Lectures on General Equilibrium Theory Michaelmas 2008

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Financial Assets

Assume that there are two dates, Today (date 0) and Tomorrow (date 1). There are L states of the world tomorrow.

Security/Asset is a promise of payment (positive or negative), conditional on the realization of the state.

We write payoff of security s as a column vector, called the payoff vector of security s:

$$\begin{pmatrix} d_{1s} \\ d_{2s} \\ d_{3s} \\ \cdot \\ \cdot \\ d_{Ls} \end{pmatrix}$$

Payoff Matrix

If economy has S securities (called 1, 2,..., S), then the payoff matrix is

$$D = \begin{pmatrix} d_{11} & d_{12} & . & . & d_{1S} \\ d_{21} & d_{22} & . & . & d_{2S} \\ d_{31} & d_{32} & . & . & d_{3S} \\ . & . & . & . & . \\ . & . & . & . & . \\ d_{L1} & d_{L2} & . & . & d_{LS} \end{pmatrix}.$$

Trade in securities/portfolio represented by the column vector z in R^S.

If $z_s > 0$ then agent is buying asset s;
if $z_s < 0$ then agent is selling asset s.

Contingent consumption

A portfolio z (in R^S) gives a payoff in state i of

$$z_1 d_{i1} + z_2 d_{i2} + ... z_S d_{iS}.$$

This is the ith entry in the column vector Dz in R^L since

$$Dz = z_1 \begin{pmatrix} d_{11} \\ d_{21} \\ . \\ . \\ d_{L1} \end{pmatrix} + z_2 \begin{pmatrix} d_{12} \\ d_{22} \\ . \\ . \\ d_{L2} \end{pmatrix} + ... z_S \begin{pmatrix} d_{1S} \\ d_{2S} \\ . \\ . \\ d_{LS} \end{pmatrix}.$$

So Dz is the payoff vector of the portfolio z.

The subspace $\{Dz; z \in R^S\}$, denoted by Span(D), is called the span of the security payoffs or the asset span.

Incomplete Markets

If Span(D) is a *strict* subspace of R^L, then the economy has incomplete markets.

Suppose agent has endowment $\omega = (\omega_0, \omega_1, ..., \omega_L) \in R_+^{L+1}$,

where ω_0 in R_+ is agent's endowment at $t = 0$ and $\omega_{-0} = (\omega_1, \omega_2, ..., \omega_L)$ is his endowment at $t = 1$.

With a portfolio $\bar{z}$, the agent's contingent consumption at $t = 1$ is $\omega_{-0} + D\bar{z}$.

With incomplete markets, there are contingent consumption bundles that the agent cannot achieve even if he could assemble any portfolio he likes, i.e.,

there is $x^* \in R_+^L$ such that there is no z with $\omega_{-0} + Dz = x^*$.

Incomplete Markets

Example: Suppose there are three states of the world at $t = 1$ and just two securities, 1 and 2, with payoff vectors $(1, 1, 0)$ and $(0, 1, 2)$ respectively.

An agent's endowment at $t = 1$ is $\omega_{-0} = (2, 3, 0)$. The portfolio $z = (z_1, z_2)$ gives this agent contingent consumption of

$$\omega_{-0} + Dz = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Note that $M = \{\omega_{-0} + Dz : z \in R^2\}$ is a plane in R^3 passing through the point ω_{-0}.

In particular $R^3_+ \not\subseteq M$ – so market is incomplete.

Agent’s utility maximization problem

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$$U(x_0, x_{-0}) = u(x_0) + \delta \left[\pi_1 v(x_1) + \pi_2 v(x_2) + ... + \pi_L v(x_L) \right].$$

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His budget set is

$$B(q, \omega, D) = \left\{ \begin{array}{ll} x \in R_+^{L+1} : & x_0 \leq \omega_0 - q \cdot z \\ & x_{-0} \leq \omega_{-0} + Dz \\ & \text{for some } z \in R^S \end{array} \right\}.$$

Agent's utility maximization problem

Agent maximizes $U(x)$ subject to x in $B(q, \omega, D)$.

Then $\hat{x} \in \text{argmax}_{x \in B(q,\omega,D)} U(x)$ is the agent's demand for contingent consumption and $\hat{z}$ such that

$$\begin{aligned} \hat{x}_0 &\leq \omega_0 - q \cdot \hat{z} \text{ and} \\ \hat{x}_{-0} &\leq \omega_{-0} + D\hat{z} \end{aligned}$$

is the agent's demand for securities.

Note: if U is strictly monotone (obeys (P2)), then the inequalities above are equalities.

Example

Agent's utility is $U(x_0, x_1, x_2) = \ln x_0 + \frac{1}{2}\ln x_1 + \frac{1}{2}\ln x_2$

Two securities with payoff vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

Price is $q = (1, 2)$. Endowment $\omega = (3, 0, 0)$.

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$$\text{Budget Set} = \left\{ \begin{array}{ll} x \in R^3_+ : & x_0 \leq 3 - z_1 - 2z_2 \\ & x_1 \leq z_1 + z_2 \\ & x_2 \leq z_2 \text{ for some } z = (z_1, z_2) \end{array} \right\}.$$

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Solution: $\bar{z}_1 = 0$, $\bar{z}_2 = \frac{3}{4}$,

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Maximize $\ln(3 - z_1 - 2z_2) + \frac{1}{2} \ln(z_1 + z_2) + \frac{1}{2} \ln z_2$.

Solution: $\bar{z}_1 = 0$, $\bar{z}_2 = \frac{3}{4}$, $\bar{x}_0 = \frac{3}{2}$, $\bar{x}_1 = \frac{3}{4}$, $\bar{x}_2 = \frac{3}{4}$.

The Financial Economy

A financial economy $\mathcal{F}$ consists of a payoff matrix D and a set A of agents, each of whom has an endowment $\omega^a = (\omega^a_0, \omega^a_{-0})$ in R^{L+1}_+ and a utility function $U^a : R^{L+1}_+ \rightarrow R$.

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(i) for each agent a, there is $\hat{z}^a$ such that $D\hat{z}^a$ maximizes $U^a(x)$ in $B(q^*, \omega^a, D)$ and

(ii) $\sum_{a \in A} \hat{z}^a = 0$.

Equilibrium in a financial economy

Proposition: *Suppose that q^* is an equilibrium price of $\mathcal{F}$ and let $\hat{z}^a$ and $\hat{x}^a$ be agent a's asset and consumption demand respectively. Then, provided U^a obeys (P2),*

$$\sum_{a \in A} \hat{x}^a = \sum_{a \in A} \omega^a.$$

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$$\sum_{a \in A} \hat{x}^a = \sum_{a \in A} \omega^a.$$

Proof: For agent a, we have $\hat{x}^a_0 = \omega_0 - q^* \cdot \hat{z}^a$. Sum across a, we obtain

$$\sum_{a \in A} \hat{x}^a_0 = \sum_{a \in A} \omega^a_0.$$

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Proof: For agent a, we have $\hat{x}_0^a = \omega_0 - q^* \cdot \hat{z}^a$. Sum across a, we obtain

$$\sum_{a \in A} \hat{x}_0^a = \sum_{a \in A} \omega_0^a.$$

Since $\hat{x}^a{}_{-0} = \hat{\omega}^a{}_{-0} + D\hat{z}^a$, summing across a gives us

$$\sum_{a \in A} \hat{x}^a{}_{-0} = \sum_{a \in A} \omega^a{}_{-0}.$$

.

QED

Constrained Pareto optimality

The allocation $\{x^a\}_{a \in A}$ is feasible if $\sum_{a \in A} x^a = \sum_{a \in A} \omega^a$.

It is a constrained feasible allocation if it is feasible and there is $\{z^a\}_{a \in A}$ such that $\sum_{a \in A} z^a = 0$ and

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An allocation $\{x^a\}_{a\in A}$ is constrained Pareto optimal if there does not exist a constrained feasible allocation $\{\bar{x}^a\}_{a\in A}$ that is Pareto superior, i.e.,

$$U^a(\bar{x}^a) \geq U^a(x^a)$$

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Theorem [First welfare theorem for financial economies]:
Suppose that U^a obeys (P2) for all a. Then every equilibrium allocation is constrained Pareto optimal.

Constrained Pareto optimality

Proof: Let q^* be the equilibrium price and $\{\hat{x}^a\}_{a \in A}$ the equilibrium allocation. Assume that $\{\bar{x}^a\}_{a \in A}$ is a Pareto superior constrained feasible allocation.

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For some agent $\tilde{a}$, we have $U^{\tilde{a}}(\bar{x}^{\tilde{a}}) > U^{\tilde{a}}(\hat{x}^{\tilde{a}})$.
So the bundle $\bar{x}^{\tilde{a}}$ cannot be in $\tilde{a}$'s budget set - if it were, agent $\tilde{a}$ would have chosen this bundle instead of $\hat{x}^{\tilde{a}}$.

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$$\bar{x}_0^{\tilde{a}} > \omega_0^{\tilde{a}} - q^* \cdot \bar{z}^{\tilde{a}}.$$

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Suppose not, then there is $\epsilon > 0$ such that $\bar{x}_0^a + \epsilon < \omega_0^a - q^* \cdot \bar{z}^a$. So $\tilde{x}^a = \bar{x}^a + (\epsilon, 0, ..., 0)$ is in $B(q^*, \omega, D)$ and

$$U^a(\tilde{x}^a) > U^a(\bar{x}^a) \geq U^a(\hat{x}^a).$$

This cannot happen since $\hat{x}^a$ is a's demand.

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Summing across a, we obtain

$$\begin{aligned} \sum_{a \in A} \bar{x}_0^a &> \sum_{a \in A} \omega_0^a - q^* \cdot \left(\sum_{a \in A} \bar{z}^a \right) \\ &= \sum_{a \in A} \omega_0^a, \end{aligned}$$

which is a contradiction. **QED**

Invariance

Consider a financial economy and keep endowments and preferences fixed. Then a change in securities that leaves the span unchanged does not change the equilibrium in an essential way.

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Lemma: Let $\{1, 2, ..., S\}$ be a set of securities with no redundant securities and let D be its payoff matrix. Let there be another set of $|S|$ securities, with payoff matrix D' such that Span (D)=Span (D').

Then there is invertible matrix $S \times S$ matrix K such that $DK = D'$.

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Proof: Exercise!

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Example: Suppose $D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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$$\begin{aligned} D' = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= DK \end{aligned}$$

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Theorem: Suppose q^* is an equilibrium price of $\mathcal{F}(D)$. At that price, let $\hat{z}^a$ be agent a's equilibrium portfolio, which achieves consumption of $\hat{x}^a$.

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Then the economy $\mathcal{F}(D')$, where $D' = DK$ for some invertible K, has the equilibrium price q^*K. The equilibrium portfolio of agent a is $K^{-1}\hat{z}^a$ and his consumption is $\hat{x}^a$.

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Proof: Note that $\sum_{a\in A} \hat{z}^a = 0$ implies $\sum_{a\in A} K^{-1}\hat{z}^a = 0$. Furthermore,

if $x \in B(q^*, \omega^a, D)$ and achieved by z, then $x \in B(q^*K, \omega^a, D')$ and achieved by $K^{-1}z$.

And vice versa. So $B(q^*, \omega^a, D) = B(q^*K, \omega^a, D')$.

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And vice versa. So $B(q^*, \omega^a, D) = B(q^*K, \omega^a, D')$.
It follows that $\hat{x}^a \in \text{argmax}_{x \in B(q^*K, \omega^a, D')} U^a(x)$ and is achieved by $K^{-1}\hat{z}^a$.

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To show: if $x \in B(q^*, \omega^a, D)$ and achieved by z, then $x \in B(q^* K, \omega^a, D')$ and achieved by $K^{-1}z$.

Since $x = (x_0, x_{-0})$ is in $B(q^*, \omega^a, D)$ and achieved by z,

$$\begin{aligned} x_0 &\leq \omega_0^a - q^* \cdot z \\ x_{-0} &\leq \omega_{-0}^a + Dz \end{aligned}$$

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$$\begin{array}{rcl} x_0 & \leq & \omega_0^a - q^* \cdot z \\ x_{-0} & \leq & \omega_{-0}^a + Dz \end{array}$$

Given that $D' = DK$, we have

$$\begin{array}{rcl} x_0 & \leq & \omega_0^a - q^*K \cdot (K^{-1}z) \\ x_{-0} & \leq & \omega_{-0}^a + D'K^{-1}z \end{array}$$

. **QED**

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Security prices $q \in R^S$ admits arbitrage if there is $\bar{z} \in R^S$ such that $q \cdot \bar{z} \leq 0$ and $D\bar{z} \geq 0$, with either inequality strict.

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An equilibrium price cannot admit arbitrage because if q^* admits arbitrage the problem $\max_{x \in B(q^*, \omega^a, D)} U^a(x)$ has no solution:

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An equilibrium price cannot admit arbitrage because if q^* admits arbitrage the problem $\max_{x \in B(q^*, \omega^a, D)} U^a(x)$ has no solution:

for any $x = (x_0, x_{-0})$ in $B(q^*, \omega, D)$, the bundle $x + (-q \cdot \bar{z}, D\bar{z})$ is also in $B(q^*, \omega, D)$ and, by (P2), has a higher utility.

Invariance

Security prices $q \in R^S$ admits arbitrage if there is $\bar{z} \in R^S$ such that $q \cdot \bar{z} \leq 0$ and $D\bar{z} \geq 0$, with either inequality strict.

An equilibrium price cannot admit arbitrage because if q^* admits arbitrage the problem $\max_{x \in B(q^*, \omega^a, D)} U^a(x)$ has no solution:

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Fundamental Theorem: If $q^* \in R^S$ admits no arbitrage, then there is $p = (p_1, p_2, ..., p_L) \gg 0$ (state prices) such that

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More succinctly, $q = pD$.