whenever $t_2 > t_1$, $\lambda(\cdot, t_2)$ first order stochastically dominates $\lambda(\cdot, t_1)$.

Decisions Under Uncertainty

Theorem 7: Suppose $\{f(\cdot, s)\}_{s\in S}$ (for $S \subseteq \mathbb{R}$) is an SCP family. Then the family $\{F(\cdot, t)\}_{t\in T}$ given by

$$F(x,t) = \int_{s\in S} f(x,s)\,\lambda(s,t)\,ds$$

is also an SCP family if $\{\lambda(\cdot, t)\}_{t\in T}$ is MLR-ordered.

[Consequently, $\text{argmax}_{x\in X} F(x, t'') \geq \text{argmax}_{x\in X} F(x, t')$ if $t'' > t'$.]

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Interpretation: Since $\{f(\cdot, s)\}_{s\in S}$ is an SCP family, Theorem 1 guarantees that $\text{argmax}_{x\in X} f(x, s'') \geq \text{argmax}_{x\in X} f(x, s')$, i.e., the optimal action is increasing in s if s is known.

Even though the action is taken *before* s is realized, we would expect the optimal action to be higher if higher states are more likely. Theorem 7 formalizes this intuition.

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$$F(x,t) = \int_{s\in S} f(x,s)\,\lambda(s,t)\,ds$$

is also an SCP family if $\{\lambda(\cdot, t)\}_{t\in T}$ is MLR-ordered.

Proof: The function $\Delta(s) = f(x'', s) - f(x', s)$ has property ($\star$). By Theorem 4, the function ϕ given by

$$\begin{aligned} \phi(t) = F(x'',t) - F(x',t) &= \int_{s\in S} [f(x'',s) - f(x',s)]\,\lambda(s,t)\,ds \\ &= \int_{s\in S} \Delta(s)\,\lambda(s,t)\,ds \end{aligned}$$

also obeys property ($\star$). In other words, $\{F(\cdot, t)\}_{t\in T}$ is an SCP family. QED

Decisions Under Uncertainty

Theorem 8: Suppose $\{f(\cdot, s)\}_{s\in S}$ (for $S \subseteq \mathbb{R}$) satisfies increasing differences. Then the family $\{F(\cdot, t)\}_{t\in T}$ given by

$$F(x,t) = \int_{s\in S} f(x,s)\,\lambda(s,t)\,ds$$

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Decisions Under Uncertainty

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also satisfies increasing differences if $\{\lambda(\cdot, t)\}_{t\in T}$ is FOSD-ordered.

Proof: Since $\{f(\cdot, s)\}_{s\in S}$ satisfies increasing differences, for $x'' > x'$, the function $\Delta(s) = f(x'', s) - f(x', s)$ is increasing. By a standard result (Rothschild-Stiglitiz),

$$\int_S \Delta(s) f(s,t_2) ds \geq \int_S \Delta(s) f(s,t_1) ds \text{ if } t_2 > t_1$$

Re-writing this expression, we obtain

$$V(x'', t_2) - V(x', t_2) \geq V(x'', t_1) - V(x', t_1),$$

as required by increasing differences. QED

Decisions Under Uncertainty

Application: In the standard portfolio problem, $\{f(\cdot, s)\}_{s\in S}$ is given by

$$f(x, s) = u((w - x)r + xs) = u(wr + x(s - r))$$

and is an SCP family:

for $s < r$, $f(\cdot, s)$ is decreasing in x;
for $s = r$, $f(\cdot, r) \equiv u(wr)$; and
for $s > r$, $f(\cdot, s)$ is increasing in x.

By Theorem 7, if $t'' > t'$, then
$\text{argmax}_{x\in[0,w]} F(x, t'') \geq \text{argmax}_{x\in[0,w]} F(x, t')$ where

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Interpretation: an MLR shift in the payoff distribution $\lambda(\cdot, t)$ means that higher payoffs for the risky asset are more likely, which raises investment in the risky asset.

An information structure

Setting: Agent chooses action after observing a signal $z \in \mathbb{R}$, but before realization of state.
Distribution over signals at a given state is $h(z|s)$.

The family $\{h(\cdot|s)\}_{s \in S}$ is the information structure H.

Assume that distributions are MLR-ordered, i.e., for $z'' > z'$

$$\frac{h(z''|s)}{h(z'|s)} \text{ is increasing in } s.$$

Higher states make higher signals more likely.

An information structure

Let P be agent's prior on S. Given P, agent can work out the posterior distributions $\{\tilde{h}_P(\cdot|z)\}_{z\in Z}$.

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Fact: If $\{h(\cdot|s)\}_{s\in S}$ is an MLR-ordered family then $\{\tilde{h}_P(\cdot|z)\}_{z\in Z}$ is also an MLR-ordered family.
Higher signals imply that higher states more likely.

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Higher signals imply that higher states more likely.

Theorem 9: Suppose $\{h(\cdot|s)\}_{s\in S}$ is MLR-ordered and $\{f(\cdot,s)\}_{s\in S}$ is an SCP family of functions. Then, for any prior P, agent has an increasing decision rule, i.e., there is

$$\phi(z) \in \text{argmax}_{x\in X} \int_{s\in S} f(x,s)\tilde{h}_P(s,z)ds$$

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such that ϕ is increasing in z.

Proof: Follows immediately from Theorem 7 and Fact stated above.

Comparing information structures

Assume prior is P and consider information structure H. If ϕ_H is the optimal decision rule, then agent's ex ante utility is

$$\mathcal{V}(H,P) = \int_{z \in Z} \left[\int_{s \in S} f(\phi_H(z), s) d\tilde{h}_P(s|z) \right] d\nu_H$$

where ν_H is the marginal distribution of z.

We wish to compare H with another information structure $G = \{g(\cdot|s)\}_{s \in S}$. Suppose its optimal decision rule is ϕ_G, so

$$\mathcal{V}(G,P) = \int_{z \in Z} \left[\int_{s \in S} f(\phi_G(z), s) d\tilde{g}_P(s|z) \right] d\nu_G.$$

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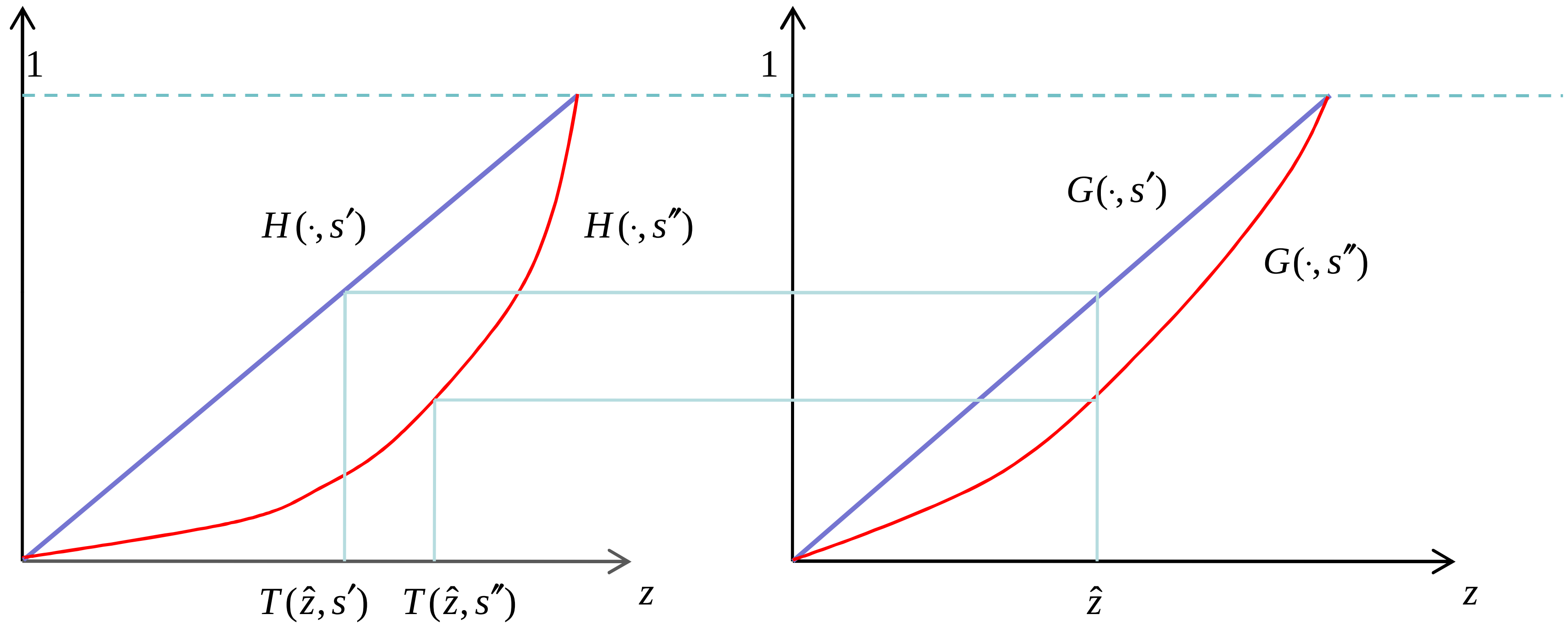
When is $\mathcal{V}(H,P) \geq \mathcal{V}(G,P)$ for all P?

Comparing information structures

Let G and H be two information structures

$H(\cdot|s)$ is distribution of signal z conditional on state s for H.
Similarly, we have $G(\cdot|s)$ is the distribution of z conditional on s for G.
Let $T(z,s)$ satisfy $H(T(z,s)|s) = G(z|s)$.

Definition: (Lehmann) H is *more informative* than G if $T(z,\cdot)$ is increasing in s.



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(ii) $\{f(\cdot, s)\}_{s \in S}$ is an SCP family
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Then $\mathcal{V}(H, P) \geq \mathcal{V}(G, P)$.

Comparing information structures

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(ii) $\{f(\cdot, s)\}_{s \in S}$ is an SCP family
(iii) at the prior P, the optimal decision rule for G, ϕ_G, is increasing in z

Then $\mathcal{V}(H, P) \geq \mathcal{V}(G, P)$.

Note: By Theorem 9, (iii) holds if $G = \{g(\cdot|s)\}_{s \in S}$ is an MLR-ordered family.

Comparing information structures

A version of Theorem 10 was first proved by Lehmann. Lehmann also obtains $\mathcal{V}(H, P) \geq \mathcal{V}(G, P)$, but under a subtly different set of assumptions.

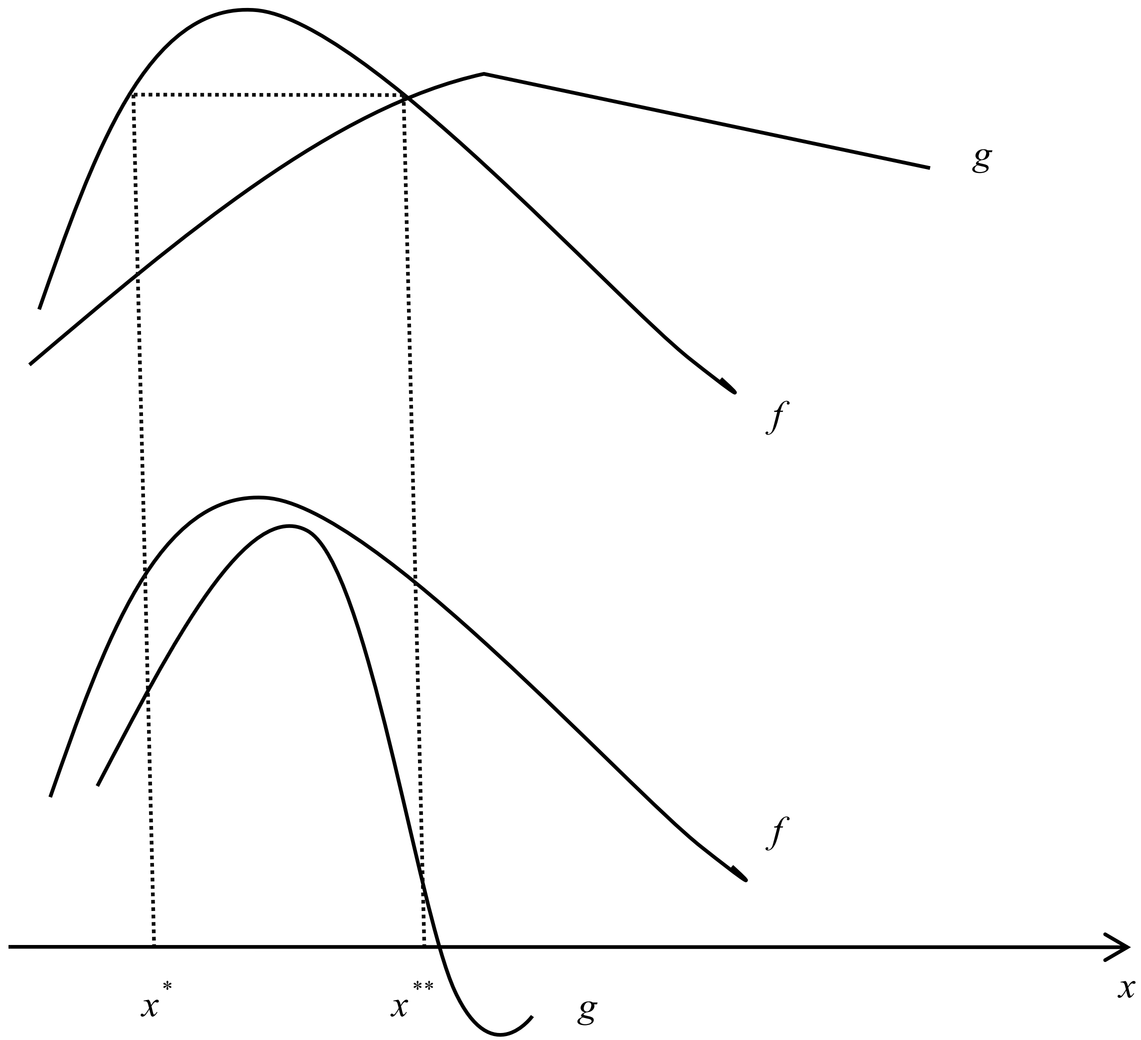
Instead of (ii), Lehmann assumes the following:
(ii)' $\{f(\cdot, s)\}_{s \in S}$ is a family of quasiconcave functions with $\text{argmax}_{x \in X} f(x, s)$ increasing in s.

Must a family of functions satisfying (ii)' be an SCP family.

It seems not...

Comparing information structures

In both pictures, the functions are concave and the peak of g is to the right of the peak of f, so Lehmann's condition (ii)' is satisfied. But g does not SCP-dominate f in the lower picture.



Comparing information structures

Quah and Strulovici (to appear, Econometrica) develop a new way of ordering functions, called the interval dominance order, which generalizes both the SCP family and Lehmann's condition (ii)'.

In other words, an SCP family and a family obeying (ii)' will both satisfy the interval dominance order.

Virtually all the results we have surveyed remain true if the SCP assumption is replaced with the weaker assumption that objective functions are ordered according to the interval dominance order.